Warm Up - Limit definition of derivatives

Thursday, December 10, 2020 8:16 AM

Let $f(x) = x^3 + x - 2$. Find $\lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$.

(a) 0

(b) 1

(c) 5

(d) 8

(e) 13

$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$x=a$

$$f'(a) \begin{cases} \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ \lim_{x\to a} \left(\frac{f(x)-f(a)}{x-a}\right) \end{cases}$$

$f'(2) = ?$

$f'(x) = 3x^2+1$; $f'(2) = 3\cdot 2^2 + 1 = 3\cdot 4 + 1 = 13$

(Alternatively)

Use limit def. of derivative: $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

$f(x) = x^3 + x - 2$

$a = 2$

$f(2) = 2^3 + 2 - 2 = 8$

$$\lim_{x\to 2} \frac{\overbrace{(x^3+x-2)-8}^{(x^3+x-10)}}{x-2}$$

$$\begin{array}{r} x^2+2x+5 \\ x-2 \,\overline{)\, x^3+x-10} \\ -\,(x^3-2x^2) \\ \hline 2x^2+x-10 \\ -\,(2x^2-4x) \\ \hline 5x-10 \\ -\,5x-10 \\ \hline \emptyset \end{array}$$

re-write as:

$$\lim_{x\to 2} \frac{\cancel{(x-2)}(x^2+2x+5)}{\cancel{(x-2)}}$$

DSP

$= 2^2 + 2\cdot 2 + 5$

$= 4 + 4 + 5$

$= 13$

u-substitution method for Integrals

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For this problem, you will explore the substitution rule for two different integrals. Consider the first (definite) integral:

$$J_1 = \int_{e^{-3}}^{e^2} \frac{2\ln(x) - 3}{5x}\, dx$$

Use the substitution $u = 2\ln(x) - 3$ to compute this integral. After you do the the substitution and translate the integral from being in terms of x to be being in terms of u, you have an integral of the following form:

$$J_1 = \int_a^b g(u)\, du$$

where $a < b$ and there is no number to the left of the integral sign.

(a) After the substitution, what is the integrand $g(u)$?

(b) After the substitution, what is the lower limit of integration? upper limit of integration?

Now use the fundamental theorem of calculate J_1, giving the following:

$$J_1 = \int_a^b g(u)\, du = G(b) - G(a)$$

(c) What is the relationship between g and G? G is the atiderivative of g.

(d) Calculate J_1.

Now consider the second (indefinite) integral:

$$J_2 = \int \frac{\ln(x)}{3x^2}\, dx$$

Use the substitution $u = \ln(x)$. After you do the the substitution and translate the integral from being in terms of x to be being in terms of u, you have an integral of the following form:

$$J_2 = \int f(u)\, du$$

where there is no number to the left of the integral sign.

(e) After the substitution, what is the integrand $f(u)$?

a) $u = 2\cdot\ln(x) - 3$

$du = 2\cdot\frac{1}{x}\cdot dx$

$\frac{1}{10}\cdot du = 2\cdot\frac{dx}{x}\cdot\frac{1}{2}\cdot\frac{1}{5}$

$\frac{du}{10} = \frac{1}{5}\cdot\frac{dx}{x}$

$J_1 = \int_a^b u\cdot\frac{du}{10}$

$g(u) = \frac{u}{10}$

b) $x = e^2 \Rightarrow u = 2\cdot\ln(x) - 3 \Rightarrow u = 2\cdot\ln(e^2) - 3 = 2\cdot 2 - 3 = 1$

$x = e^{-3} \Rightarrow u = 2\cdot\ln(x) - 3 \Rightarrow u = 2\cdot\ln(e^{-3}) - 3 = 2\cdot(-3) - 3 = -9$

Lower limit of integration is -9, upper limit of integration is 1.

d) $J_1 = \int_{-9}^{1} \frac{u}{10}\cdot du = \frac{u^2}{20}\Big|_{-9}^{1} = \left(\frac{1^2}{20} - \left(\frac{(-9)^2}{20}\right)\right) = \frac{1-81}{20} = \frac{-80}{20} = -4$

e) $J_2 = \int \frac{\ln(x)}{3x^2}\cdot dx$

$u = \ln_e x \Rightarrow du = \frac{1}{x}\cdot dx$

$u = \ln_e x$

$J_2 = \int \frac{\ln x}{3x\cdot x}\cdot dx$

$x = e^u$

$3x = 3\cdot e^u$

$J_2 = \int \frac{u}{3\cdot e^u}\cdot du$; $f(u) = \frac{u}{3e^u}$

Continuity

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Lim exists? LL=RL } 2-sided lim. exists

Let

$$f(x) = \begin{cases} \dfrac{\sin(2x)}{x}, & x \neq 0, \\ k, & x = 0, \end{cases}$$

where k is a constant. Determine the value of k that makes f continuous at $x = 0$.

