Two Special Limits

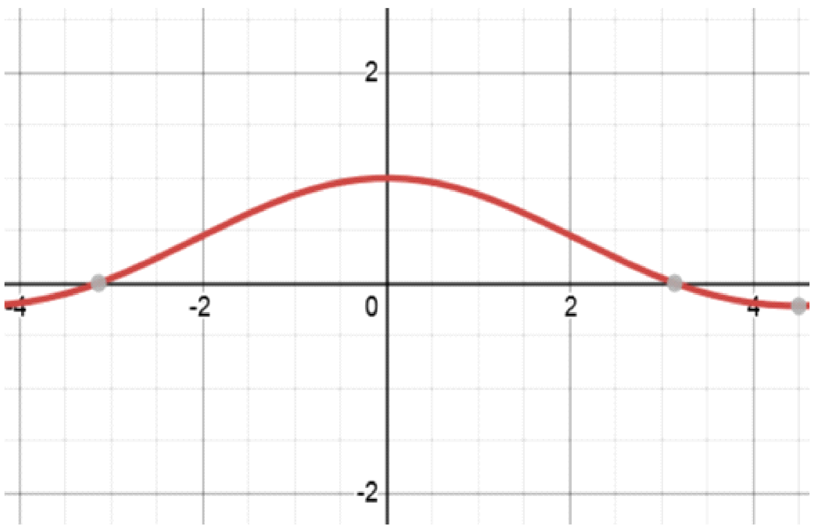
Our principal goal is to determine derivative formulas for $\sin x$ and $\cos x$. To do this, we use two special limits.

THEOREM 3.10 Trigonometric Limits

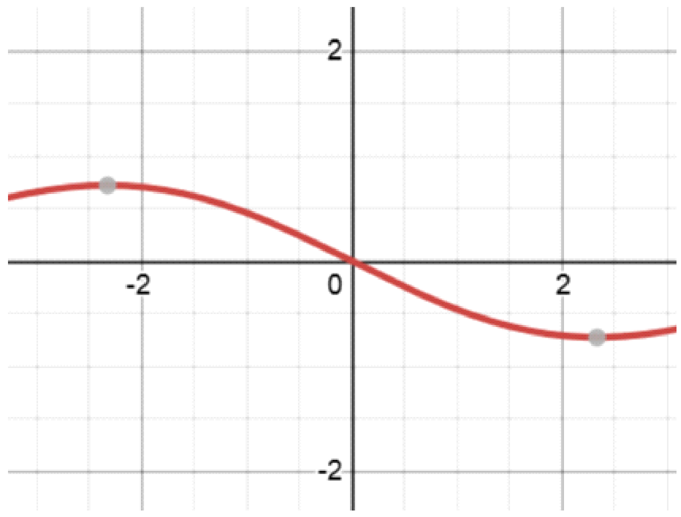
$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$

Put it on the formula sheet!

Graph of f(x)= sin(x) / (x)



Graph of f(x) = (cos(x) -1)/ (x)



EXAMPLE 1 **Calculating trigonometric limits** Evaluate the following limits.

a. $\lim_{x \to 0} \frac{\sin 4x}{x}$ **b.** $\lim_{x \to 0} \frac{\sin 3x}{\sin 5x}$

a) Try "DSP"

$$\lim_{x \to 0} \frac{\sin 4x}{x} \overset{\text{"DSP"}}{=} \frac{\sin(4 \cdot 0)}{0} = \text{"}\frac{0}{0}\text{"}$$ indeterminate form

Use the special trig limit theorem:

$$\lim_{x \to 0} \frac{\sin 4x}{x} \cdot \frac{4}{4} = \lim_{x \to 0} 4 \cdot \frac{\cancel{\sin 4x}}{4x}^{1} = 4$$

b) Try "DSP"

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 5x} \overset{\text{"DSP" " "}}{=} \frac{0}{0}$$ indet. form; use special trig limit th:

