A Short Proof of BEC in the GP Regime and Beyond

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AMS-UMI Meeting
Palermo, July 25, 2024

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Introduction

- consider N non-interacting BOSONS in $\Lambda_L = \mathbb{R}^3/L\mathbb{Z}^3$ with Hamiltonian

$$H_{N,L}^{\text{free}} = \sum_{i=1}^{N} -\Delta_{x_i},$$

acting in a dense subspace of $L_s^2(\Lambda_L^N) = \bigotimes_{\text{SYM}}^N L^2(\Lambda_L)$

- the spectrum $\sigma(H_{N,L}^{\text{free}})$ consists of finite sums of the form

$$\sum_{p\in\Lambda^*} n_p|p|^2$$

for $n_p \in \mathbb{N}, \Lambda^* = \frac{2\pi}{L}\mathbb{Z}^3$ with orthonormal product eigenstates

$$\varphi_{p_1} \otimes_s \cdots \otimes_s \varphi_{p_N},$$

built up from plane waves $x \mapsto \varphi_p(x) = L^{-3/2}e^{ipx}$, for $p \in \Lambda^*$

- the normalized ground state vector ψ_N with energy $E_{N,L}^{\text{free}} = 0$ equals

$$\psi_N = \varphi_0^{\otimes N}, \qquad \varphi_0 = L^{-3/2} \in L^2(\Lambda_L)$$

- in particular, if one measures $\mathcal{O} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$ in the ground state ψ_N, for $\mathcal{O} = \mathcal{O}^* \in \mathcal{B}(L^2_s(\Lambda_L^k))$ and for some fixed $1 \leq k \leq N$, then

$$\langle \psi_N, \mathcal{O} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}\, \psi_N \rangle = \langle \varphi_0^{\otimes k}, \mathcal{O} \varphi_0^{\otimes k} \rangle$$

- expectations of observables are thus fully determined by $\varphi_0 \in L^2(\Lambda_L)$: we say that the system exhibits BOSE-EINSTEIN CONDENSATION (BEC)
- BEC has been verified experimentally in 1995, leading to the Nobel Prize in Physics for Eric Cornell, Carl Wieman and Wolfgang Ketterle
- a fundamental question in mathematical physics is thus to understand the spectrum and BEC for INTERACTING MODELS with energy

$$H_{N,L} = \sum_{i=1}^{N} -\Delta_{x_i} + \sum_{1 \leq i < j \leq N} V(x_i - x_j)$$

- a natural problem is to consider the thermodynamic limit in which the density $\rho = N/L^3$ is small, but fixed and the particle number $N \to \infty$

- if we naively view the interaction as a perturbation, we may expect

$$\frac{E_{N,L}}{N} \approx \frac{\langle \varphi_0^{\otimes N}, H_{N,L}\varphi_0^{\otimes N}\rangle}{N} \approx \frac{\rho}{2}\int_\Lambda V = \frac{\widehat{V}(0)}{2}\rho$$

- this ignores PARTICLE CORRELATIONS and turns out to be wrong, overestimating the energy - the correct formula at low density is

$$\frac{E_{N,L}}{N} \approx 4\pi\mathfrak{a}\rho, \tag{1}$$

where $\mathfrak{a}$ denotes the SCATTERING LENGTH OF V, characterized by

$$\mathfrak{a} = \frac{1}{8\pi}\inf\left\{\int_{\mathbb{R}^3}\left(2|\nabla f|^2 + V|f|^2\right), \lim_{|x|\to\infty} f(x) = 1\right\}$$

- a heuristic idea that suggests (1) at low density is that

$$E_{N,L} \approx \#(\text{pairs of particles}) \times E_{2,L} \approx \frac{N(N-1)}{2}\frac{8\pi\mathfrak{a}}{L^3} \approx 4\pi\mathfrak{a}\rho N$$

