Final Exam Review - Supplementary Problems

Q) Find an equation of the normal line to the curve: $x^2 \cdot \sqrt{y-2} = y^2 - 3x - 5$ at $(1,3)$.

$$2x \cdot \sqrt{y-2} + x^2 \cdot \frac{1}{2} \cdot (y-2)^{-\frac{1}{2}} \cdot \boxed{\frac{dy}{dx}} = 2y \cdot \boxed{\frac{dy}{dx}} - 3$$

$$2 \cdot 1 \cdot \sqrt{3-2} + 1^2 \cdot \frac{1}{2} (3-2)^{-1/2} \cdot \frac{dy}{dx} = 2 \cdot 3 \cdot \frac{dy}{dx} - 3$$

$$2 + \frac{1}{2} \cdot 1 \cdot \frac{dy}{dx} = 6 \cdot \frac{dy}{dx} - 3$$

$$5 = \frac{6}{(\frac{1}{2})} \cdot \frac{dy}{dx} - \frac{1}{2} \cdot \frac{dy}{dx}$$

$$5 = \left(\frac{12}{2} - \frac{1}{2}\right) \cdot \frac{dy}{dx}$$

$$\frac{2}{11} \cdot 5 = \frac{11}{2} \cdot \frac{dy}{dx} \cdot \frac{2}{11}$$

$$\underbrace{\frac{dy}{dx} = \frac{10}{11}}_{m_{tan}}$$

$$m_{normal} = \frac{-1}{m_{tan}} = \frac{-1}{10/11} = \frac{-11}{10}$$

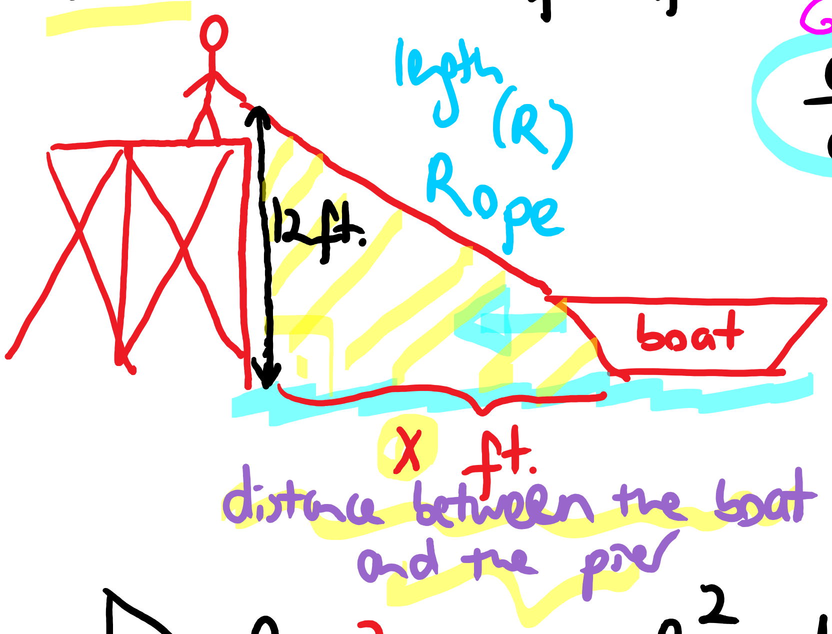
$(1,3)$

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{-11}{10}(x-1)$$

Q)

A person is standing at the end of a pier 12 ft. above the water and is pulling in a rope attached to a rowboat at the waterline at the rate of 6 ft. of rope per min. How fast is the boat moving in the water when it is 16 ft. from the pier?