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 5x} = \lim_{x\to 0}\frac{\cancelto{1}{\frac{\sin 3x}{3x}}\cdot 3x}{\cancelto{1}{\frac{\sin 5x}{5x}}\cdot 5x}$$

(they all cancel out, however, this lets you to use the special trig limit th.)

$$= \lim_{x\to 0}\frac{3x}{5x}$$ (cancel out x's)

$$= \lim_{x\to 0}\frac{3}{5} = \frac{3}{5}$$

3.5 Official Textbook Problems (even numbered questions only)

Practice Exercises

11–22. Trigonometric limits *Use Theorem 3.10 to evaluate the following limits.*

11. $\lim_{x\to 0} \frac{\sin 3x}{x}$

12. $\lim_{x\to 0} \frac{\sin 5x}{3x}$

13. $\lim_{x\to 0} \frac{\sin 7x}{\sin 3x}$

14. $\lim_{x\to 0} \frac{\sin 3x}{\tan 4x}$

15. $\lim_{x\to 0} \frac{\tan 5x}{x}$

16. $\lim_{\theta\to 0} \frac{\cos^2\theta - 1}{\theta}$

17. $\lim_{x\to 0} \frac{\tan 7x}{\sin x}$

18. $\lim_{\theta\to 0} \frac{\sec\theta - 1}{\theta}$

19. $\lim_{x\to 2} \frac{\sin(x-2)}{x^2-4}$

20. $\lim_{x\to -3} \frac{\sin(x+3)}{x^2+8x+15}$

21. $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$, where a and b are constants with $b \neq 0$

22. $\lim_{x\to 0} \frac{\sin ax}{bx}$, where a and b are constants with $b \neq 0$

Use special trig. limit theorems:

12) $$\lim_{x\to 0}\left(\frac{\sin 5x}{3x}\right) = \lim_{x\to 0}\left(\frac{\sin 5x}{3x}\cdot\frac{5}{5}\right) = \lim_{x\to 0}\left(\frac{5\cdot \overbrace{\sin 5x}^{1}}{3\cdot 5x}\right)$$

group these

$$= \frac{5}{3}$$

14) $$\lim_{x\to 0}\left(\frac{\sin 3x}{\tan 4x}\right) = \lim_{x\to 0}\left(\frac{\sin 3x}{\frac{\sin 4x}{\cos 4x}}\right) = \lim_{x\to 0}\left(\frac{\sin 3x\cdot\cos 4x}{\sin 4x}\right)$$

$$= \lim_{x\to 0}\left(\overbrace{\frac{\sin 3x}{1\cdot 3x}}^{1}\cdot 3x\cdot\frac{\cos 4x}{1}\cdot\overbrace{\frac{1\cdot 4x}{\sin 4x\cdot 4x}}^{1}\right) = \lim_{x\to 0}\left(3\cancel{x}\cdot\cos 4x\cdot\frac{1}{4\cancel{x}}\right)$$

$$= \lim_{x\to 0}\left(\frac{3\cdot\cos 4x}{4}\right) \overset{\text{"DSP"}}{=} \frac{3\cdot\cos 0}{4} = \frac{3}{4}$$

recall:
$$\lim_{x\to 0}\frac{x}{\sin x} = 1$$

check graph!

16) $\lim\limits_{\theta \to 0} \left(\dfrac{\cos^2\theta - 1}{\theta} \right)$

Recall: $\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta - 1 = -\sin^2\theta$

$$\lim_{\theta \to 0} \left(\frac{-\sin^2\theta}{\theta} \right) = \lim_{\theta \to 0} \left(\cancelto{1}{\frac{-\sin\theta}{\theta}} \cdot \frac{\sin\theta}{1} \right)$$

$$= \lim_{\theta \to 0} \left(-\sin\theta \right) \overset{\text{"DSP"}}{=} -\sin 0 = 0$$