- for $\psi_N \in L^2_s(\Lambda_L^N)$ define its ONE-PARTICLE DENSITY $\gamma_N^{(1)}$ through

$$\gamma_N^{(1)}(x,y) = \int_{\Lambda_L^{N-1}} dx_2 \dots dx_N \, \psi_N(x; x_2, \dots, x_N) \overline{\psi}_N(y; x_2, \dots, x_N)$$
$$= \big(\mathrm{tr}_{2,\dots,N} |\psi_N\rangle\langle\psi_N|\big)(x,y)$$

so that, assuming $\|\psi_N\| = 1$, we have that

$$\gamma_N^{(1)} \in \mathcal{B}(L^2(\Lambda_L)), \qquad 0 \le \gamma_N^{(1)} \le 1, \qquad \mathrm{tr}\, \gamma_N^{(1)} = 1$$

- we say that $(\psi_N)_{N\in\mathbb{N}}$ EXHIBITS COMPLETE BEC INTO $\varphi \in L^2(\Lambda_L)$ if

$$\lim_{N\to\infty} \langle \varphi, \gamma_N^{(1)} \varphi \rangle = 1 \qquad \leftrightarrow \qquad \lim_{N\to\infty} \mathrm{tr} \left| \gamma_N^{(1)} - |\varphi\rangle\langle\varphi| \right| = 0$$

- proving that the ground state ψ_N exhibits Bose-Einstein condensation is a CHALLENGING OPEN PROBLEM IN MATHEMATICAL PHYSICS

- since a proof of BEC in the thermodynamic limit is currently out of reach, it is natural to study STRONGLY DILUTED SYSTEMS where $\rho = \rho_N \overset{N\to\infty}{\to} 0$
- set $L = L_N = N^{1-\kappa}$, $\kappa \geq 0$, and study rescaled system in $\Lambda = \mathbb{R}^3/\mathbb{Z}^3$ with

$$H_N = \sum_{i=1}^{N} -\Delta_{x_i} + \sum_{1\leq i<j\leq N} N^{2-2\kappa} V\big(N^{1-\kappa}(x_i - x_j)\big),$$

- by scaling, the scattering length of $V_N = N^{2-2\kappa}V(N^{1-\kappa}.)$ is equal to $\mathfrak{a}/N^{1-\kappa}$ and, similarly, the solution f_N of the zero energy scattering equation for V_N is obtained by scaling as $f_N(.) = f(N^{1-\kappa}.)$, where

$$(-2\Delta + V)f = 0, \qquad \lim_{|x|\to\infty} f(x) = 1$$

- the thermodynamic limit corresponds to $\kappa = \frac{2}{3}$ (at density $\rho = 1$) while the choice $\kappa = 0$ describes the well-known GROSS-PITAEVSKII (GP) LIMIT
- notice that in the Gross-Pitaevskii regime, the total kinetic and potential energies are typically both of the SAME ORDER $\mathcal{O}(N)$
- in the GP regime, if V is sufficiently small, e.g. in the sense that $\mathfrak{a} \ll 1$, one can use simple PERTURBATIVE ARGUMENTS to derive BEC

Theorem (B.-Brooks-Caraci-Oldenburg '24)

Let $V \in L^1(\mathbb{R}^3)$ be pointwise non-negative, radially symmetric and of compact support. Moreover, let $\kappa \in [0, \frac{1}{20})$, $\varphi_0 = 1_{|\Lambda} \in L^2(\Lambda)$ and denote by ψ_N the normalized ground state of H_N with one-particle density $\gamma_N^{(1)}$. Then

$$\lim_{N\to\infty} \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle = 1.$$

