Time-dependent density functional theory

E.K.U. Gross

Fritz Haber Center for Molecular Dynamics



האוניברסיטה העברית בירושלים THE HEBREW UNIVERSITY OF JERUSALEM الجامعة العبرية في اورشليم القدس

DAY 1

- Basic theorems of TDDFT
- <u>TDDFT in the linear response regime:</u>
 -- Dyson-like equation for the response function
 -- Calculation of photo-absorption spectra

DAY 2

- **TDDFT beyond the linear regime:**
 - -- Ultrafast laser-driven spin dynamics
 - -- xc functional for non-collinear magnetism

What do we want to describe?

System in laser field: Generic situation



$$\hat{H}(t) = \hat{T}_{e} + \hat{T}_{n} + \hat{W}_{ee} + \hat{W}_{nn} + \sum_{j,\alpha} - \frac{Z_{\alpha} e^{2}}{|r_{j} - R_{\alpha}|} + \vec{r}_{j} \cdot \vec{E}(t) \cdot \sin \omega t$$

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Electronic transport: Generic situation left lead L central region C right lead R

Bias between L and R is turned on: U(t) \longrightarrow V

Why don't we just solve the many-particle SE?

Example: Aluminium atom (13 electrons)

 $\Psi(\vec{r}_1,...,\vec{r}_{13})$ depends on 39 coordinates

rough table of the wavefunction

10 entries per coordinate: $\Rightarrow 10^{39}$ entries1 byte per entry: $\Rightarrow 10^{39}$ bytes 10^{12} bytes per SSD: $\Rightarrow 10^{27}$ SSDs20 g per SSD: $\Rightarrow 2 \times 10^{28}$ g SSDs (4x earth)

For Ti atom the required mass of SSDs exceeds mass of the universe

ESSENCE OF DENSITY-FUNTIONAL THEORY

- Every observable quantity of a quantum system can be calculated from the density of the system ALONE
- The density of particles interacting with each other can be calculated as the density of an auxiliary system of <u>non</u>-interacting particles

$$\hat{H}(t) = \hat{T}_{e} + \hat{W}_{ee} + \sum_{j,\alpha} - \frac{Z_{\alpha} e^{2}}{|r_{j} - R_{\alpha}|} + \vec{r}_{j} \cdot \vec{E}(t) \cdot \sin \omega t$$

Strong laser $(v_{laser}(t) \ge v_{en})$:

Non-perturbative solution of full TDSE required

Weak laser ($v_{laser}(t) \ll v_{en}$) :Calculate1. Linear density response $\rho_1(\vec{r} t)$

2. Dynamical polarizability $\alpha(\omega) = -\frac{e}{F} \int z \rho_1(\vec{r}, \omega) d^3 r$

3. Photo-absorption cross section
$$\sigma(\omega) = -\frac{4\pi\omega}{c} \text{Im}\alpha$$

 $\Lambda \pi \omega$

Photo-absorption in weak lasers



Basic theorems of TDDFT

(E. Runge, E.K.U.G., PRL 52, 997 (1984))

1-1 correspondence (TD analogue of Hohenberg-Kohn theorem):

The time-dependent density determines uniquely $v(rt) \xleftarrow{1-1}{\leftarrow} \rho(rt)$ the time-dependent external potential and hence all physical observables for fixed initial state.

TDKS theorem:

The time-dependent density of the <u>interacting</u> system of interest can be calculated as density 12

$$\varphi(\mathbf{rt}) = \sum_{j=1}^{N} \left| \varphi_{j}(\mathbf{rt}) \right|^{2}$$

of an auxiliary non-interacting (KS) system

$$i\hbar\frac{\partial}{\partial t}\varphi_{j}(rt) = \left(-\frac{\hbar^{2}\nabla^{2}}{2m} + v_{s}[\rho](rt)\right)\varphi_{j}(rt)$$

with the local potential

$$\mathbf{v}_{s}\left[\rho(\mathbf{r}'\mathbf{t}')\right](\mathbf{rt}) = \mathbf{v}(\mathbf{rt}) + \int d^{3}\mathbf{r}' \frac{\rho(\mathbf{r}'\mathbf{t})}{|\mathbf{r}-\mathbf{r}'|} + \mathbf{v}_{xc}\left[\rho(\mathbf{r}'\mathbf{t}')\right](\mathbf{rt})$$

