## 2.4 Group Activity Problems

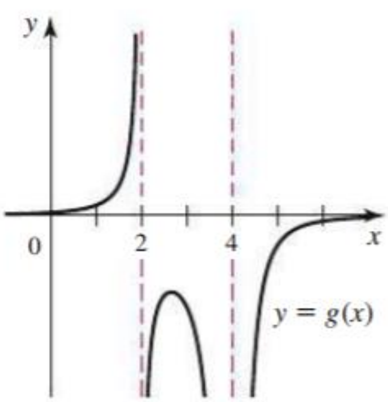

**4.** Consider the function $F(x) = f(x)/g(x)$ with $g(a) = 0$. Does $F$ necessarily have a vertical asymptote at $x = a$? Explain your reasoning.

**2.4.4** No. For example, if $f(x) = x^2 - 4$ and $g(x) = x - 2$ and $a = 2$, we would have $\lim_{x \to 2} \frac{f(x)}{g(x)} = 4$, even though $g(2) = 0$.

**8.** The graph of $g$ in the figure has vertical asymptotes at $x = 2$ and $x = 4$. Analyze the following limits.

**a.** $\lim_{x \to 2^-} g(x)$ **b.** $\lim_{x \to 2^+} g(x)$ **c.** $\lim_{x \to 2} g(x)$

**d.** $\lim_{x \to 4^-} g(x)$ **e.** $\lim_{x \to 4^+} g(x)$ **f.** $\lim_{x \to 4} g(x)$



**2.4.8**

a. $\lim_{x\to 2^-} g(x) = \infty$.

b. $\lim_{x\to 2^+} g(x) = -\infty$.

c. $\lim_{x\to 2} g(x)$ does not exist.

d. $\lim_{x\to 4^-} g(x) = -\infty$.

e. $\lim_{x\to 4^+} g(x) = -\infty$.

f. $\lim_{x\to 4} g(x) = -\infty$.

**15.** Verify that the function $f(x) = \dfrac{x^2 - 4x + 3}{x^2 - 3x + 2}$ is undefined at $x = 1$ and at $x = 2$. Does the graph of $f$ have vertical asymptotes at both these values of $x$? Explain.

**2.4.15** Note that $\lim_{x\to 1} \frac{x^2-4x+3}{x^2-3x+2} = \lim_{x\to 1} \frac{(x-3)(x-1)}{(x-2)(x-1)} = \lim_{x\to 1} \frac{x-3}{x-2} = \frac{-2}{-1} = 2$. So there is *not* a vertical asymptote at $x = 1$. On the other hand, $\lim_{x\to 2^+} \frac{x^2-4x+3}{x^2-3x+2} = \lim_{x\to 2^+} \frac{(x-3)(x-1)}{(x-2)(x-1)} = \lim_{x\to 2^+} \frac{x-3}{x-2} = -\infty$, so there is a vertical asymptote at $x = 2$.

**Determine the limits analytically.**

**22.** **a.** $\lim_{x\to 3^+} \dfrac{2}{(x-3)^3}$ **b.** $\lim_{x\to 3^-} \dfrac{2}{(x-3)^3}$ **c.** $\lim_{x\to 3} \dfrac{2}{(x-3)^3}$

**2.4.22**

a. $\lim_{x\to 3^+} \dfrac{2}{(x-3)^3} = \infty.$

b. $\lim_{x\to 3^-} \dfrac{2}{(x-3)^3} = -\infty.$

c. $\lim_{x\to 3} \dfrac{2}{(x-3)^3}$ does not exist.

**28.** **a.** $\lim_{t\to -2^+} \dfrac{t^3 - 5t^2 + 6t}{t^4 - 4t^2}$ **b.** $\lim_{t\to -2^-} \dfrac{t^3 - 5t^2 + 6t}{t^4 - 4t^2}$

