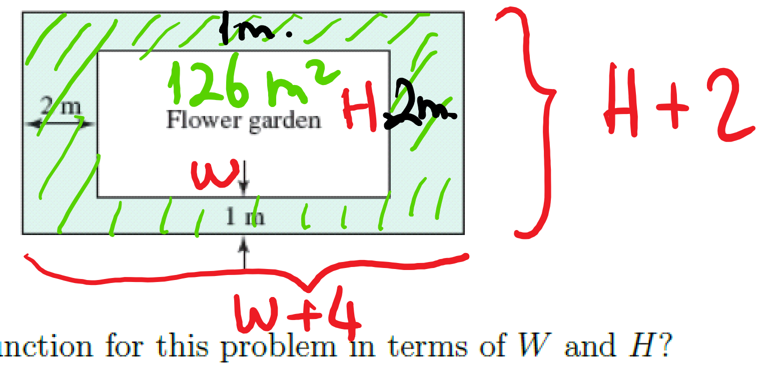
Spring 2020 Final Exam Question

A local park has hired you to construct a rectangular flower garden surrounded by a grass border that is 1 m wide on two sides and 2 m wide on the other two sides. (See the figure blow.) The area of the garden only (the small rectangle) must be 126 m^2. Your primary task is to find the dimensions of the garden that give the smallest possible combined area of the garden and the grass border. For this problem, let W be the horizontal width of the garden and let H be the vertical height of the garden.



$W \cdot H = 126$

$W = 252$ m.

$H = \frac{1}{2}$ m.

H ↓ W ↑

(a) What is the objective function for this problem in terms of W and H?

(b) What is the constraint equation for this problem in terms of W and H?

(c) Find the objective function in terms of W only.

(d) What is the interval of interest for the objective function? $W \to (0, \infty)$

(e) Find the values of W and H that minimize the total combined area.

(f) What horizontal width W of the garden will *maximize* the total area? None

a) Obj: MIN. combined area

Obj. F: $A(W,H) = (W+4)\cdot(H+2) \to A(W)$, $A(H)$

$W \cdot H = 126$

$H = \frac{126}{W}$

b) Constraint eq: $W \cdot H = 126$

c) $A(W,H) \to A(W)$

$$A(W) = (W+4)\cdot\left(\frac{126}{W} + 2\right) = 126 + 2W + \frac{504}{W} + 8$$

$$= 134 + 2W + 504 \cdot W^{-1}$$

e) MIN. $A(W)$

$A'(W) = 0$ or DNE ; $A'(W) = (134 + 2W + 504 \cdot W^{-1})'$

$= 0 + 2 - 1 \cdot W^{-2} \cdot 504$

$= 2 - 504 \cdot W^{-2} = 0$

$= 2 - \frac{504}{W^2} = \frac{2W^2 - 504}{W^2}$

crit. P: $2W^2 - 504 = 0$, $W = 0$

$W^2 = 252$

$W = 6\sqrt{7}$

$(0, \infty)$

e) min. $A(w)$

$A'(w) = 0$ or DNE ; $A'(w) = (134 + 2w + 504 \cdot w^{-1})'$

crit. P: $2w^2 - 504 = 0$, ~~$w = 0$~~

$w^2 = 252$

$w = 6\sqrt{7}$

$(0, \infty)$

$= 0 + 2 - 1 \cdot w^{-2} \cdot 504$

$= 2 - 504 \cdot w^{-2} = 0$

$= 2 - \dfrac{504}{w^2} = \dfrac{2w^2 - 504}{w^2}$

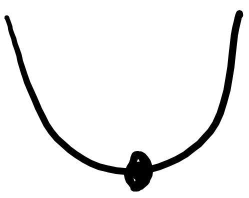
Sign chart

	0	$\overset{2}{\downarrow}$	$6\sqrt{7}$	$\overset{504}{\downarrow}$	∞
$A'(w)$		$-$		$+$	
$A(w)$		$\searrow$	$\boxed{\text{local min}}$	$\nearrow$	

local min at $w = 6\sqrt{7}$

$A''(w) = (2 - 504 \cdot w^{-2})' = 0 + \underbrace{504 \cdot (+2) \cdot w^{-3}}_{\text{pos.}}$

$A''(w) > 0$



$A(w)$

$\hookrightarrow w = 6\sqrt{7}$

~~local~~ global min

When $w = 6\sqrt{7}$ m. $A(w)$ is min.

$W \cdot H = 126 \Rightarrow H = \dfrac{\cancel{126}^{21}}{\cancel{6}\sqrt{7}} = \dfrac{\cancel{21}^{3}\sqrt{7}}{\cancel{7}} = 3\sqrt{7}$ m.

f) None.