4.4 Group Activity Problems – Solutions

Graphing Guidelines for $y = f(x)$

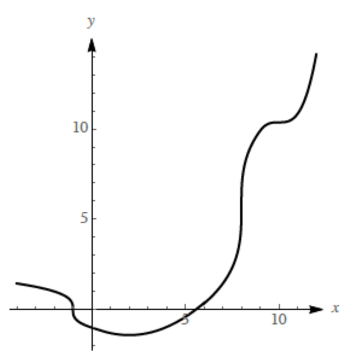
1. **Identify the domain or interval of interest.** On what interval(s) should the function be graphed? It may be the domain of the function or some subset of the domain.
2. **Exploit symmetry.** Take advantage of symmetry. For example, is the function *even* $(f(-x) = f(x))$, *odd* $(f(-x) = -f(x))$, or neither?
3. **Find the first and second derivatives.** They are needed to determine extreme values, concavity, inflection points, and intervals of increase and decrease. Computing derivatives—particularly second derivatives—may not be practical, so some functions may need to be graphed without complete derivative information.
4. **Find critical points and possible inflection points.** Determine points at which $f'(x) = 0$ or f' is undefined. Determine points at which $f''(x) = 0$ or f'' is undefined.
5. **Find intervals on which the function is increasing/decreasing and concave up/down.** The first derivative determines the intervals of increase and decrease. The second derivative determines the intervals on which the function is concave up or concave down.
6. **Identify extreme values and inflection points.** Use either the First or Second Derivative Test to classify the critical points. Both x- and y-coordinates of maxima, minima, and inflection points are needed for graphing.
7. **Locate all asymptotes and determine end behavior.** Vertical asymptotes often occur at zeros of denominators. Horizontal asymptotes require examining limits as $x \to \pm\infty$; these limits determine end behavior. Slant asymptotes occur with rational functions in which the degree of the numerator is one more than the degree of the denominator.
8. **Find the intercepts.** The y-intercept of the graph is found by setting $x = 0$. The x-intercepts are found by solving $f(x) = 0$; they are the real zeros (or roots) of f.
9. **Choose an appropriate graphing window and plot a graph.** Use the results of the previous steps to graph the function. If you use graphing software, check for consistency with your analytical work. Is your graph *complete*—that is, does it show all the essential details of the function?

7–8. *Sketch a graph of a function f with the following properties.*

8. $f' < 0$ and $f'' < 0$, for $x < -1$
$f' < 0$ and $f'' > 0$, for $-1 < x < 2$
$f' > 0$ and $f'' > 0$, for $2 < x < 8$
$f' > 0$ and $f'' < 0$, for $8 < x < 10$
$f' > 0$ and $f'' > 0$, for $x > 10$

4.4.8

The function sketched should be decreasing on $(-\infty, 2)$ and increasing on $(2, \infty)$. It should be concave down on $(-\infty, -1)$ and on $(8, 10)$. It should be concave up on $(-1, 8)$ and on $(10, \infty)$.



13. Let $f(x) = (x - 3)(x + 3)^2$.

a. Verify that $f'(x) = 3(x - 1)(x + 3)$ and $f''(x) = 6(x + 1)$.

b. Find the critical points and possible inflection points of f.

c. Find the intervals on which f is increasing or decreasing.

d. Determine the intervals on which f is concave up or concave down.

e. Identify the local extreme values and inflection points of f.

f. State the x- and y-intercepts of the graph of f.

g. Use your work in parts (a) through (f) to sketch a graph of f.

4.4.13

a. $f'(x) = 1 \cdot (x+3)^2 + (x-3)(2)(x+3) = (x+3)(x+3+2x-6) = (x+3)(3x-3) = 3(x-1)(x+3)$. Also, $f''(x) = 3(x+3) + 3(x-1) = 3x + 9 + 3x - 3 = 6x + 6 = 6(x+1)$.

b. $f'(x) = 0$ for $x = 1$ and $x = -3$ (so those are the critical points), while $f''(x) = 0$ for $x = -1$, so that is the location of a potential inflection point.

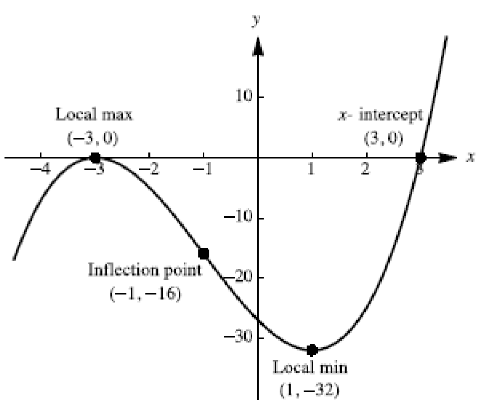
c. $f' > 0$ on $(-\infty, -3)$ and $(1, \infty)$, so f is increasing there. $f' < 0$ on $(-3, 1)$ so f is decreasing there.

d. $f'' > 0$ on $(-1, \infty)$ so f is concave up there; $f'' < 0$ on $(-\infty, -1)$, so f is concave down there.

e. f has a local max of $f(-3) = 0$ at $x = -3$ and a local minimum of $f(1) = -32$ at $x = 1$. The point $(-1, -16)$ is an inflection point.

f. The y-intercept is $f(0) = -27$. The x-intercepts are $x = 3, -3$.

g.