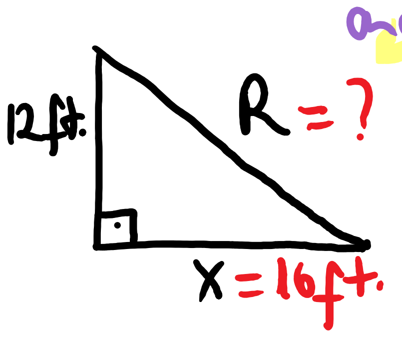


Given:

$$\frac{dR}{dt} = -6 \ \frac{ft}{min}$$

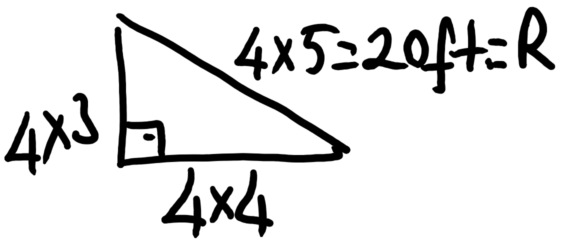
Find $\frac{dx}{dt}$?

when $x = 16ft$



$$R^2 = 12^2 + x^2$$

$$\underbrace{2R\cdot\frac{dR}{dt} = 0 + 2x\cdot\frac{dx}{dt}}$$



$$\cancel{2}\cdot\cancel{20}^{\,5}\cdot(\cancel{-6}^{\,-3}) = \cancel{2}\cdot\cancel{16}^{\,\cancel{4}\,2}\cdot\frac{dx}{dt}$$

$$-15 = 2\cdot\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = -7.5 \ \frac{ft}{min.}$$

Q)

$C(x) = \frac{1}{7}x^2 + 4x + 100$ Cost f.

$p(x) = \frac{1}{4}(80-x)$ price per item

Calc

a) What's the marginal cost?

b) What's the price when the marginal cost is 10?

c) Estimate the cost of producing the 11th item

d) Find the actual cost of producing the 11th item.

Alg

a) $MC = C'(x) = \left(\frac{1}{7}x^2 + 4x + 100\right)' = \frac{2}{7}x + 4$

b) $MC = 10 \Rightarrow p(x) = ?$ $C'(x) = 10 = \frac{2}{7}x + 4$

$$\begin{array}{rcl} 10 & = & \frac{2}{7}x + 4 \\ -4 & & -4 \\ \hline \frac{6}{1} & = & \frac{2}{7}x \Rightarrow x = 21 \end{array}$$

$p(21) = \frac{1}{4}(80-21)$

$= \frac{1}{4} \cdot 59 = 14\frac{3}{4} = \$14.75$

c) $MC(10) = \frac{2}{7} \cdot 10 + 4 = \frac{20}{7} + \frac{4}{1}\frac{(7)}{(7)} = \frac{48}{7} \approx \$6.90$

d) $C(11) - C(10)$

$C(x) = \frac{1}{7}x^2 + 4x + 100$

d) $C(11) - C(10)$

$$\frac{1}{7}\cdot 11^2 + 4\cdot 11 + 100 - \left(\frac{1}{7}\cdot 10^2 + 4\cdot 10 + 100\right)$$

$$\frac{1}{7}\cdot\underbrace{(11^2 - 10^2)} + \underbrace{4\cdot 11 - 4\cdot 10}$$

$$\frac{1}{7}(11-10)(11+10) + 4$$

$$\frac{1}{7}\cdot 1\cdot 21 + 4 = 3 + 4 = \$7$$

actual

$6.9

estimate

Q)

$$f(x)=\frac{x^3+1}{x^3-8},\quad f'(x)=\frac{-27x^2}{(x^3-8)^2},\quad f''(x)=\frac{108x(x^3+4)}{(x^3-8)}$$

