Chapter 2 - Limits

Tuesday, September 8, 2020 8:06 AM

Intro: All of calculus is based on the idea of a limit.

Limits underlie the two fundametal operations

1. Differentiation
2. Integration

$d = t \cdot \underline{\underline{v}} \Rightarrow v = \frac{d}{t}$

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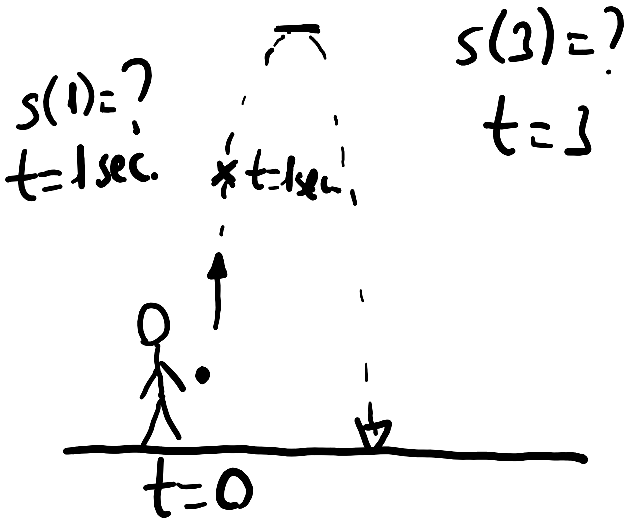


EXAMPLE 1 **Average velocity** A rock is launched vertically upward from the ground with a speed of 96 ft/s. Neglecting air resistance, a well-known formula from physics states that the position of the rock after t seconds is given by the function

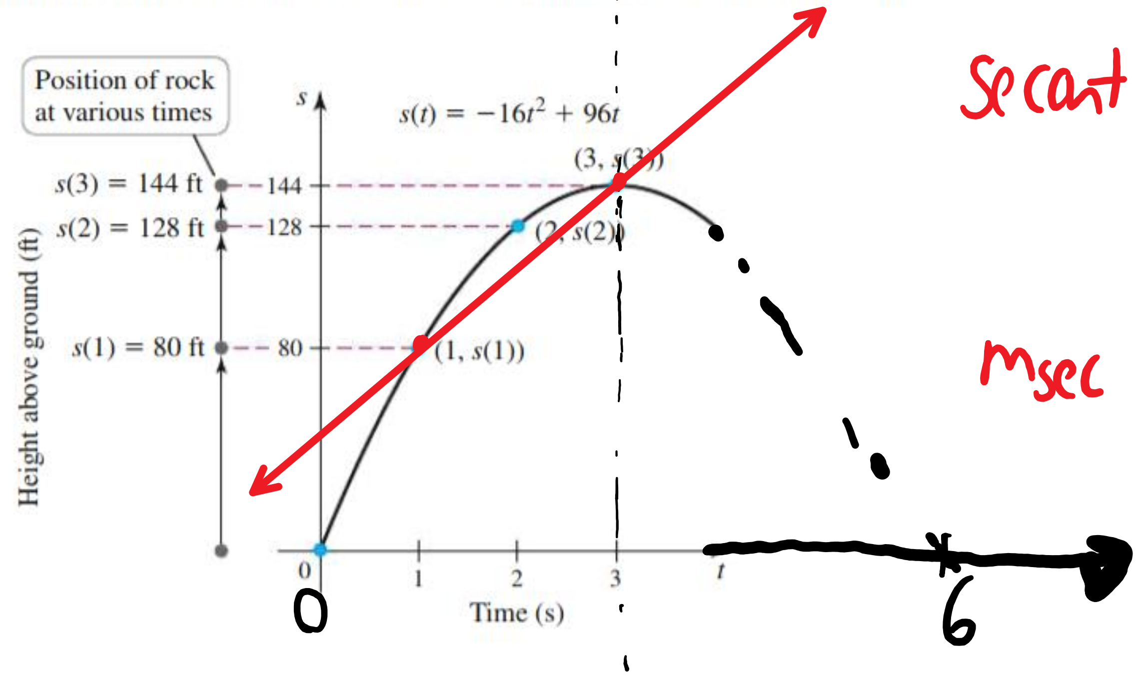
$$s(t) = -16t^2 + 96t.$$

The position s is measured in feet with $s = 0$ corresponding to the ground. Find the average velocity of the rock between each pair of times.