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i.e. $v(\vec{r} t) \neq v'(\vec{r} t) + c(t) \implies \rho(\vec{r} t) \neq \rho'(\vec{r} t)$

$\frac{\text{proof (basic idea):}}{v(\vec{r} t) \longrightarrow \vec{j}(\vec{r} t) \longrightarrow \rho(\vec{r} t)}$ $v'(\vec{r} t) \longrightarrow \vec{j}'(\vec{r} t) \longrightarrow \rho'(\vec{r} t)$

$$i\frac{\partial \vec{j}(\vec{r},t)}{\partial t} = \left\langle \Psi(t) \left| \begin{bmatrix} \hat{j}(\vec{r}) , \hat{H}(t) \end{bmatrix} \right| \Psi(t) \right\rangle \quad \text{and} \quad \frac{\partial \rho(\vec{r},t)}{\partial t} = -\operatorname{div} j(\vec{r},t)$$
equation of motion for \vec{j}

to show that there exists an integer k>0 such that



 $\Rightarrow \rho$ and ρ' will become different from each other infinitesimally later than t₀



Simplest possible approximation for $v_{xc}[\rho](\vec{r}t)$

Adiabatic Local Density Approximation (ALDA)

$$\mathbf{v}_{xc}^{\text{ALDA}}(\vec{r} t) \coloneqq \mathbf{v}_{xc,\text{stat}}^{\text{hom}}(n) \Big|_{n=\rho(\vec{r} t)}$$

 $V_{xc,stat}^{hom}$ = xc potential of static homogeneous e-gas

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Any approximate ground-state-DFT xc functional can be used to generate an adiabatic approximation for TDDFT

ı.

$$\mathbf{v}_{xc}^{\text{adiab}}\left(\vec{r} \ t\right) \coloneqq \mathbf{v}_{xc,GS}^{\text{approx}}[\mathbf{n}] \mid_{\mathbf{n}=\rho(\vec{r} \ t)}$$

LINEAR RESPONSE THEORY

 $t = t_0 : \text{Interacting system in ground state of potential } v_0(r) \text{ with density } \rho_0(r)$ $t > t_0 : \text{Switch on perturbation } v_1(r t) \text{ (with } v_1(r t_0)=0\text{)}.$ Density: $\rho(r t) = \rho_0(r) + \delta\rho(r t)$

Consider functional $\rho[v](r t)$ defined by solution of interacting TDSE

Functional Taylor expansion of $\rho[v]$ around v_o :

$$\begin{split} \rho[\mathbf{v}](\mathbf{r} t) &= \rho[\mathbf{v}_0 + \mathbf{v}_1](\mathbf{r} t) \\ &= \rho[\mathbf{v}_0](\mathbf{r} t) & \longrightarrow \rho_o(\mathbf{r}) \\ &+ \int \frac{\delta \rho[\mathbf{v}](\mathbf{r} t)}{\delta \mathbf{v} (\mathbf{r}' t')} \Big|_{\mathbf{v}_0} \mathbf{v}_1(\mathbf{r}' t') d^3 \mathbf{r}' dt' & \longrightarrow \rho_1(\mathbf{r} t) \\ &+ \frac{1}{2} \int \int \frac{\delta^2 \rho[\mathbf{v}](\mathbf{r} t)}{\delta \mathbf{v} (\mathbf{r}' t') \delta \mathbf{v} (\mathbf{r}'' t'')} \Big|_{\mathbf{v}_0} \mathbf{v}_1(\mathbf{r}', t') \mathbf{v}_1(\mathbf{r}'', t'') d^3 \mathbf{r}' d^3 \mathbf{r}'' dt' dt'' & \longrightarrow \rho_2(\mathbf{r} t) \\ &\vdots \end{split}$$

$$\rho_{1}(\mathbf{r},\mathbf{t}) = \text{linear density response of interacting system}$$
$$\chi(\mathbf{r} t, \mathbf{r}' t') \coloneqq \frac{\delta \rho[\mathbf{v}](\mathbf{r} t)}{\delta \mathbf{v}(\mathbf{r}' t')} \bigg|_{\mathbf{v}_{0}} = \text{density-density response function of interacting system}$$

