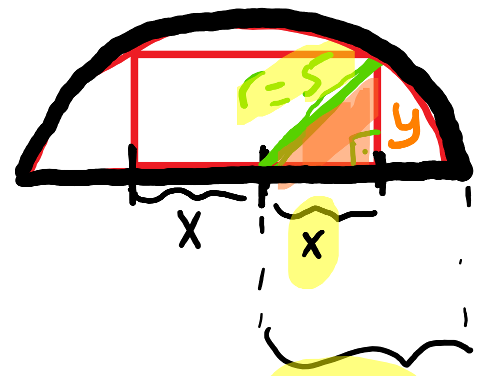
Rectangle Area Optimization

17. Rectangles beneath a semicircle A rectangle is constructed with its base on the diameter of a semicircle with radius 5 and its two other vertices on the semicircle. What are the dimensions of the rectangle with maximum area?

$r=5$

Draw:



Obj. : MAX $A_{rectangle} = 2x \cdot y$ $r=5$

Constraints: right Δ: $x^2+y^2=5^2$

constraint eq: $x^2+y^2=5^2 \Rightarrow y^2=25-x^2$

Obj. F: $A(x,y) = 2x \cdot y$ MAX

$\hookrightarrow A(x),\ A(y)$

$y^2=25-x^2 \Rightarrow y=\sqrt{25-x^2}$ $\rightarrow (25-x^2)^{1/2}$

Re-write obj. F: $A(x)=2x\cdot\sqrt{25-x^2}$ MAX

Critical P. of $A(x) \rightarrow$ MAX $A(x)$

$A'(x)=0$ or DNE

$A'(x) = 2\cdot\sqrt{25-x^2} + \cancel{2}x\cdot\frac{1}{\cancel{2}}(25-x^2)^{-1/2}\underbrace{(-2x)}_{\text{chain rule!}}$

$= \frac{2\sqrt{25-x^2}}{1}\left(\sqrt{25-x^2}\right) - \frac{2x^2}{\sqrt{25-x^2}} = \frac{2(25-x^2)-2x^2}{\sqrt{25-x^2}}$

Rectangle Area Optimization

$I: (0, 5)$

$$A'(x) = \frac{50-2x^2-2x^2}{\sqrt{25-x^2}} = \boxed{\frac{\overbrace{50-4x^2}^{}}{\sqrt{25-x^2}}} \begin{matrix} \to 0 \\ \to \text{DNE} \end{matrix}; \quad \boxed{x = \sqrt{\frac{50}{4}}} \quad \cancel{x = \pm 5}$$

critical P.

$\hookrightarrow$ not in I

Sign chart for $A'(x)$

	$(0$ $\quad 3 \quad$	$x = \sqrt{\frac{50}{4}} \approx 3.53$ $\quad 4 \quad 5)$
A'	$+$	$-$
MAX $\leftarrow$ A	local max at $x = \sqrt{\frac{50}{4}}$	

$$A'(3) = \frac{50-4\cdot 3^2}{\cancel{\sqrt{}} \to \oplus} = \frac{50-36}{\oplus} = \frac{\oplus}{\oplus} \to \oplus$$

$$A'(4) = \frac{50-4\cdot 4^2}{\oplus} \to \frac{\ominus}{\oplus} \to \ominus$$

Since, $x = \sqrt{\frac{50}{4}}$ is the ONLY local MAX;

the $x = \sqrt{\frac{50}{4}}$ is also the GLOBAL MAX.

Use the constraint eq: $(x^2+y^2=25)$ to find y.

$$\left(\sqrt{\frac{50}{4}}\right)^2 + y^2 = 25 \Rightarrow y^2 = 25 - \frac{50}{4} = \frac{50}{4} \Rightarrow y = \sqrt{\frac{50}{4}}$$