(a) 0

(b) $\frac{1}{2}$

(c) 1

(d) 2

(e) No value of k makes f continuous at $x = 0$.

3 conditions

1) LL

2) RL

3) f(0)

Suppose that $f(x)$ is continuous at $x = 2$. Then which of the following is **NOT** necessarily true?

(a) $f'(2)$ exists.

(b) $f(2)$ is defined.

(c) $\lim_{x\to 2} f(x)$ exists.

(d) $\lim_{x\to 2} f(x) = f(2)$.

(e) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$.

x=2

f'(2) DNE

LL = RL ✓

✓

$$\underbrace{\lim_{x\to 0} \frac{\sin(ax)}{(ax)} = 1}_{\text{special trig. limit}}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\lim_{x\to 0^-} \frac{\sin(2x)\cdot 2}{2\cdot x} = \lim_{x\to 0^+} \frac{\sin(2x)\cdot 2}{2\cdot x} = k \cdot \frac{\cancel{2}}{\cancel{2}}$$

$$\lim_{x\to 0^-}\left(\cancel{\frac{\sin(2x)}{2x}}^{\,1}\cdot 2\right) = \lim_{x\to 0^+}\left(\cancel{\frac{\sin(2x)}{2x}}^{\,1}\cdot 2\right) = k$$

$$2 = 2 = k$$

$$\boxed{k=2}$$

Local min/max

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A cubic function f is defined by $f(x) = x^3 + ax^2 + bx + 1$ where a and b are constants.

(a) Find $f'(x)$.

$$f'(x) = 3x^2 + 2ax + b$$

(b) Find $f''(x)$.

$$f''(x) = 6x + 2a$$

(c) For what values of a and b does the function have a local minimum at $x = 1$ and the graph of the function have a point of inflection at $x = -1$?

$$\underbrace{f'(x)=0 \text{ or DNE}}_{\text{crit. P.}}$$

Local min at $x=1$

PoI at $x=-1$

crit. P. $f''(x)=0$ or DNE

concavity changes

First-order crit. P:

$f'(x)=0$ or $\cancel{\underbrace{\text{DNE}}_{\text{polynomial}}}$

$f'(x) = 3x^2 + 2ax + b$

$f'(1) = 0 = 3\cdot 1^2 + 2a\cdot 1 + b = 3 + 2a + b$

second-order crit. P.

$f''(x) = 0$ or $\cancel{\text{DNE}}$ polynomial

$f''(x) = 6x + 2a = 0$

$f''(-1) = 0 \Rightarrow -6 + 2a = 0 \Rightarrow \boxed{a=3}$

$0 = 3 + 2a + b$

$0 = 3 + 2\cdot 3 + b$

$0 = 9 + b$

$\boxed{b = -9}$

Optimization

Thursday, December 10, 2020 8:14 AM

We wish to construct a Norman window that has the shape of a rectangle surmounted by a semicircle. (See the diagram below.) Suppose that the perimeter of the window, i.e., the distance around the outside of the window, is 12 feet. We will determine the dimensions of the window so that the **rectangular** part of the window has the largest possible area. (**Caution! We are NOT maximizing the area of the entire window.**)

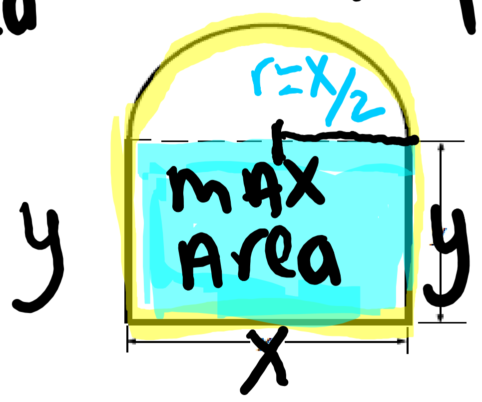
$C = 2\pi r$

$C_{semicircle} = \pi r$

$r = \frac{x}{2}$

$C_{semicircle} = \pi \cdot \frac{x}{2}$

Obj → MAX Area of Rect.



half the circumference of the circle

constraint eq.

