Flipturning Polygons*

(Extended abstract)

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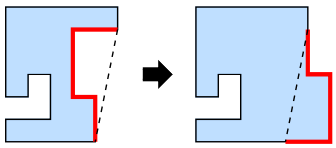

Figure 1. A flipturn. The pocket is bold (red), and its lid is dashed.

A central problem in polymer physics and molecular biology is the reconfiguration of large molecules (modeled as polygons) such as circular DNA. Most of the research in this area involves computer-intensive Monte-Carlo simulations. One efficient method frequently used to generate random chains or polygons is to modify one such object into another using a simple *pivot* operation. This paper is concerned with a pivot of central concern in polymer physics research called a *flipturn*, first defined in an unpublished 1973 paper of Joss and Shannon [4] as follows. A *pocket* of a nonconvex polygon P is a maximal connected sequence of polygon edges disjoint from the convex hull of P except at its endpoints. The line segment joining the endpoints of a pocket is called the *lid*. A flipturn rotates a pocket 180 degrees about the midpoint of its lid, or equivalently, reverses the order of the edges of a pocket without changing their lengths or orientations. Figure 1 shows the effect of a single flipturn on a nonconvex orthogonal polygon.

Joss and Shannon proved that any simple polygon with n sides can be convexified by a sequence of at most $(n-1)!$ flipturns, by observing that each flipturn produces a new cyclic permutation of the edges. They also conjectured that $n^2/4$ flipturns are always sufficient. Biedl [2] discovered a family of polygons that are convexified only after $(n-2)^2/4$ badly chosen flipturns, nearly matching Joss and Shannon's conjectured upper bound. Dubins *et al.* [3] showed that simple lattice polygon in the plane can be convexified with $n-4$ well-chosen flipturns [5]. Until very recently this was the best upper bound known. Ahn *et al.* [1] recently proved that any polygon with s distinct edge slopes is convexified after at most $\lceil n(s-1)/2-s \rceil$ modified flipturns; in particular, $n/2-2$ modified flipturns suffice to convexify any orthogonal polygon, and $n(n-3)/2$ modified flipturns suffice to convexify any simple polygon. For related reultss, see [7].

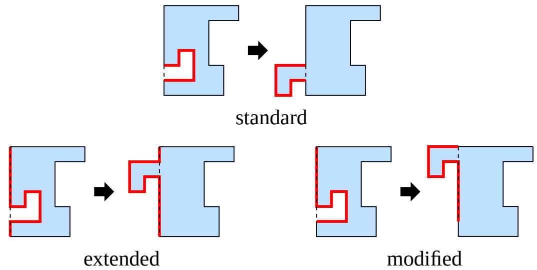

Figure 2. Three types of degenerate flipturns.

Our results depend critically on the behavior of flipturns in the degenerate case where the polygon edges just outside the pocket lie on the same line as the lid. We offer three alternate definitions, illustrated in Figure 2. A *standard* pocket touches the polygons convex hull only at its endpoints; this is closest to the original definition of Joss and Shannon. An *extended* pocket starts and ends at convex hull vertices. Ahn *et al.* [1] define a *modified* pocket to be the same as a standard pocket, except that it includes one extra edge if that edge is colinear with the lid. Modified flipturns seem to have been defined specifically to avoid the 'interesting' properties of degenerate flipturns. For orthogonal polygons, for example, *every* modified flipturn removes exactly two vertices, but standard and extended flipturns only remove vertices if the lid is not horizontal or vertical.

We derive new worst-case upper and lower bounds on both the shortest and longest convexifying flipturn sequences for orthogonal polygons. The bounds for the shortest sequence tell us how quickly we can convexify a polygon if we choose flipturns intelligently; the longest sequence bounds tell us how many flipturns we can perform even if we choose flipturns blindly. Our results are summarized in the first two rows of Table 1; the last row of each table gives the corresponding results of Ahn *et al.* for modified flipturns [1]. We also show that the shortest and longest flipturn sequences

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Flipturn type	Shortest flipturn sequence orthogonal polygons	Longest flipturn sequence orthogonal polygons	Longest flipturn sequence s-oriented polygons	Longest flipturn sequence arbitrary polygons
standard	$\lfloor 3(n-4)/4 \rfloor \le ?? \le \lfloor 5(n-4)/6 \rfloor$	$\lfloor 5(n-4)/6 \rfloor \le ?? \le n-5$	$\le ns - \lfloor 3(n+s)/2 \rfloor - 1$	$\le n^2 - 4n + 1$
extended	$\lfloor 3(n-4)/4 \rfloor$	$\lfloor 3(n-4)/4 \rfloor \le ?? \le n-5$	$\le ns - \lfloor 3(n+s)/2 \rfloor - 1$	$\le n^2 - 4n + 1$
modified [1]	$(n-4)/2$	$(n-4)/2$	$\le \lceil n(s-1)/2 \rceil - s$	$\le n(n-3)/2$

Table 1. Bounds for shortest and longest flipturn sequences for various types of flipturns and polygons.