Lehmann representation of the full response function

$$\chi(\mathbf{r},\mathbf{r}';\omega) = \lim_{\eta \to 0^{+}} \sum_{\mathbf{m}} \left(\frac{\langle 0|\hat{\rho}(\mathbf{r})|\mathbf{m}\rangle\langle \mathbf{m}|\hat{\rho}(\mathbf{r}')|0\rangle}{\omega - (E_{\mathbf{m}} - E_{0}) + i\eta} - \frac{\langle 0|\hat{\rho}(\mathbf{r}')|\mathbf{m}\rangle\langle \mathbf{m}|\hat{\rho}(\mathbf{r})|0\rangle}{\omega + (E_{\mathbf{m}} - E_{0}) + i\eta} \right)$$

with the exact many-body eigenfunctions and energies of the initial unperturbed interacting system Hamiltnian $H(t_0)|m\rangle = E_m |m\rangle$

 $\Rightarrow The exact linear density response$ $\rho_1 (\omega) = \chi (\omega) v_1$ $has poles at the exact excitation energies <math>\Omega = E_m - E_0$

Analogous function $\rho_s[v_s](r t)$ for <u>non</u>-interacting system

$$\rho_{\rm S}[v_{\rm S}](r\,t) = \rho_{\rm S}[v_{\rm S,0} + v_{\rm S,1}](r\,t) = \rho_{\rm S}[v_{\rm S,0}](r\,t) + \int \frac{\delta\rho_{\rm S}[v_{\rm S}](r\,t)}{\delta v_{\rm S}(r'\,t')} \bigg|_{v_{\rm S,0}} v_{\rm S,1}(r'\,t')\,d^3r'\,dt' + \cdots$$

 $\chi_{s}(r t, r't') \coloneqq \frac{\delta \rho_{s}[v_{s}](r t)}{\delta v_{s}(r't')} \bigg|_{v_{s,0}} = \frac{\text{density-density response function of}}{\underline{\text{non-interacting system}}}$

 χ_{s} (r,r', ω) has also poles as function of ω , but at the non-interacting single-particle (KS) excitation energies.

<u>GOAL</u>: Find a way to calculate $\rho_1(r t)$ without explicitly evaluating $\chi(r t, r't')$ of the <u>interacting</u> system

starting point: Definition of xc potential

$$\mathbf{v}_{xc}[\rho](\mathbf{r} t) \coloneqq \mathbf{v}_{s}[\rho](\mathbf{r} t) - \mathbf{v}_{ext}[\rho](\mathbf{r} t) - \mathbf{v}_{H}[\rho](\mathbf{r} t)$$

v_{xc} is well-defined through the non-interacting and the interacting 1-1 mapping.



$$\frac{\delta v_{xc}[\rho](r t)}{\delta \rho(r' t')}\Big|_{\rho_0} = \left.\frac{\delta v_s[\rho](r t)}{\delta \rho(r' t')}\right|_{\rho_0} - \left.\frac{\delta v_{ext}[\rho](r t)}{\delta \rho(r' t')}\right|_{\rho_0} - \left.\frac{\delta(t-t')}{|r-r'|}\right|_{\rho_0}$$





$$f_{xc} + W_C = \chi_S^{-1} - \chi^{-1}$$



$$\chi_{\rm S} \bullet \left[f_{\rm xc} + W_{\rm C} = \chi_{\rm S}^{-1} - \chi^{-1} \right] \bullet \chi$$



$$\chi_{\rm S} \bullet \left| f_{\rm xc} + W_{\rm C} = \chi_{\rm S}^{-1} - \chi^{-1} \right| \bullet \chi$$

$$\chi_{\rm S} (f_{\rm xc} + W_{\rm C}) \chi = \chi - \chi_{\rm S}$$

$$\chi = \chi_{s} + \chi_{s} \left(W_{_{ee}} + f_{_{xc}} \right) \chi$$

Act with this operator equation on arbitrary $v_1(r t)$ and use $\chi v_1 = \rho_1$:

$$\rho_{1}(\mathbf{r} \mathbf{t}) = \int \mathbf{d}^{3}\mathbf{r}' \mathbf{d}\mathbf{t}' \chi_{s}(\mathbf{r} \mathbf{t}, \mathbf{r}' \mathbf{t}') \left[\mathbf{v}_{1}(\mathbf{r} \mathbf{t}) + \int \mathbf{d}^{3}\mathbf{r}'' \mathbf{d}\mathbf{t}'' \left\{ \mathbf{W}_{ee}(\mathbf{r}' \mathbf{t}', \mathbf{r}'' \mathbf{t}'') + \mathbf{f}_{xe}(\mathbf{r}' \mathbf{t}', \mathbf{r}'' \mathbf{t}'') \right\} \rho_{1}(\mathbf{r}'' \mathbf{t}'') \right]$$

- Exact integral equation for $\rho_1(r t)$, to be solved iteratively
- Need approximation for $f_{xc}(r't', r''t'') = \frac{\delta v_{xc}[\rho](r't')}{\delta \rho(r''t'')}\Big|_{\rho_0}$ (either for f_{xc} directly or for v_{xc})

Total photoabsorption cross section of the Xe atom versus photon energy in the vicinity of the 4d threshold.



Solid line: self-consistent time-dependent KS calculation [A. Zangwill and P. Soven, Phys. Rev. A 21, 1561 (1980)]; crosses: experimental data [R. Haensel, G. Keitel, P. Schreiber, and C. Kunz, Phys. Rev. 188, 1375 (1969)].

Photo-absorption in weak lasers



Photo-absorption in weak lasers



Looking at those frequencies, Ω , for which $\rho_1(\omega)$ has poles, leads to a (non-linear) eigenvalue equation

M. Petersilka, U. J. Gossmann, E.K.U.G., PRL <u>76</u>, 1212 (1996) T. Grabo, M. Petersilka, EKUG, J. Mol. Struc. (Theochem) <u>501</u>, 353 (2000) M.E. Casida, Recent Advances in Density Functional Methods I, 155 (1996)

$$\sum_{q'} \left(A_{qq'}(\Omega) + \omega_q \delta_{qq'} \right) \beta_{q'} = \Omega \beta_q$$

where

$$\begin{split} A_{qq'} &= \alpha_{q'} \int d^3 r \int d^3 r' \, \Phi_q(r) \Biggl(\frac{1}{|r-r'|} + f_{xc}(r,r',\Omega) \Biggr) \Phi_{q'}(r') \\ q &= (j,a) \text{ double index} \qquad \alpha_q = f_a - f_j \end{split}$$

$$\Phi_{q}(\mathbf{r}) = \varphi_{a}^{*}(\mathbf{r})\varphi_{j}(\mathbf{r})$$

 $\omega_{q} = \varepsilon_{a} - \varepsilon_{j}$

Atom	Experimental Excitation Energies ¹ S→ ¹ P	KS energy differences	TDDFT
	(in Ry)	$\Delta \in_{\mathrm{KS}} (\mathrm{Ry})$	
Be	0.388	0.259	0.391
Mg	0.319	0.234	0.327
Ca	0.216	0.157	0.234
Zn	0.426	0.315	0.423
Sr	0.198	0.141	0.210
Cd	0.398	0.269	0.391

from: M. Petersilka, U. J. Gossmann, E.K.U.G., PRL <u>76</u>, 1212 (1996)

Excitation energies of CO molecule [mH]

State	$\Omega_{ m expt~KS}$	KS-transition	$\Delta\epsilon_{ m KS}$	TDDFT
$A \ ^1\Pi$	312.7	$5\sigma \rightarrow 2\pi$	252.3	310.2
a ³∏	232.3			221.4
Ι ¹ Σ-	363.1	$1\pi \rightarrow 2\pi$	362.6	362.6
e ³ Σ-	363.1			362.6
a' ${}^{3}\Sigma^{+}$	312.7			314.9
D $^{1}\Delta$	375.9			380.7
d $^{3}\Delta$	344.0			339.6

Molecular excitation energies from time-dependent density-functional theory T Grabo, M Petersilka, EKU Gross, J Mol Struc-Theochem 501, 353 (2000).