15–46. Graphing functions *Use the guidelines of this section to make a complete graph of f.*

34. $f(x) = \dfrac{4x}{x^2 + 3}$

4.4.34 The domain of f is $(-\infty, \infty)$. There is odd symmetry because

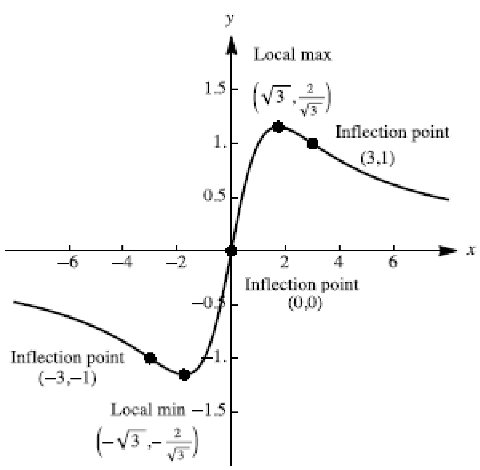
$$f(-x) = \frac{-4x}{(-x)^2 + 3} = -\frac{4x}{x^2 + 3} = -f(x).$$

The only intercept is $(0, 0)$. We have $\lim_{x\to\infty} \frac{4x}{x^2+3} = \lim_{x\to\infty} \frac{4/x}{1 + 3/x^2} = 0$, so $y = 0$ is a horizontal asymptote.

$f'(x) = \frac{(x^2+3)4 - 4x(2x)}{(x^2+3)^2} = \frac{4(3 - x^2)}{(x^2+3)^2}$. This is zero for $x = \pm\sqrt{3}$. $f' > 0$ on $(-\sqrt{3}, \sqrt{3})$ so f is increasing there, while $f' < 0$ on $(-\infty, -\sqrt{3})$ and on $(\sqrt{3}, \infty)$, so f is decreasing there. There is therefore a local maximum at $x = \sqrt{3}$ and a local minimum at $x = -\sqrt{3}$. The y value of the local maximum is $\frac{2}{\sqrt{3}}$ and for the minimum it is $-\frac{2}{\sqrt{3}}$

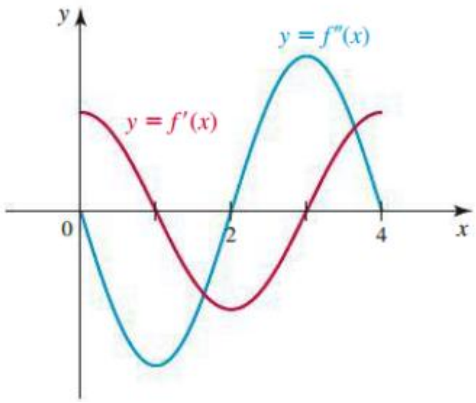
$$\begin{aligned}
f''(x) &= \frac{(x^2+3)^2(-8x) - 4(3 - x^2)(2)(x^2+3)(2x)}{(x^2+3)^4} \\
&= \frac{-8x(x^2+3)(x^2+3+2(3-x^2))}{(x^2+3)^4} \\
&= \frac{-8x(9 - x^2)}{(x^2+3)^3} \\
&= \frac{8x(x-3)(x+3)}{(x^2+3)^3}.
\end{aligned}$$

This is zero for $x = -3, 0, 3$, $f'' < 0$ on $(-\infty, -3)$ and on $(0, 3)$ so f is concave down there, while $f' > 0$ on $(-3, 0)$ and on $(3, \infty)$, so f is concave up there. There are inflection points at $(-3, -1)$, $(0, 0)$, and $(3, 1)$.



47–48. *Use the graphs of f' and f'' to find the critical points and inflection points of f, the intervals on which f is increasing and decreasing, and the intervals of concavity. Then graph f assuming $f(0) = 0$.*

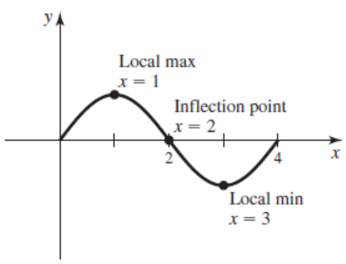
47.



4.4.47 $f'(x)$ is 0 at $x = 1$ and $x = 3$.

$f'(x) > 0$ on $(0, 1)$ and on $(3, 4)$, so f is increasing on those intervals. $f'(x) < 0$ on $(1, 3)$, so f is decreasing on that interval. There is a local maximum at $x = 1$ and a local minimum at $x = 3$.

$f''(x)$ changes sign at $x = 2$ from negative to positive, so $x = 2$ is an inflection point where the concavity of f changes from down to up. An example of such a function is sketched.



55. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. If the zeros of f' are -3, 1, and 4, then the local extrema of f are located at these points.

b. If the zeros of f'' are -2 and 4, then the inflection points of f are also located at these points.

c. If the zeros of the denominator of f are -3 and 4, then f has vertical asymptotes at these points.

4.4.55

a. False. Maxima and minima can also occur at points where $f'(x)$ doesn't exist. Also, it is possible to have a zero of f' which doesn't correspond to an extreme point.

b. False. Inflection points can also occur at points where $f''(x)$ doesn't exist, and a zero of f'' might not correspond to an inflection point.

c. False. For example, $f(x) = \dfrac{(x^2-9)(x^2-16)}{(x+3)(x-4)}$ doesn't have a vertical asymptote at $x = -3$ or $x = 4$.

56–59. Functions from derivatives *Use the derivative f' to determine the x-coordinates of the local maxima and minima of f, and the intervals of increase and decrease. Sketch a possible graph of f (f is not unique).*

56. $f'(x) = (x - 1)(x + 2)(x + 4)$

4.4.56 $f'(x)$ is 0 at $x = -4$, $x = -2$, and $x = 1$. $f'(x) > 0$ on $(-4, -2)$ and on $(1, \infty)$, so f is increasing there, while $f'(x) < 0$ on $(-\infty, -4)$ and on $(-2, 1)$, so f is decreasing on those intervals. There must be a local maximum at $x = -2$ and local minimums at $x = -4$ and $x = 1$. An example of such a function is sketched.