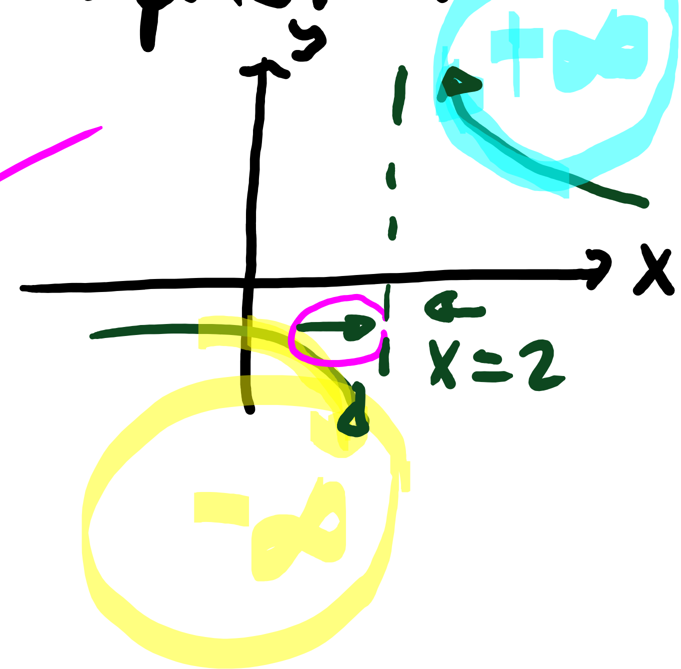
Domain of $f(x)$: $(-\infty, 2) \cup (2,\infty)$ → AND,

$x^3-8=0 \Rightarrow x^3=8 \Rightarrow x=2$

Function is NOT defined at $x=2$

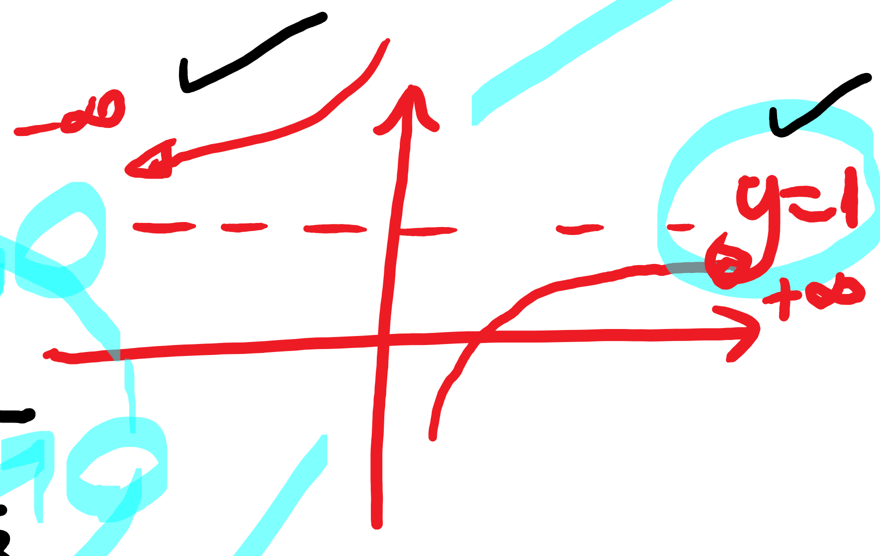
Vertical Asymptote: $x=2$ is a potential V.A.

$$\lim_{x\to 2^+}\frac{x^3+1}{\underbrace{x^3-8}_{x^3-2^3}} \overset{\text{"DSP"}}{=} \frac{2^3+1}{\text{"very tiny pos. \#"}} = +\infty$$



$$\lim_{x\to 2^-}\frac{x^3+1}{\underbrace{x^3-8}_{x^3-2^3}} \overset{\text{"DSP"}}{=} \frac{2^3+1}{\text{"very tiny neg. \#"}} = -\infty$$

Horizontal Asymptotes



$$\lim_{x\to\infty}\frac{x^3+1}{x^3-8} = \lim_{x\to\infty}\frac{\frac{x^3}{x^3}+\frac{1}{x^3}}{\frac{x^3}{x^3}-\frac{8}{x^3}}$$