**c.** $\lim_{t\to -2} \dfrac{t^3 - 5t^2 + 6t}{t^4 - 4t^2}$ **d.** $\lim_{t\to 2} \dfrac{t^3 - 5t^2 + 6t}{t^4 - 4t^2}$

**2.4.28**

a. $\lim_{x\to -2^+} \dfrac{x^3 - 5x^2 + 6x}{x^4 - 4x^2} = \lim_{x\to -2^+} \dfrac{x(x-2)(x-3)}{x^2(x-2)(x+2)} = \lim_{x\to -2^+} \dfrac{x-3}{x(x+2)} = \infty.$

b. $\lim_{x\to -2^-} \dfrac{x^3 - 5x^2 + 6x}{x^4 - 4x^2} = \lim_{x\to -2^-} \dfrac{x(x-2)(x-3)}{x^2(x-2)(x+2)} = \lim_{x\to -2^-} \dfrac{x-3}{x(x+2)} = -\infty.$

c. Because the two one-sided limits differ, $\lim_{x\to -2} \dfrac{x^3 - 5x^2 + 6x}{x^4 - 4x^2}$ does not exist.

d. $\lim_{x\to 2} \dfrac{x^3 - 5x^2 + 6x}{x^4 - 4x^2} = \lim_{x\to 2} \dfrac{x-3}{x(x+2)} = \dfrac{-1}{8}.$

**30.** **a.** $\lim_{x\to 1^+} \dfrac{x-3}{\sqrt{x^2 - 5x + 4}}$ **b.** $\lim_{x\to 1^-} \dfrac{x-3}{\sqrt{x^2 - 5x + 4}}$

**c.** $\lim_{x\to 1} \dfrac{x-3}{\sqrt{x^2 - 5x + 4}}$

**2.4.30**

a. $\lim_{x\to1^+}\frac{x-3}{\sqrt{x^2-5x+4}}$ does not exist. Note that $x^2-5x+4=(x-4)(x-1)$ so the domain of the function is $(-\infty,1)\cup(4,\infty)$.

b. ] $\lim_{x\to1^-}\frac{x-3}{\sqrt{x^2-5x+4}}=-\infty$.

c. $\lim_{x\to1}\frac{x-3}{\sqrt{x^2-5x+4}}$ does not exist.

**Finding vertical asymptotes** *Find all vertical asymptotes* $x=a$ *of the following functions. For each value of a, determine* $\lim_{x\to a^+}f(x)$, $\lim_{x\to a^-}f(x)$, *and* $\lim_{x\to a}f(x)$.

$$f(x)=\frac{x+1}{x^3-4x^2+4x}$$

**2.4.49** $f(x)=\frac{x+1}{x^3-4x^2+4x}=\frac{x+1}{x(x-2)^2}$. There are vertical asymptotes at $x=0$ and $x=2$. We have $\lim_{x\to0^-}f(x)=\lim_{x\to0^-}\frac{x+1}{x(x-2)^2}=-\infty$, while $\lim_{x\to0^+}f(x)=\lim_{x\to0^+}\frac{x+1}{x(x-2)^2}=\infty$, and thus $\lim_{x\to0}f(x)$ doesn't exist.

Also we have $\lim_{x\to2^-}f(x)=\lim_{x\to2^-}\frac{x+1}{x(x-2)^2}=\infty$, while $\lim_{x\to2^+}f(x)=\lim_{x\to2^+}\frac{x+1}{x(x-2)^2}=\infty$, and thus $\lim_{x\to2}f(x)=\infty$ as well.

## 2.5 Group Activity Problems

**29.** $\lim_{w \to \infty} \frac{15w^2 + 3w + 1}{\sqrt{9w^4 + w^3}}$

**2.5.29** Note that for $w > 0$, $w^2 = \sqrt{w^4}$. We have

$$\lim_{w\to\infty} \frac{(15w^2+3w+1)}{\sqrt{9w^4+w^3}} \cdot \frac{1/w^2}{1/\sqrt{w^4}} = \lim_{w\to\infty} \frac{15+(3/w)+(1/w^2)}{\sqrt{9+(1/w)}} = \frac{15}{\sqrt{9}} = 5.$$