- first proof of BEC by [Lieb-Seiringer '02], valid for $\kappa \in [0, \frac{1}{10})$
- [Fournais '21] proves BEC up to $\kappa \in [0, \frac{2}{5} + \epsilon)$, extending ideas previously developed in [Brietzke-Fournais-Solovej '20, Fournais-Solovej '20]; detailed review in [Fournais-Girardot-Junge-Morin-Olivieri '23]
- [Lieb-Seiringer '02], [Fournais '21] employ localization arguments to reduce the analysis to systems with GP scaling and small scattering length
- [Adhikari-B.-Schlein '21] provides alternative proof for $\kappa \in [0, \frac{1}{43})$ avoiding localization, but extending operator expansion methods developed in [B.-Schlein '19], [Boccato-B.-Cenatiempo-Schlein '18, '19, '20]
- in [BBCO '24] our goal is to simplify the proof of [Adhikari-B.-Schlein '21] and to clarify its structure, combining previous ideas with [Brooks '23]

- [Adhikari-B.-Schlein '21] is BASED ON THE HEURISTICS that

$$\begin{aligned}\psi_N \approx C \prod_{1\le i<j\le N} f_N^{(ij)}\,\varphi_0^{\otimes N} &= \prod_{1\le i<j\le N} \big(1-(1-f_N^{(ij)})\big)\varphi_0^{\otimes N}\\ &\approx C\Big(1-\sum_{1\le i<j\le N}(1-f_N^{(ij)})+\dots\Big)\varphi_0^{\otimes N}\\ &\approx C\exp\bigg(-\frac{1}{2}\sum_{p,q,r\in\Lambda^*}\eta_r\, a^*_{p+r}a^*_{q-r}a_pa_q\bigg)\varphi_0^{\otimes N}\\ &\approx e^{-B_\eta}\varphi_0^{\otimes N},\end{aligned}$$

where $f_N^{(ij)} = f(N^{1-\kappa}(x_i-x_j))$, $\eta_r = (\widehat{1-f_N})(r)$ and

$$B_\eta = \frac{1}{2}\sum_{\substack{p,q,r\in\Lambda^*:\\ |r|>N^\alpha;|p|,|q|\le N^\alpha}} \eta_r\, a^*_{p+r}a^*_{q-r}a_pa_q - \text{h.c.}$$

- recall that, by the scattering equation, $(1-f_N)$ solves

$$\big(-2\Delta+V_N\big)(1-f_N) = V_N$$

- one can then derive a suitable coercivity bound on

$$e^{B_\eta} H_N e^{-B_\eta} = H_N - [H_N, B_\eta] + \frac{1}{2}\big[[H_N, B_\eta], B_\eta\big] + \dots$$

  and use that $e^{\pm B_\eta}\mathcal{N}_+ e^{\mp B_\eta}$ is comparable to $\mathcal{N}_+ = \sum_{p\neq 0} a_p^* a_p$
- the main effect of $e^{B_\eta}(\,\cdot\,)e^{-B_\eta}$ is to renormalize the potential for $|p| \le N^\alpha$:

$$\widehat{V}(p/N^{1-\kappa}) \;\;\mapsto\;\; \widehat{V}_{\text{ren}}(p) = \widehat{Vf}(p/N^{1-\kappa}) \approx 8\pi\mathfrak{a}$$

- if $\kappa > 0$ is small enough, one then finds $0 < c < 1$ so that

$$\begin{aligned} e^{B_\eta} H_N e^{-B_\eta} &\ge c \sum_{p\in\Lambda^*} |p|^2 a_p^* a_p + \frac{N^\kappa}{2N} \sum_{\substack{p,q,r\in\Lambda^*:\\ |r|\le N^\alpha}} \widehat{V}_{\text{ren}}(r) a_{p+r}^* a_{q-r}^* a_p a_q + o(N) \\ &\ge 4\pi\mathfrak{a} N^{1+\kappa} + c\mathcal{N}_+ - CN^{\kappa+3\alpha} + o(N) \end{aligned}$$

- QUESTION: the crucial renormalizations are essentially due to commutators linear in B_η - is there a simpler proof without operator expansions?
- QUESTION: how explain the emergence of V_{ren} in a more structural way?

