3.1 Group Activity Problems – Solutions

DEFINITION **Rate of Change and the Slope of the Tangent Line**

The **average rate of change** in f on the interval $[a, x]$ is the slope of the corresponding secant line:

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

The **instantaneous rate of change** in f at a is

$$m_{\text{tan}} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \qquad (1)$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists. The **tangent line** is the unique line through $(a, f(a))$ with slope m_{tan}. Its equation is

$$y - f(a) = m_{\text{tan}}(x - a).$$

ALTERNATIVE DEFINITION **Rate of Change and the Slope of the Tangent Line**

The **average rate of change** in f on the interval $[a, a + h]$ is the slope of the corresponding secant line:

$$m_{\text{sec}} = \frac{f(a + h) - f(a)}{h}.$$

The **instantaneous rate of change** in f at a is

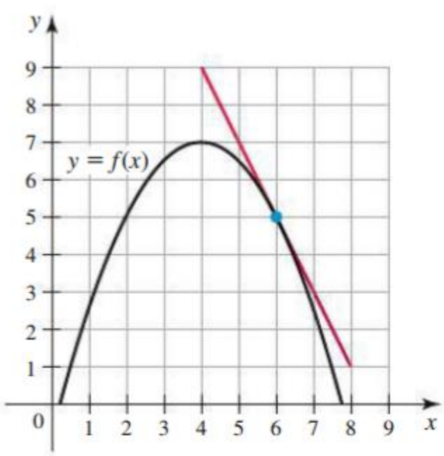
$$m_{\text{tan}} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}, \qquad (2)$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists.

4. Explain the relationships among the slope of a tangent line, the instantaneous rate of change, and the value of the derivative at a point.

3.1.4 The slope of the tangent line, the instantaneous rate of change, and the value of the derivative of a function at a given point are all the same.

6. The following figure shows the graph of f and a line tangent to the graph of f at $x = 6$. Find $f(6)$ and $f'(6)$.



3.1.6 $f(6)$ is the y-value of the point on the graph associated with $x = 6$, so $f(6) = 5$. $f'(6)$ is the slope of the tangent line at the point $(6, 5)$, so is $f'(6) = -2$.

8. An equation of the line tangent to the graph of g at $x = 3$ is $y = 5x + 4$. Find $g(3)$ and $g'(3)$.

9. If $h(1) = 2$ and $h'(1) = 3$, find an equation of the line tangent to the graph of h at $x = 1$.

3.1.8 $g(3) = 5(3) + 4 = 19$. $g'(3)$ is the slope of the tangent line, so $g'(3) = 5$.

3.1.9 Using the point-slope form of the equation of a line, we have $y - 2 = 3(x - 1)$, or $y = 3x - 1$.

Use the definition (1) and definition (2) to:

a) **Find the slope of the line tangent to the graph of f at P.**
b) **Determine an equation of the tangent line at P.**
c) **Plot the graph of f and the tangent line at P.**

Model Question – Q18 (Refer to the problem-solving session)

18. $f(x) = \dfrac{4}{x^2}$; $P(-1, 4)$

You Try It!

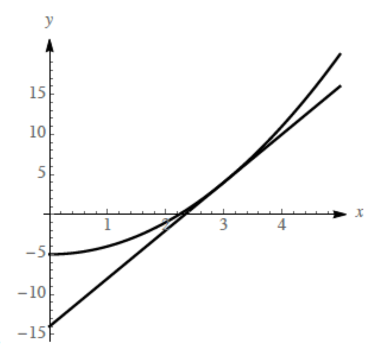
15. $f(x) = x^2 - 5$; $P(3, 4)$

3.1.15

a. $m_{\tan} = \lim_{x \to 3} \dfrac{x^2 - 5 - 4}{x - 3} = \lim_{x \to 3} \dfrac{x^2 - 9}{x - 3} = \lim_{x \to 3} \dfrac{(x-3)(x+3)}{x - 3} = \lim_{x \to 3} (x + 3) = 6.$

b. Using the point-slope form of the equation of a line, we obtain $y - 4 = 6(x - 3)$, or $y = 6x - 14$.

c.



Determine which definition (1 or 2) is more practical to use for the following question and then solve the problem.

a) **Find the slope of the line tangent to the graph of f at P.**
b) **Determine an equation of the tangent line at P.**

30. $f(x) = \sqrt{x-1};\ P(2, 1)$

3.1.30

a. $m_{\tan} = \lim_{h\to 0} \frac{\sqrt{h+2-1}-1}{h} = \lim_{h\to 0} \frac{(\sqrt{h+1}-1)(\sqrt{h+1}+1)}{h(\sqrt{h+1}+1)} = \lim_{h\to 0} \frac{h+1-1}{h(\sqrt{h+1}+1)} =$
$\lim_{h\to 0} \frac{1}{\sqrt{h+1}+1} = \frac{1}{2}.$

b. $y - 1 = \frac{1}{2}(x-2)$, or $y = \frac{1}{2}x.$

33–42. Derivatives and tangent lines

a. *For the following functions and values of a, find* $f'(a)$.