**30.** $\lim\limits_{x\to-\infty} \dfrac{40x^4 + x^2 + 5x}{\sqrt{64x^8 + x^6}}$

**2.5.30** Note that $\sqrt{x^8} = x^4$ (even for $x < 0$). We have

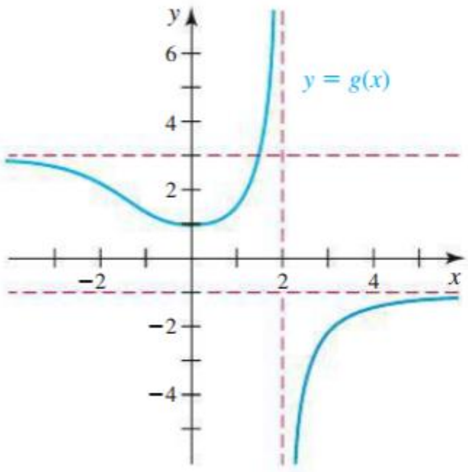
$$\lim_{x\to-\infty} \frac{(40x^4 + x^2 + 5x)}{\sqrt{64x^8 + x^6}} \cdot \frac{1/x^4}{1/\sqrt{x^8}} = \lim_{x\to-\infty} \frac{40 + (1/x^2) + (5/x^3)}{\sqrt{64 + (1/x^2)}} = \frac{40}{\sqrt{64}} = \frac{40}{8} = 5.$$

**15.** Suppose the function $g$ satisfies the inequality $3 - \dfrac{1}{x^2} \le g(x) \le 3 + \dfrac{1}{x^2}$, for all nonzero values of $x$. Evaluate $\lim\limits_{x\to\infty} g(x)$ and $\lim\limits_{x\to-\infty} g(x)$.

**2.5.15** Because $\lim\limits_{x\to\infty} 3 - \dfrac{1}{x^2} = 3$ and $\lim\limits_{x\to\infty} 3 + \dfrac{1}{x^2} = 3$, by the Squeeze Theorem we must have $\lim\limits_{x\to\infty} g(x) = 3$.

Similarly, because $\lim\limits_{x\to-\infty} 3 - \dfrac{1}{x^2} = 3$ and $\lim\limits_{x\to-\infty} 3 + \dfrac{1}{x^2} = 3$, by the Squeeze Theorem we must have $\lim\limits_{x\to-\infty} g(x) = 3$.

**16.** The graph of $g$ has a vertical asymptote at $x = 2$ and horizontal asymptotes at $y = -1$ and $y = 3$ (see figure). Determine the following limits: $\lim_{x\to-\infty} g(x)$, $\lim_{x\to\infty} g(x)$, $\lim_{x\to 2^-} g(x)$, and $\lim_{x\to 2^+} g(x)$.




$$\textbf{2.5.16} \quad \lim_{x\to-\infty} g(x) = 3, \ \lim_{x\to\infty} g(x) = -1, \ \lim_{x\to-2^-} g(x) = \infty, \ \lim_{x\to 2^+} g(x) = -\infty.$$

**Determine the following limits.**

**24.** $\lim_{x\to-\infty} (2x^{-8} + 4x^3)$

**26.** $\lim_{x\to\infty} \dfrac{9x^3 + x^2 - 5}{3x^4 + 4x^2}$

**28.** $\lim_{x\to\infty} \dfrac{x^4 + 7}{x^5 + x^2 - x}$

**32.** $\lim_{x\to\infty} \dfrac{6x^2}{4x^2 + \sqrt{16x^4 + x^2}}$

**2.5.24** $\lim_{x \to -\infty} (2x^{-8} + 4x^3) = 0 + \lim_{x \to -\infty} 4x^3 = -\infty.$

**2.5.26** $\lim_{x \to \infty} \frac{(9x^3 + x^2 - 5)}{(3x^4 + 4x^2)} \cdot \frac{1/x^4}{1/x^4} = \lim_{x \to \infty} \frac{(9/x) + (1/x^2) - (5/x^4)}{3 + (4/x^2)} = \frac{0}{3} = 0.$

**2.5.28** $\lim_{x \to \infty} \frac{(x^4 + 7)}{(x^5 + x^2 - x)} \cdot \frac{1/x^5}{1/x^5} = \lim_{x \to \infty} \frac{(1/x) + (7/x^5)}{1 + (1/x^3) - (1/x^4)} = \frac{0 + 0}{1 + 0 - 0} = 0.$

**2.5.32** Note that $x^2 = \sqrt{x^4}$ for all $x$. We have

$$\lim_{x \to \infty} \frac{6x^2}{(4x^2 + \sqrt{16x^4 + x^2})} \cdot \frac{1/x^2}{1/x^2} = \lim_{x \to \infty} \frac{6}{(4 + \sqrt{16 + (1/x^2)})} = \frac{6}{4 + \sqrt{16}} = \frac{3}{4}.$$