Ideas from the Proof: Two-Body Case

- consider the two-body Hamiltonian that acts on $L^2_s(\Lambda^2)$ by

$$\mathcal{H}_2 = -\Delta_{x_1} - \Delta_{x_2} + V_N = -\Delta_{x_1} - \Delta_{x_2} + N^{2-2\kappa} V(N^{1-\kappa}(x_1 - x_2))$$

  and let us recall a simple proof of the fact that $\inf \sigma(\mathcal{H}_2) \approx \frac{8\pi\mathfrak{a}}{N^{1-\kappa}}$

- key observation: BEC is trivial in this case, in the sense that

$$4\pi^2 \langle \zeta_2, \mathcal{N}_+ \zeta_2 \rangle \leq \inf \sigma(\mathcal{H}_2) \leq \langle \varphi_0 \otimes \varphi_0, \mathcal{H}_2 \varphi_0 \otimes \varphi_0 \rangle \leq \frac{C}{N^{1-\kappa}}$$

- given this a priori information, let's compare $\mathcal{H}_2$ to a Schrödinger operator with eigenstate $\varphi_0 \otimes \varphi_0$ based on the Schur complement formula
- setting $\Pi_{\mathrm{L}} = |\varphi_0 \otimes \varphi_0\rangle\langle \varphi_0 \otimes \varphi_0|$ and $\Pi_{\mathrm{H}} = \mathbf{1} - \Pi_{\mathrm{L}}$, one finds that

$$\mathcal{H}_2 = (1+\eta^*)\big(-\Delta_{x_1} - \Delta_{x_2} + \Pi_{\mathrm{L}} V_{\mathrm{ren}} \Pi_{\mathrm{L}} + \Pi_{\mathrm{H}} V_N \Pi_{\mathrm{H}} \big)(1+\eta),$$

  where η and V_{ren} are defined by

$$\eta = \Pi_{\mathrm{H}} \big(\Pi_{\mathrm{H}} \mathcal{H}_2 \Pi_{\mathrm{H}} \big)^{-1} \Pi_{\mathrm{H}} V_N \Pi_{\mathrm{L}},$$
$$V_{\mathrm{ren}} = V_N - V_N \Pi_{\mathrm{H}} \big(\Pi_{\mathrm{H}} \mathcal{H}_2 \Pi_{\mathrm{H}} \big)^{-1} \Pi_{\mathrm{H}} V_N$$

- anticipating that approximately

$$\eta \approx \big((-2\Delta + V_N)^{-1} V_N\big)(x_1 - x_2) \approx (1 - f)\big(N^{1-\kappa}(x_1 - x_2)\big),$$

  it is not hard to prove that

$$\langle \varphi_0 \otimes \varphi_0, V_{\mathrm{ren}} \varphi_0 \otimes \varphi_0 \rangle = \langle \varphi_0 \otimes \varphi_0, V_N (1 - \eta) \varphi_0 \otimes \varphi_0 \rangle \approx \frac{8\pi\mathfrak{a}}{N^{1-\kappa}}$$

- combined with the identity $2 = a_0^* a_0 + \mathcal{N}_+$ in $L^2_s(\Lambda^2)$ and the fact that

$$(1 + \eta^*) \Pi_{\mathrm{L}} V_{\mathrm{ren}} \Pi_{\mathrm{L}} (1 + \eta) = \Pi_{\mathrm{L}} V_{\mathrm{ren}} \Pi_{\mathrm{L}},$$

  we obtain the lower bound

$$\mathcal{H}_2 \geq \frac{4\pi\mathfrak{a}}{N^{1-\kappa}} \big(a_0^* a_0 a_0^* a_0 - a_0^* a_0 + o(1)\big) \approx \frac{8\pi\mathfrak{a}}{N^{1-\kappa}}$$