b. *Determine an equation of the line tangent to the graph of f at the point* $(a, f(a))$ *for the given value of a.*

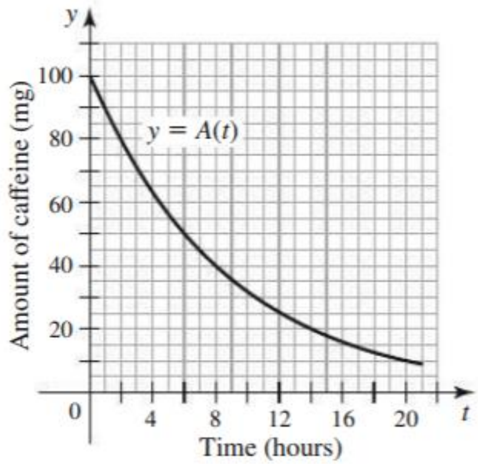
37. $f(x) = \frac{1}{\sqrt{x}};\ a = \frac{1}{4}$

3.1.37

a. $f'\left(\frac{1}{4}\right) = \lim_{h\to 0} \frac{\frac{1}{\sqrt{\frac{1}{4}+h}} - 2}{h} = \lim_{h\to 0} \frac{1 - 2\sqrt{\frac{1}{4}+h}}{h\sqrt{\frac{1}{4}+h}} = \lim_{h\to 0} \frac{\left(1 - 2\sqrt{\frac{1}{4}+h}\right)\left(1 + 2\sqrt{\frac{1}{4}+h}\right)}{h\sqrt{\frac{1}{4}+h}\left(1 + 2\sqrt{\frac{1}{4}+h}\right)} =$
$\lim_{h\to 0} \frac{1 - 4\left(\frac{1}{4}+h\right)}{h\sqrt{\frac{1}{4}+h}\left(1 + 2\sqrt{\frac{1}{4}+h}\right)} = \lim_{h\to 0} -\frac{4}{\sqrt{\frac{1}{4}+h}\left(1 + 2\sqrt{\frac{1}{4}+h}\right)} = -4.$

b. $y - 2 = -4\left(x - \frac{1}{4}\right)$, or $y = -4x + 3.$

54. Caffeine levels Let $A(t)$ be the amount of caffeine (in mg) in the bloodstream t hours after a cup of coffee has been consumed (see figure). Estimate the values of $A'(7)$ and $A'(15)$, rounding answers to the nearest whole number. Include units in your answers and interpret the physical meaning of these values.

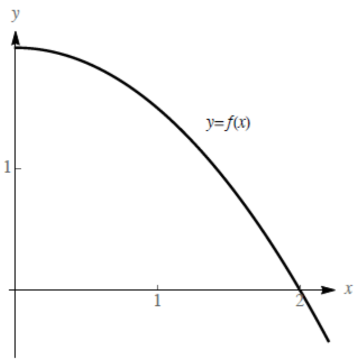


3.1.54 $A'(7) \approx -5\,\text{mg/hour}$. $A'(15) \approx -2\,\text{mg/hour}$. The caffeine levels are dropping at about 5 mg/hour after 7 hours and at 2 mg/hr after 15 hours.

3.2 Group Activity Problems– Solutions

6. Sketch a graph of a function f, where $f(x) > 0$ and $f'(x) < 0$ for all x in $(0, 2)$.

3.2.6



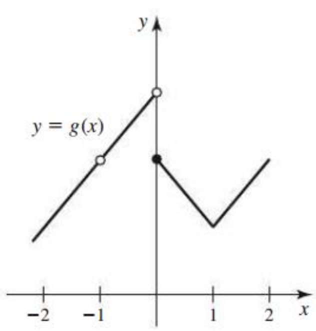
9. Describe the graph of f if $f(0) = 1$ and $f'(x) = 3$, for $-\infty < x < \infty$.

3.2.9 The graph is a line with y-intercept of 1 and a slope of 3.

20. Use the graph of g in the figure to do the following.

a. Find the values of x in $(-2, 2)$ at which g is not continuous.

b. Find the values of x in $(-2, 2)$ at which g is not differentiable.



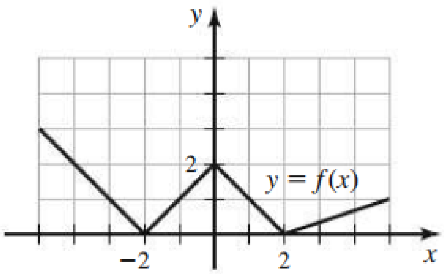
3.2.20

a. g is not continuous at $x = -1$ and $x = 0$.

b. g is not differentiable at $x = -1$ and at $x = 0$ and at $x = 1$.

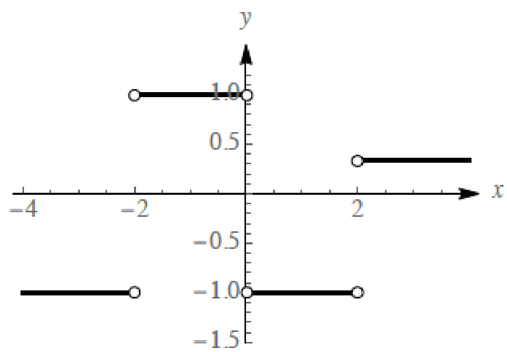
17–18. Sketching derivatives *Reproduce the graph of f and then sketch a graph of f′ on the same axes.*

17.