a. $t = 1$ s and $t = 3$ s **b.** $t = 1$ s and $t = 2$ s

Graph $s(t) = -16t^2 + 96t = \underbrace{-16t}_{t=0\text{ s.}} \cdot \underbrace{(t-6)}_{t=6\text{ s.}}$

SOLUTION Figure 2.1 shows the position function of the rock on the time interval $0 \le t \le 3$. The graph is *not* the path of the rock. The rock travels up and down on a vertical line.



Secant line requires 2-points

$m_{sec} \to V_{ave}$

Figure 2.1

The average velocity of the rock over any time interval $[t_0, t_1]$ is the change in position divided by the elapsed time:

$$v_{av} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}.$$

$s(t) = -16t^2 + 96t$

$s(3) = -16 \cdot 3^2 + 96 \cdot 3 = 144\text{ ft}$

$\rightarrow s(1) = -16 \cdot 1^2 + 96 \cdot 1 = 80\text{ ft}$

$s(2) = -16 \cdot 2^2 + 96 \cdot 2$ ✓

$= -16 \cdot 2(2-6)$ ✓

$= -16 \cdot 2 \cdot -4 = 128\text{ ft.}$

$1 \le t \le 3$

$V_{av} = \frac{s(3) - s(1)}{3-1}$

$= \frac{144-80}{2} = \frac{64}{2} = 32 \frac{ft}{sec.}$

b) $V_{av} = \frac{s(2) - s(1)}{2-1} \left(\frac{\Delta s}{\Delta t}\right) \Rightarrow V_{av} = \frac{128-80}{1} = 48 \frac{ft}{sec.}$

Instantaneous Velocity

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Instantaneous Velocity

To compute the average velocity, we use the position of the object at *two* distinct points in time. How do we compute the instantaneous velocity at a *single* point in time? As illustrated in Example 2, the instantaneous velocity at a point $t = t_0$ is determined by computing average velocities over intervals $[t_0, t_1]$ that decrease in length. As t_1 approaches t_0, the average velocities typically approach a unique number, which is the instantaneous velocity. This single number is called a **limit**.

Left Right

t=1sec

[0.99, 1] Vave.

[0.9999, 1] Vave

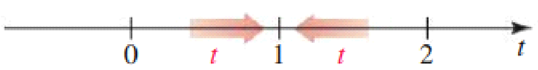
EXAMPLE 2 **Instantaneous velocity** Estimate the instantaneous velocity of the rock in Example 1 at the *single* point $t = 1$.

Table 2.1

Time interval	Average velocity
[1, 2] wider	48 ft/s
[1, 1.5]	56 ft/s
[1, 1.1]	62.4 ft/s
[1, 1.01]	63.84 ft/s
[1, 1.001]	63.984 ft/s
[1, 1.0001]	63.9984 ft/s

L

➤ The same instantaneous velocity is obtained as t approaches 1 from the left (with $t < 1$) and as t approaches 1 from the right (with $t > 1$).