(a) Express the perimeter of the window in terms of x and y.

$P = x + 2y +$ Circumference of semi-circle $\Rightarrow P(x,y) = x + 2y + \frac{\pi x}{2} = 2y + x\left(1 + \frac{\pi}{2}\right)$

(b) Use the result of (a) and the fact that the perimeter of the window is 12 feet to express y in terms of x.

$P(x,y) = 12$ (Given) $\Rightarrow 2y + x\left(1 + \frac{\pi}{2}\right) = 12 \Rightarrow \frac{2y}{2} = \frac{12 - x\left(1 + \frac{\pi}{2}\right)}{2}$

(c) Use the results of (a) and (b) to express the area of the **rectangular** part of the window in terms of x.

$A(x,y) = x \cdot y \Rightarrow A(x) = x \cdot \left(\frac{12 - x\left(1 + \frac{\pi}{2}\right)}{2}\right)$

Obj. f. MAX.

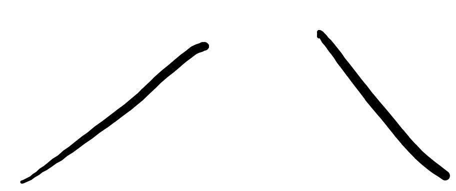
$$\boxed{\text{b) } y = \frac{12 - x\left(1 + \frac{\pi}{2}\right)}{2}}$$

(d) Finally, find the dimensions of x and y so that the area of the **rectangular** part of the window is maximized.

$$A(x) = x\left(\frac{12}{2} - \frac{x\left(\frac{2+\pi}{2}\right)}{2}\right)$$

$$= x\left(6 - \frac{x}{2}\left(\frac{2+\pi}{2}\right)\right)$$

MAX: $A'(x) = 0$ or DNE



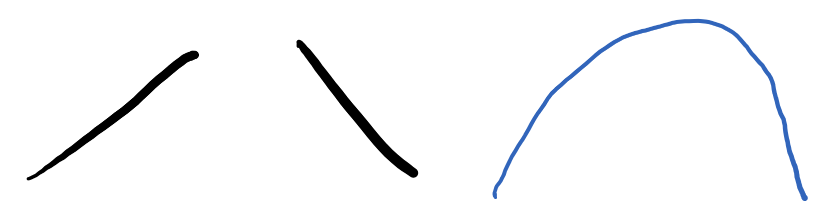
(c) Use the results of (a) and (b) to express the area of the **rectangular** part of the window in terms of x.

(d) Finally, find the dimensions of x and y so that the area of the **rectangular** part of the window is maximized.

$$A(x) = x\left(\frac{12}{2} - \frac{x\left(\frac{2+\pi}{2}\right)}{2}\right)$$

$$= x\left(6 - \frac{x}{2}\left(\frac{2+\pi}{2}\right)\right) = 6x - \frac{x^2}{2}\left(\frac{2+\pi}{2}\right)$$

MAX: $A'(x) = 0$ or DNE



$$A'(x) = \left(6x - \frac{x^2(2+\pi)}{4}\right)' = 0 \quad \text{or} \quad \cancel{\text{DNE}} \ \text{poly.}$$

$$A'(x) = 6 - \frac{\cancel{2}^{1}x(2+\pi)}{\cancel{4}_{2}} = \left(6 - \frac{x}{2}(2+\pi)\right) = 0$$

$$\frac{6}{1}_{(2)} - \frac{x}{2}(2+\pi) = 0 \Rightarrow \frac{12 - x(2+\pi)}{2} = 0$$

$$12 - x(2+\pi) = 0$$

$$\boxed{x = \frac{12}{2+\pi}}$$

Verify:

$$A''(x) = -\frac{(2+\pi)}{2} < 0 \rightarrow \text{local max}$$

global

(c) Use the results of (a) and (b) to express the area of the **rectangular** part of the window in terms of x.

(d) Finally, find the dimensions of x and y so that the area of the **rectangular** part of the window is maximized.