$$= \lim_{x\to\infty}\frac{1}{1} = 1$$

$$\lim_{x\to-\infty}\frac{1+\frac{1}{x^3}}{1-\frac{8}{x^3}} = 1$$

∞ $1-\frac{8}{x^3} \to 0$ — $-$

Include the x-coordinates of V.A in the sign charts for $f'(x)$, $f''(x)$

$$f'(x) = \frac{-27x^2}{(x^3-8)^2}$$

First-order critical #s

$f'(x) = 0$

$-27x^2 = 0$

$x = 0$

$f'(x)$ DNE

$x = 2$

Construct sign chart for $f'(x)$

	$-\infty$ … 0	0 … 2	2 … $+\infty$
sign of $f'(x)$	$-$	$-$	$-$
incr/decr.	↘	↘	↘

$f(x)$ is decreasing on $(-\infty, 2) \cup (2, \infty)$

NONE → local min
local max

$f(x)$ is NOT defined at $x=2$

Sign chart for $f''(x) = \dfrac{108x\,(x^3+4)}{(x^3-8)^3}$

include $x=2$ in the sign chart

$f''(x)=0$ or DNE

$108x\,(x^3+4)=0$

$x=0$, $\quad x^3+4=0$

$x^3 = -4$

$x = \sqrt[3]{-4} \approx -1.58$

$\sqrt[3]{-8} \qquad \sqrt[3]{-1}$

sign chart for $f''(x)$	$-\infty$	$\overset{-10}{\downarrow}$	$\sqrt[3]{-4}$	$\overset{-1}{\downarrow}$	0	$\overset{1}{\downarrow}$	$\cancel{2}$	$\overset{5}{\downarrow}$	$+\infty$
sign of $f''(x)$		$-$	|	$+$	|	$-$	|	$+$	
Concave up/down		$\cap$	|	$\cup$	|	$\cap$	|	$\cup$	

$f''(x) = \dfrac{108x\,(x^3+4)}{(x^3-8)^3}$

Test points:

$f''(-10) = \dfrac{\ominus\cdot(-10^3+4)}{(-10^3-8)^3} = \dfrac{\ominus\cdot\ominus}{\ominus} = \dfrac{\oplus}{\ominus} = \ominus$

$f''(-1) = \dfrac{\ominus\;(-1^3+4)}{(-1^3-8)^3} = \dfrac{\ominus\,\oplus}{\ominus} = \oplus$

$f''(1) = \dfrac{\oplus\cdot\oplus}{\ominus} = \dfrac{\oplus}{\ominus}$ | $f''(5) = \dfrac{\oplus\,\oplus}{\oplus} = \oplus$

The points of inflection are: (x, y)

$\left(0, f(0)\right) \Rightarrow f(0) = \frac{0^3+1}{0^3-8} = -\frac{1}{8}$ $\left(0, -\frac{1}{8}\right)$

$\left(\sqrt[3]{-4}, f\left(\sqrt[3]{-4}\right)\right) \Rightarrow f\left(\sqrt[3]{-4}\right) = \frac{-4+1}{-4-8} = \frac{-3}{-12} = \frac{1}{4}$

$f(x) = \frac{x^3+1}{x^3-8}$ $\left(\sqrt[3]{-4}, \frac{1}{4}\right)$

$f(x)$ is concave up on $\left(\sqrt[3]{-4}, 0\right) \cup (2, +\infty)$

$f(x)$ is concave down on $\left(-\infty, \sqrt[3]{-4}\right) \cup (0, 2)$

L'Hôpital's Rule

Evaluate the limit $\lim_{x \to 0} \frac{x^2 + \sin(x^2)}{x^2 + x^3}$

$$\text{"DSP"} = \frac{0^2 + \sin(0^2)}{0^2 + 0^3} = \text{"}\frac{0}{0}\text{"}$$

$$\text{L.R.} = \lim_{x \to 0} \frac{2x + \cos(x^2)\cdot(2x)}{2x + 3x^2} \overset{\text{"DSP"}}{=} \text{"}\frac{0 + \cos 0 \cdot 0}{0 + 0}\text{"}$$

$$\text{L.R} = \lim_{x \to 0} \frac{2 + (-\sin(x^2))\cdot 2x \cdot 2x + \cos(x^2)\cdot 2}{2 + 6x}$$

$$= \lim_{x \to 0} \frac{2 + (-\sin(x^2))\cdot 2x \cdot 2x + \cos(x^2)\cdot 2}{2 + 6x}$$

$$\overset{\text{"DSP"}}{=} \frac{2 + (-\sin 0)\cdot 0 + \cos(0^2)\cdot 2}{2 + 0}$$

$$= \frac{2 + 0 + 2}{2} = \frac{4}{2} = 2$$