3.2.17

The function f is not differentiable at $x = -2, 0, 2$, so f' is not defined at those points. Elsewhere, the slope is constant.



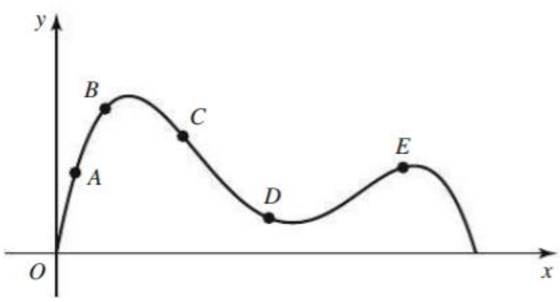
45–46. Analyzing slopes *Use the points A, B, C, D, and E in the following graphs to answer these questions.*

a. *At which points is the slope of the curve negative?*

b. *At which points is the slope of the curve positive?*

c. *Using A–E, list the slopes in decreasing order.*

45.



3.2.45

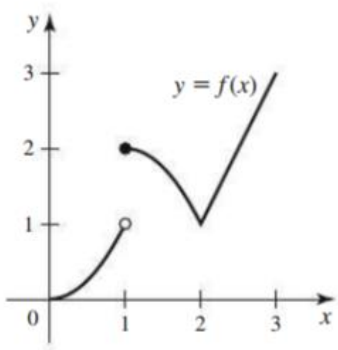
a. At C and D, the slope of the tangent line (and thus of the curve) is negative.

b. At A, B, and E, the slope of the curve is positive.

c. The graph is in its steepest ascent at A followed by B. At E it barely increases, at D it slightly decreases and at C it is decreasing the most, so the points in decreasing order of slope are A, B, E, D, C.

53. Where is the function continuous? Differentiable? Use the graph of f in the figure to do the following.

a. Find the values of x in $(0, 3)$ at which f is not continuous.
b. Find the values of x in $(0, 3)$ at which f is not differentiabl
c. Sketch a graph of f'.

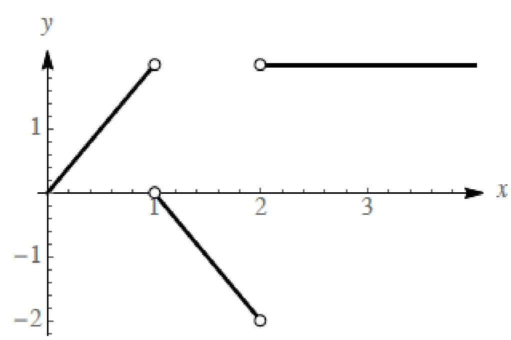


3.2.53

a. The function f is not continuous at $x = 1$, because the graph has a jump there.

b. The function f is not differentiable at $x = 1$ because it is not continuous at that point (Theorem 3.1 Alternate Version), and it is also not differentiable at $x = 2$ because the graph has a corner there.

c.



21–30. Derivatives

a. *Use limits to find the derivative function f' for the following functions f.*

b. *Evaluate $f'(a)$ for the given values of a.*

(You may pick one to solve during the recitation, however, complete both problems after the recitation)

26. $f(x) = \dfrac{x}{x+2}; a = -1, 0$

28. $f(w) = \sqrt{4w - 3}; a = 1, 3$

3.2.26

a. $f'(x) = \lim_{h\to 0} \dfrac{\frac{x+h}{x+h+2} - \frac{x}{x+2}}{h} = \lim_{h\to 0} \dfrac{(x+2)(x+h) - (x+h+2)(x)}{h(x+h+2)(x+2)} =$

$\lim_{h\to 0} \dfrac{x^2 + hx + 2x + 2h - x^2 - hx - 2x}{h(x+h+2)(x+2)} = \lim_{h\to 0} \dfrac{2}{(x+h+2)(x+2)} = \dfrac{2}{(x+2)^2}.$

b. $f'(-1) = 2$ and $f'(0) = \frac{1}{2}$.

3.2.28

a. $f'(w) = \lim_{h\to 0} \dfrac{(\sqrt{4(w+h)-3} - \sqrt{4w-3})}{h} \cdot \dfrac{(\sqrt{4(w+h)-3} + \sqrt{4w-3})}{(\sqrt{4(w+h)-3} + \sqrt{4w-3})} =$

$\lim_{h\to 0} \dfrac{4(w+h) - 3 - (4w-3)}{h(\sqrt{4(w+h)-3} + \sqrt{4w-3})} = \lim_{h\to 0} \dfrac{4}{\sqrt{4(w+h)-3} + \sqrt{4w-3}} = \dfrac{2}{\sqrt{4w-3}}.$

b. $f'(1) = 2$ and $f'(3) = \frac{2}{3}$.