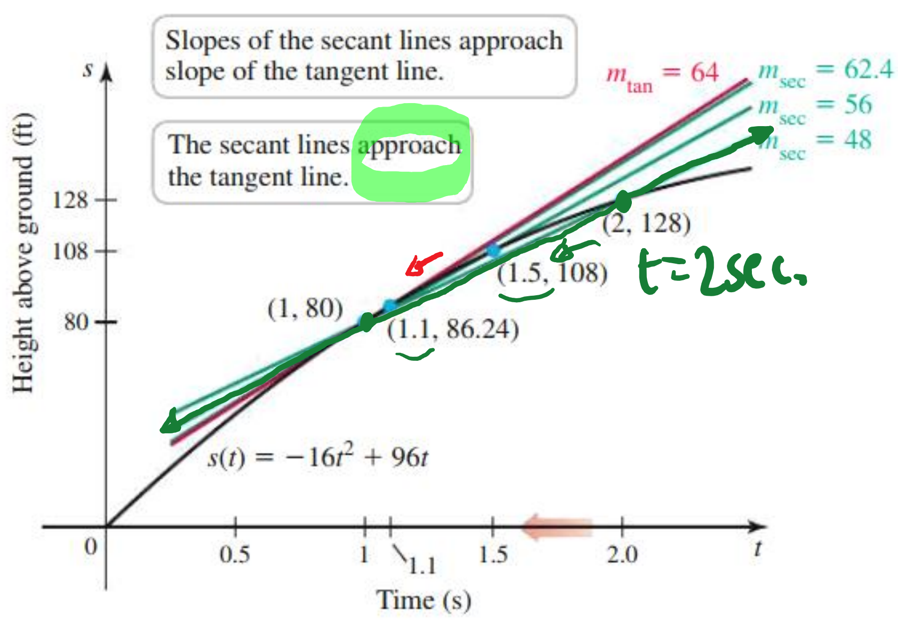
Slope of the Tangent Line

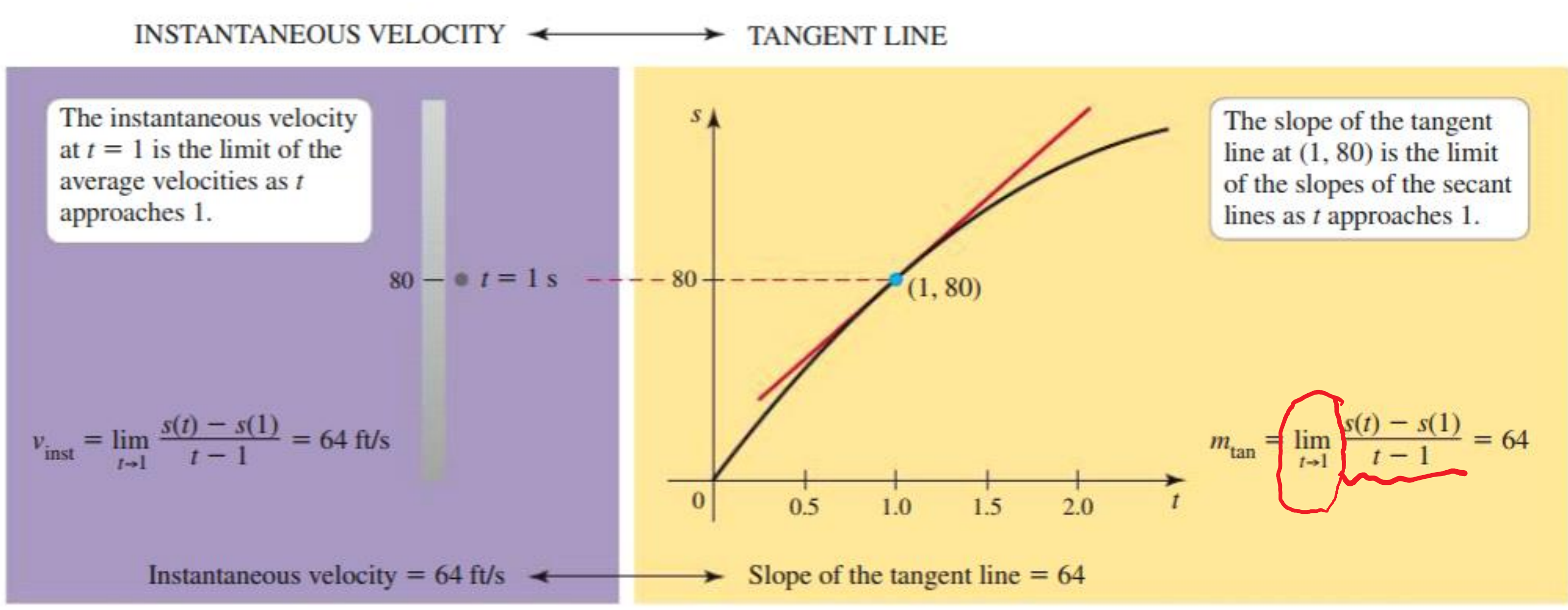
Several important conclusions follow from Examples 1 and 2. Each average velocity in Table 2.1 corresponds to the slope of a secant line on the graph of the position function (Figure 2.5). Just as the average velocities approach a limit as t approaches 1, the slopes of the secant lines approach the same limit as t approaches 1. Specifically, as t approaches 1, two things happen:

1. The secant lines approach a unique line called the **tangent line**.
2. The slopes of the secant lines m_{sec} approach the slope of the tangent line m_{tan} at the point $(1, s(1))$. Therefore, the slope of the tangent line is also expressed as a limit:

$$m_{\text{tan}} = \lim_{t \to 1} m_{\text{sec}} = \lim_{t \to 1} \frac{s(t) - s(1)}{t - 1} = 64.$$

This limit is the same limit that defines the instantaneous velocity. Therefore, the instantaneous velocity at $t = 1$ is the slope of the line tangent to the position curve at $t = 1$.





"slope"

$\frac{\Delta s}{\Delta t}$

Poll Q#1

2.2 Definition of Limits

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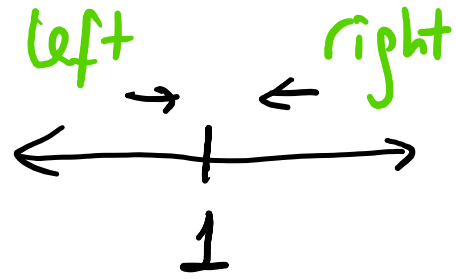
DEFINITION Limit of a Function (Preliminary)

Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = L$$

and say the limit of $f(x)$ as x approaches a equals L.



TI-84 "spider"

EXAMPLE 1 Finding limits from a graph Use the graph of f (Figure 2.7) to determine the following values, if possible.

a. $f(1)$ and $\lim_{x \to 1} f(x)$ **b.** $f(2)$ and $\lim_{x \to 2} f(x)$ **c.** $f(3)$ and $\lim_{x \to 3} f(x)$

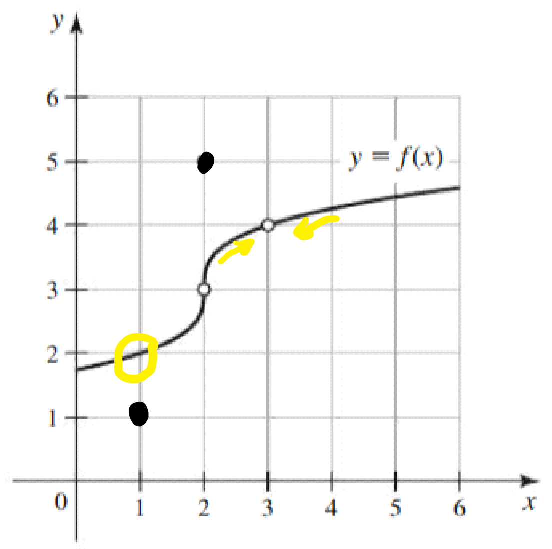


Figure 2.7

a) $f(1) = 2$

$\lim_{x \to 1} f(x) = 2$ $\}$ $L = 2$

b) $f(2) = 5$ $\lim_{x \to 2} f(x) = 3$

c) $f(3)$ is undefined $\lim_{x \to 3} f(x) = 4$

QUICK CHECK 1 In Example 1, suppose we redefine the function at one point so that $f(1) = 1$. Does this change the value of $\lim_{x \to 1} f(x)$?

NO!

EXAMPLE 2 **Finding limits from a table** Create a table of values of $f(x) = \frac{\sqrt{x}-1}{x-1}$ corresponding to values of x near 1. Then make a conjecture about the value of $\lim_{x\to 1} f(x)$.

$$\lim_{x\to 1} f(x) = 0.5$$

Table 2.2

x	0.9	0.99	0.999	0.9999	1.0001	1.001	1.01	1.1
$f(x) = \frac{\sqrt{x}-1}{x-1}$	0.5131670	0.5012563	0.5001251	0.5000125	0.4999875	0.4998751	0.4987562	0.4880885

How can we do it algebraically?