Failures of ALDA in the linear response regime

• H₂ dissociation is incorrect: $E(^{1}\Sigma_{u}^{+}) - E(^{1}\Sigma_{g}^{+}) \xrightarrow[R \to \infty]{} 0 \text{ (in ALDA)}$

(see: Gritsenko, van Gisbergen, Görling, Baerends, J. Chem. Phys. 113, 8478 (2000))

- response of long chains strongly overestimated
 (see: Champagne et al., J. Chem. Phys. <u>109</u>, 10489 (1998) and <u>110</u>, 11664 (1999))
- in periodic solids, $f_{xc}^{ALDA}(q, \omega, \rho) = c(\rho)$ whereas, for insulators, $f_{xc}^{exact} \xrightarrow[q \to 0]{} 1/q^2$ divergent.
- charge-transfer excitations not properly described (see: Dreuw et al., J. Chem. Phys. <u>119</u>, 2943 (2003))

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These difficulties have largely been solved by xc functionals more advanced than ALDA. But no single functional cures all problems!

Linear-response TDDFT is now being used to predict and to interpret experimental optical spectra in essentially all corners of physics and chemistry. Some examples:



Inorganic Chemistry

Luminescent Properties of Metal—Organic Framework MOF-5: Relativistic Time-Dependent Density Functional Theory Investigations

Min Ji, Xin Lan, Zhenping Han, Ce Hao,* and Jieshan Qiu

State Key Laboratory of Fine Chemicals, School of Chemical Engineering, Dalian University of Technology, Dalian 116024, Liaoning, China

ABSTRACT: The electronically excited state and luminescence property of metal—organic framework MOF-5 were investigated using relativistic density functional theory (DFT) and timedependent DFT (TDDFT). The geometry, IR spectra, and UV vis spectra of MOF-5 in the ground state were calculated using relativistic DFT, leading to good agreement between the experimental and theoretical results. The frontier molecular orbitals and electronic configuration indicated that the luminescence mechanism in MOF-5 follows ligand-to-ligand charge transfer (LLCT), namely, $\pi^* \rightarrow \pi$, rather than emission with the ZnO quantum dot (QD) proposed by Bordiga et al. The geometry and IR spectra of MOF-5 in the electronically excited state have been calculated using the relativistic TDDFT and compared with those for



the ground state. The comparison reveals that the Zn_4O_{13} QD is rigid, whereas the ligands BDC^{2-} are nonrigid. In addition, the calculated emission band of MOF-5 is in good agreement with the experimental result and is similar to that of the ligand H₂BDC. The combined results confirmed that the luminescence mechanism for MOF-5 should be LLCT with little mixing of the ligand-to-metal charge transfer. The reason for the MOF-5 luminescence is explained by the excellent coplanarity between the six-membered ring consisting of zinc, oxygen, carbon, and the benzene ring.

A NANOPLASMONIC SWITCH BASED ON MOLECULAR MACHINES

 Yue Bing Zheng¹, Ying-Wei Yang², Lasse Jensen¹, Lei Fang², Bala Krishna Juluri¹, Paul S. Weiss¹, J. Fraser Stoddart², Tony Jun Huang^{1*}
 ¹The Pennsylvania State University, University Park, Pennsylvania 16802 USA
 ²Northwestern University, Evanston, Illinois 60208 USA

ABSTRACT

We aim to develop a molecular-machine-driven nanoplasmonic switch for its use in future nanophotonic integrated circuits (ICs) that have applications in optical communication, information processing, biological and chemical sensing. Experimental data show that an Au nanodisk array, coated with rotaxane molecular machines, switches its localized surface plasmon resonances (LSPR) reversibly when it is exposed to chemical oxidants and reductants. Conversely, bare Au nanodisks and disks coated with mechanically inert control compounds, do not display the same switching behavior. Along with calculations based on time-dependent density functional theory (TDDFT), these observations suggest that the nanoscale movements within surface-bound "molecular machines" can be used as the active components in plasmonic devices.



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First-principles calculation of electronic spectra of light-harvesting complex II⁺

Carolin König and Johannes Neugebauer*





Lecture Notes in Physics <u>706</u> (Springer, 2006) Lecture Notes in Physics <u>837</u> (Springer, 2012)