**37–50. Horizontal asymptotes** *Determine* $\lim_{x \to \infty} f(x)$ *and* $\lim_{x \to -\infty} f(x)$ *for the following functions. Then give the horizontal asymptotes of f (if any).*

**44.** $f(x) = \frac{6x^2 + 1}{\sqrt{4x^4 + 3x + 1}}$

**46.** $f(x) = \frac{\sqrt{x^2 + 1}}{2x + 1}$

**47.** $f(x) = \dfrac{4x^3 + 1}{2x^3 + \sqrt{16x^6 + 1}}$

**2.5.44** Note that for all $x$, $\sqrt{x^4} = x^2$. Then

$$\lim_{x\to\pm\infty} \frac{(6x^2+1)}{\sqrt{4x^4+3x+1}} \cdot \frac{1/x^2}{\sqrt{1/x^4}} = \lim_{x\to\pm\infty} \frac{6+(1/x^2)}{\sqrt{4+(3/x^3)+(1/x^4)}} = \frac{6}{\sqrt{4}} = 3.$$

So $y = 3$ is the only horizontal asymptote.

**2.5.46** First note that $\sqrt{x^2} = x$ for $x > 0$, while $\sqrt{x^2} = -x$ for $x < 0$. Then $\lim_{x\to\infty} f(x)$ can be written as

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{2x+1} \cdot \frac{1/\sqrt{x^2}}{1/x} = \lim_{x\to\infty} \frac{\sqrt{1+1/x^2}}{2+1/x} = \frac{1}{2}.$$

However, $\lim_{x\to-\infty} f(x)$ can be written as

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{2x+1} \cdot \frac{1/\sqrt{x^2}}{-1/x} = \lim_{x\to-\infty} \frac{\sqrt{1+1/x^2}}{-2-1/x} = -\frac{1}{2}.$$

**2.5.47** First note that $\sqrt{x^6} = x^3$ if $x > 0$, but $\sqrt{x^6} = -x^3$ if $x < 0$. We have $\lim_{x\to\infty} \frac{4x^3+1}{(2x^3+\sqrt{16x^6+1})} \cdot \frac{1/x^3}{1/x^3} = \lim_{x\to\infty} \frac{4+1/x^3}{2+\sqrt{16+1/x^6}} = \frac{4+0}{2+\sqrt{16+0}} = \frac{2}{3}.$

However, $\lim_{x\to-\infty} \frac{4x^3+1}{(2x^3+\sqrt{16x^6+1})} \cdot \frac{1/x^3}{1/x^3} = \lim_{x\to-\infty} \frac{4+1/x^3}{2-\sqrt{16+1/x^6}} = \frac{4+0}{2-\sqrt{16+0}} = \frac{4}{-2} = -2.$

So $y = \frac{2}{3}$ is a horizontal asymptote (as $x \to \infty$) and $y = -2$ is a horizontal asymptote (as $x \to -\infty$).

**94. End behavior of exponentials** Use the following instructions to determine the end behavior of $f(x) = \dfrac{4e^x + 2e^{2x}}{8e^x + e^{2x}}$.

- **a.** Evaluate $\lim_{x\to\infty} f(x)$ by first dividing the numerator and denominator by $e^{2x}$.
- **b.** Evaluate $\lim_{x\to-\infty} f(x)$ by first dividing the numerator and denominator by $e^x$.
- **c.** Give the horizontal asymptote(s).

**2.5.94**

a. $\lim_{x\to\infty} \frac{4e^x + 2e^{2x}}{8e^x + e^{2x}} = \lim_{x\to\infty} \frac{(4e^x + 2e^{2x})}{(8e^x + e^{2x})} \cdot \frac{1/e^{2x}}{1/e^{2x}} = \lim_{x\to\infty} \frac{(4/e^x) + 2}{(8/e^x) + 1} = 2.$

b. $\lim_{x\to-\infty} \frac{4e^x + 2e^{2x}}{8e^x + e^{2x}} = \lim_{x\to-\infty} \frac{(4e^x + 2e^{2x})}{(8e^x + e^{2x})} \cdot \frac{1/e^x}{1/e^x} = \lim_{x\to-\infty} \frac{4 + 2e^x}{8 + e^x} = \frac{1}{2}.$

c. The lines $y = 2$ and $y = \frac{1}{2}$ are horizontal asymptotes.

d.