Difference of Two Squares

$$a^2 - b^2 = (a-b)(a+b)$$

$$(x-1) = (\sqrt{x})^2 - 1^2 = (\sqrt{x}-1)(\sqrt{x}+1)$$

$$\lim_{x\to 1} \frac{(\sqrt{x}-1)}{x-1}$$

0.5

$$\lim_{x\to 1} \frac{\cancel{(\sqrt{x}-1)}}{\cancel{(\sqrt{x}-1)}(\sqrt{x}+1)} = \lim_{x\to 1} \frac{1}{(\sqrt{x}+1)} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

One-sided limits

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One-Sided Limits

The limit $\lim_{x \to a} f(x) = L$ is referred to as a *two-sided* limit because $f(x)$ approaches L as x approaches a for values of x less than a *and* for values of x greater than a. For some functions, it makes sense to examine *one-sided* limits called *right-sided* and *left-sided* limits.

left a right

DEFINITION One-Sided Limits

1. **Right-sided limit** Suppose f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write

$$\lim_{x \to a^+} f(x) = L$$

and say the limit of $f(x)$ as x approaches a from the right equals L.

2. **Left-sided limit** Suppose f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write

$$\lim_{x \to a^-} f(x) = L$$

and say the limit of $f(x)$ as x approaches a from the left equals L.

Two-sided Limit

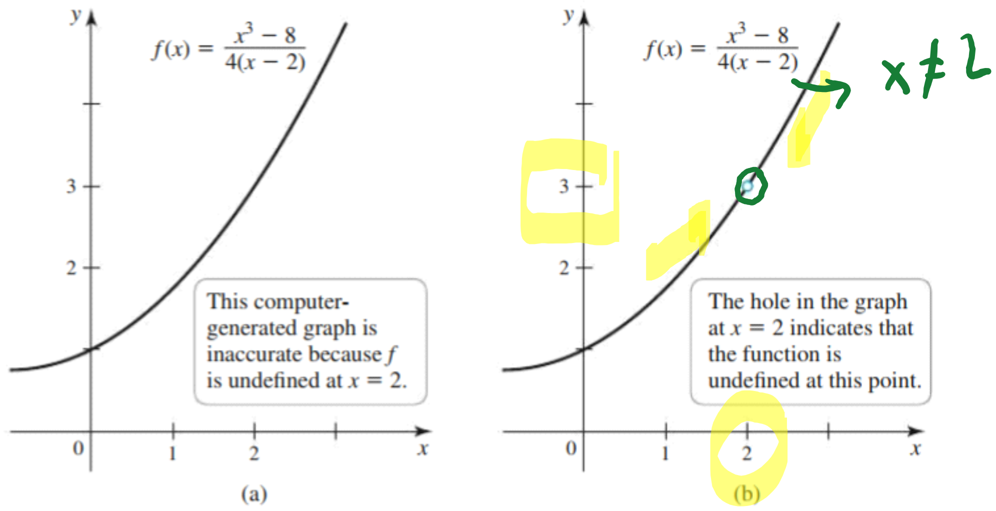
$$\lim_{x \to a} f(x)$$

One-sided limits

$\lim_{x \to a^-} f(x)$ left limit

$\lim_{x \to a^+} f(x)$ Right limit

EXAMPLE 3 **Examining limits graphically and numerically** Let $f(x) = \dfrac{x^3 - 8}{4(x-2)}$.
Use tables and graphs to make a conjecture about the values of $\lim_{x \to 2^+} f(x)$, $\lim_{x \to 2^-} f(x)$, and $\lim_{x \to 2} f(x)$, if they exist.



LL

$\lim_{x \to 2^-} f(x) = 3$

RL

$\lim_{x \to 2^+} f(x) = 3$

LL=RL

$\lim_{x \to 2} f(x) = 3$

Table 2.3

$\xrightarrow{\quad} 2 \xleftarrow{\quad}$

x	1.9	1.99	1.999	1.9999	2.0001	2.001	2.01	2.1
$f(x) = \dfrac{x^3 - 8}{4(x-2)}$	2.8525	2.985025	2.99850025	2.99985000	3.00015000	3.00150025	3.015025	3.1525

How can we do it algebraically?