$x = \frac{12}{2+\pi}$

From part b) $y = \frac{12 - x\left(1+\frac{\pi}{2}\right)}{2}$

$$y = \frac{12 - \left(\frac{12}{\cancel{2+\pi}}\right)\left(\frac{\cancel{2+\pi}}{2}\right)}{2} = \frac{12-6}{2} = \frac{6}{2} = 3$$

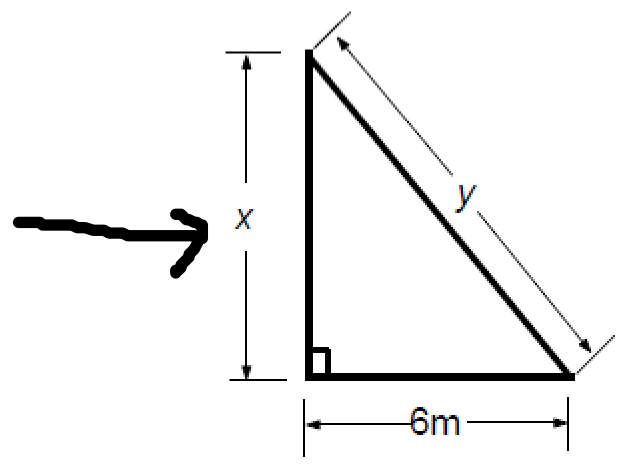
The area of the rectangular part of the window is maximized when the dimensions are:

$x = \frac{12}{2+\pi}$ ft, $y = 3$ ft.

Related Rates

Thursday, December 10, 2020 8:34 AM

A right triangle has a base of length 6 meters and a height that is increasing at a rate of 2 meters/second. At what rate is the length of the hypotenuse increasing when the height is 8 meters?



Given: base = 6m.

$$\frac{dx}{dt} = +2 \text{ m/sec.}$$

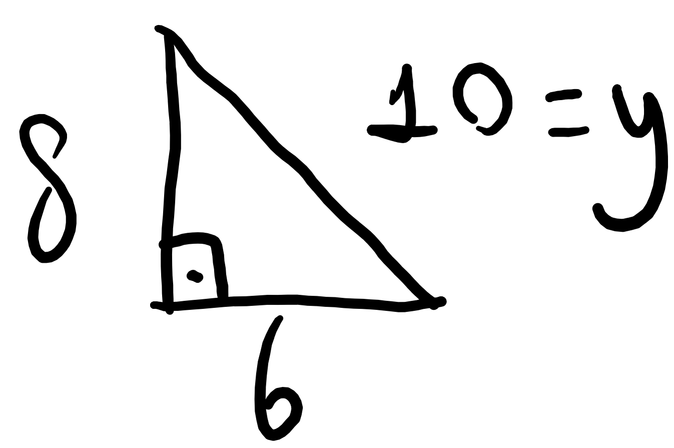
Asked:

$$\frac{dy}{dt} = ? \text{ when } \boxed{x = 8m}$$

$$x^2 + 6^2 = y^2$$

$$2x \cdot \frac{dx}{dt} + 0 = 2y \cdot \frac{dy}{dt} \Rightarrow x \cdot \frac{dx}{dt} = y \cdot \frac{dy}{dt}$$

When $x = 8m$.



$$8^2 + 6^2 = 64 + 36 = 100$$

$$y^2 = 100$$

$$x \cdot \frac{dx}{dt} = y \cdot \frac{dy}{dt} \Rightarrow 8 \cdot 2 = 10 \cdot \frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \frac{16}{10} \text{ m/sec}$$

Implicit Differentiation

Thursday, December 10, 2020 8:36 AM

Find the derivative $\frac{dy}{dx}$ at the point with $x=-2$ and $y=1$, given that

$$xy^2 - x^2 + y + 5 = 0$$

$y \to \frac{dy}{dx}$

$$1 \cdot y^2 + x \cdot 2y \cdot \frac{dy}{dx} - 2x + \frac{dy}{dx} + 0 = 0$$

$$y^2 + 2xy\frac{dy}{dx} - 2x + \frac{dy}{dx} = 0$$

$$2xy\frac{dy}{dx} + \frac{dy}{dx} = 2x - y^2$$

$$(2xy+1)\frac{dy}{dx} = 2x - y^2 \Rightarrow \frac{dy}{dx} = \frac{2x-y^2}{2xy+1}$$

$$\left.\frac{dy}{dx}\right|_{x=-2,\, y=1} = \frac{2(-2)-1^2}{2\cdot(-2)\cdot 1+1} = \frac{-5}{-3} = \frac{5}{3}$$