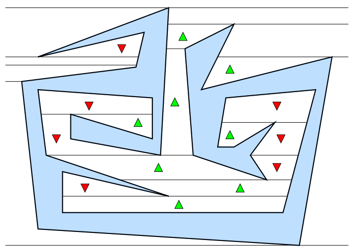

Figure 3. Decomposing the exterior of a polygon. Triangles indicate up-regions and down-regions.

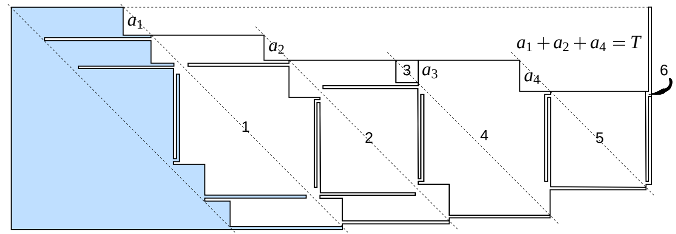

Figure 4. The reduction from a positive instance of SUBSET SUM to am orthogonal polygon (shaded), and a sequence of flipturns that leads to an orthogonal pocket.

for orthogonal polygons can differ in length by as much as $\lfloor (n-4)/4 \rfloor$ in the worst case. To prove that any orthogonal polygon is convexified after at most $n-5$ flipturns, we carefully analyze the number of *brackets*—edges with two convex or two reflex endpoints—added or removed by any flipturn. Combining this with the *discrete angle* technique of Ahn *et al.* [1], we also show that any simple polygon is convexified after at most $n^2 - 4n + 1$ standard or extended flipturns.

We describe how to maintain implicit descriptions of a simple polygon and its convex hull in $O(\log^4 n)$ time per flipturn, using a data structure of size $O(n)$. Together with our earlier results, this implies that we can compute a convexifying sequence of flipturns for any polygon in $O(n^2 \log^4 n)$ time, or for any orthogonal polygon in $O(n \log^4 n)$ time. Our data structure is a variant of the dynamic convex hull structure of Overmars and van Leeuwen [6], with some important differences. First, instead of insertions and deletions, we must support an operation that reverses an entire subsequence of edges. This requires us to store the vertices in their order of appearance around the polygon, rather than in any coordinate order. Since a linear number of vertices could be affected by a flipturn, our data structure must implicitly represent both the order and the locations of the vertices. Like Overmars and van Leeuwen's dynamic hull structure, our structure is a balanced binary tree that (implicitly) stores the convex hull of a subset of the vertices at each node. Unlike their structure, however, sibling subhulls in our structure can be disjoint, nested, or overlapping with exactly two common boundary points. Distinguishing these three cases and merging the subhulls in each case requires considerably more effort.

We also prove that for any simple polygon, every sequence of flipturns eventually leads to the same convex polygon. The fact that the *shape* of the final convex polygon is independent of the flipturn sequence is rather obvious, but the independence of the final polygon's *position* requires more effort. Our approach is to consider a horizontal trapezoidal decomposition of the exterior of the polygon. We call a finite trapezoid an *up-region* (resp. down-region) if the shortest path to infinity starts by going upward (resp. downward). We show that the total height of the up-regions is exactly the distance that the top of the polygon moves upward during during any flipturn sequence. This characterization allows us to compute the final convex polygon in $O(n \log n)$ time, without computing any actual flipturns.

Finally, we show that finding the longest flipturn sequence for a given simple polygon is NP-hard, using a linear-time reduction from the problem SUBSET SUM. The reduction is sketched in Figure 4. Given a set of integers A and a target integer T, our algorithm constructs an orthogonal polygon that can undergo exactly one orthogonal flipturn if and only if some subset of A sums to T. Since orthogonal flipturns do not remove vertices, this increases the number of flipturns by one. The overall structure of the polygon is a staircase, where the lengths of the stairs are the elements of A and the integer T, adorned with several inward and outward 'spikes' of length roughly equal to T.

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