THEOREM 2.1 Relationship Between One-Sided and Two-Sided Limits
Assume f is defined for all x near a except possibly at a. Then $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$.

EXAMPLE 4 A function with a jump Sketch the graph of $g(x) = \dfrac{2x^2 - 6x + 4}{|x - 1|}$ and use the graph to find the values of $\lim_{x \to 1^-} g(x)$, $\lim_{x \to 1^+} g(x)$, and $\lim_{x \to 1} g(x)$, if they exist.

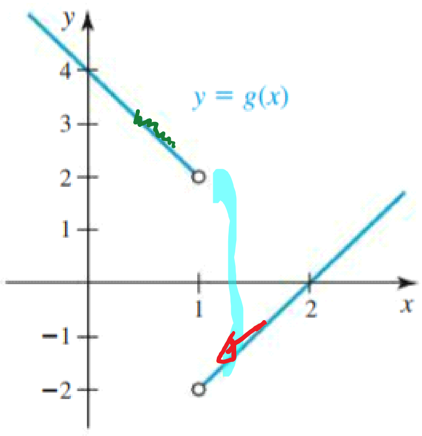


Figure 2.13

a) $\lim_{x \to 1^-} g(x) = 2$ left Limit

b) $\lim_{x \to 1^+} g(x) = -2$ Right Limit

c) since: $\lim_{x \to 1^-} g(x) \neq \lim_{x \to 1^+} g(x)$

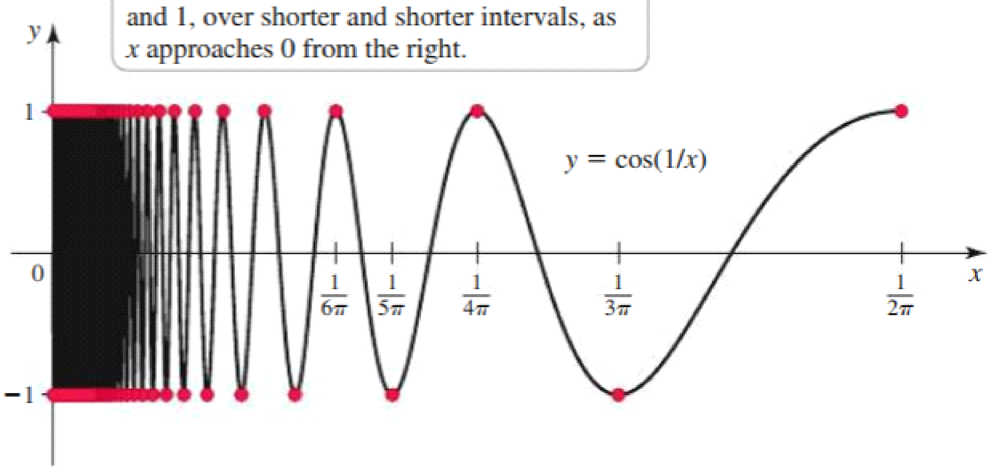
$\lim_{x \to 1} g(x)$ DOES NOT EXIST (DNE)

EXAMPLE 5 **Some strange behavior** Examine $\lim_{x \to 0} \cos(1/x)$.

Table 2.4

x	$\cos(1/x)$
0.001	0.56238
0.0001	−0.95216
0.00001	−0.99936
0.000001	0.93675
0.0000001	−0.90727
0.00000001	−0.36338

We might *incorrectly* conclude that $\cos(1/x)$ approaches -1 as x approaches 0 from the right.



$$\lim_{x \to 0} \cos\left(\frac{1}{x}\right) \quad \text{DNE}$$

Poll Q

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Q#1

Tangent lines with zero slope

a. Graph the function $f(x) = x^2 - 4x + 3$. $3 = (x-3)(x-1) = 0$ $x = 1, 3$ L.C: $1 > 0$ ∪

b. Identify the point $(a, f(a))$ at which the function has a tangent line with zero slope. (horizontal tangent)

$m_{tan} = 0$

A) $x = -2$

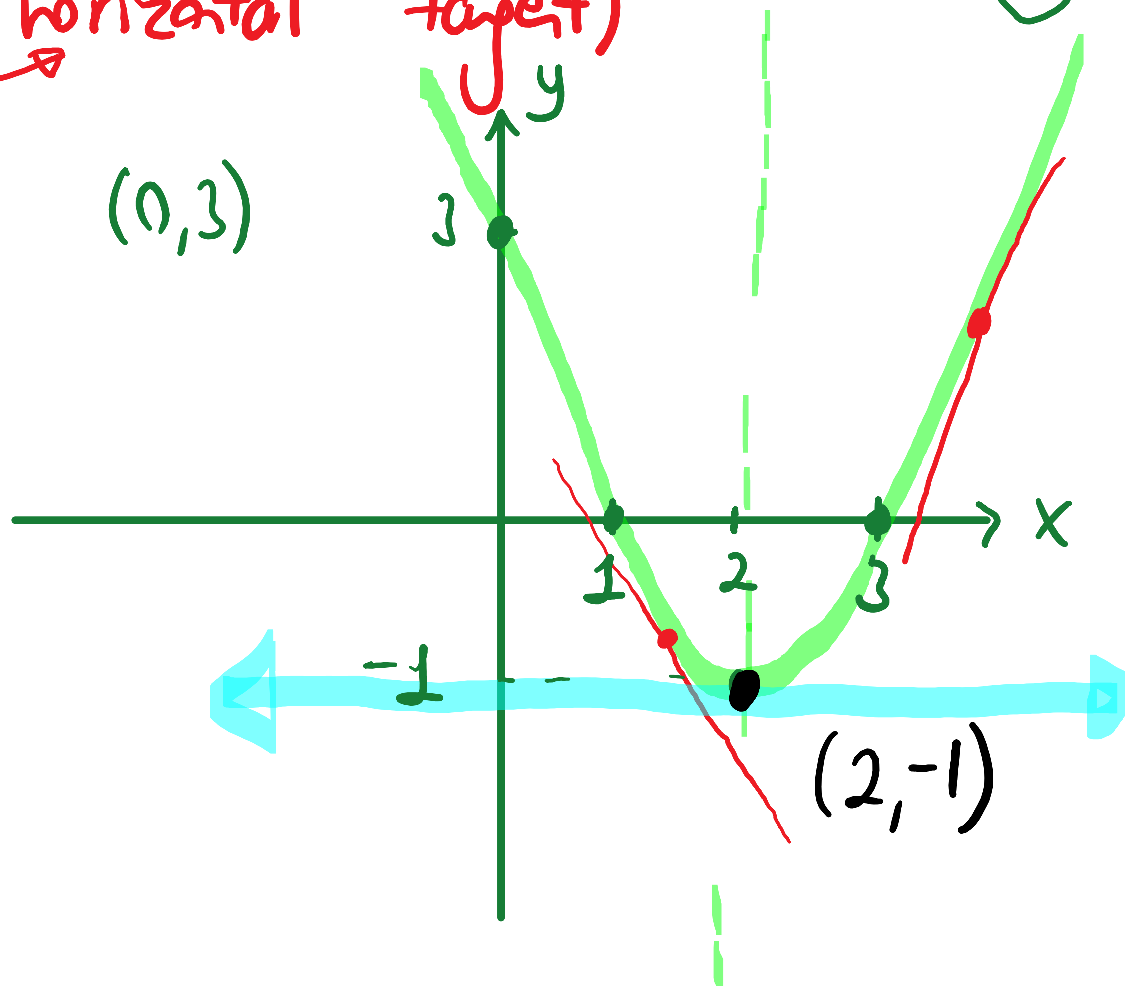
B) $x = 2$

C) $(2, 1)$

D) $(2, -1)$

E) None of the above

(0,3)



$f(2) = 2^2 - 4 \cdot 2 + 3 = -1$

Poll Q#2

Use the graph of f in the figure to find the following values or state that they do not exist. If a limit does not exist, explain why.

a. $f(1)$ **b.** $\lim_{x \to 1^-} f(x)$ **c.** $\lim_{x \to 1^+} f(x)$ **d.** $\lim_{x \to 1} f(x)$