- NOTE: using basic properties of $\eta = \mathcal{O}(1/N^{1-\kappa})$, a matching upper bound on the ground state energy can be obtained using the trial state

$$\frac{(1 + \eta)^{-1} \varphi_0 \otimes \varphi_0}{\|(1 + \eta)^{-1} \varphi_0 \otimes \varphi_0\|}$$

Ideas from the Proof: N-Body Case

- consider now the N-particle setting in $L^2_s(\Lambda^N)$ with Hamiltonian

$$H_N = \sum_{r\in\Lambda^*} |r|^2 a_r^* a_r + \frac{N^\kappa}{2N} \sum_{p,q,r\in\Lambda^*} \widehat{V}(r/N^{1-\kappa}) a^*_{p+r} a^*_{q-r} a_p a_q$$

- QUESTION: how mimic the two-body proof and what is a good substitute for the a priori information on BEC (which we would like to prove)?
- setting $\mathcal{N}_{>N^\alpha} = \sum_{|p|>N^\alpha} a_p^* a_p$, notice by Markov that

$$\langle \psi_N, \mathcal{N}_{>N^\alpha} \psi_N\rangle \leq N^{-2\alpha} \sum_{p\in\Lambda^*} |p|^2 \langle \psi_N, a_p^* a_p \psi_N\rangle \leq C N^{1+\kappa-2\alpha}$$

- in other words, the a priori information on BEC can be replaced by the weaker information that $\langle \psi_N, \mathcal{N}_{>N^\alpha} \psi_N\rangle = o(N)$, FOR EVERY $\alpha > \frac{\kappa}{2}$
- such a priori information, and generalizations thereof to observables like

$$N^{-1} \mathcal{K} \mathcal{N}_{>N^\alpha} \quad \text{for} \quad \mathcal{K} = \sum_{r\in\Lambda_+^*} |r|^2 a_r^* a_r,$$

is the main tool in [Adhikari-B.-Schlein '21] to control the errors

- based on the weaker a priori information, we can lower bound

$$V_N = \Delta_{x_1} + \Delta_{x_2} + (1+\eta^*)\big(-\Delta_{x_1} - \Delta_{x_2} + \Pi_{\mathrm{L}} V_{\mathrm{ren}} \Pi_{\mathrm{L}} + \Pi_{\mathrm{H}} V_N \Pi_{\mathrm{H}}\big)(1+\eta)$$
$$\geq (-\Delta_{x_1} - \Delta_{x_2})\,\eta + \eta^*(-\Delta_{x_1} - \Delta_{x_2}) + \eta^*\big(-\Delta_{x_1} - \Delta_{x_2}\big)\eta + \Pi_{\mathrm{L}} V_{\mathrm{ren}} \Pi_{\mathrm{L}},$$

- here, Π_{L} projects onto the low-momentum space

$$\overline{\mathrm{span}(\varphi_k \otimes \varphi_l : k, l \in P_{\mathrm{L}})} \quad \text{for} \quad P_{\mathrm{L}} = \big\{p \in \Lambda^* : |p| \leq N^\alpha\big\}$$

- this implies the simple N-BODY LOWER BOUND

$$H_N \geq \sum_{r \in \Lambda_+^*} |r|^2 c_r^* c_r + \frac{1}{2} \sum_{\substack{p,q,r \in \Lambda^*: \\ p,q,p+r,q-r \in P_{\mathrm{L}}}} \langle \varphi_{p+r} \otimes \varphi_{q-r}, V_{\mathrm{ren}} \varphi_p \otimes \varphi_q \rangle a_{p+r}^* a_{q-r}^* a_p a_q - R_N$$

for an explicit, self-adjoint error term R_N and where

$$c_r = a_r + \sum_{(p,q) \in P_{\mathrm{L}}^2} \langle \varphi_{p+q-r} \otimes \varphi_r, \eta\, \varphi_p \otimes \varphi_q \rangle a_{p+q-r}^* a_p a_q$$

- similarly as in the two-body case, one can prove that

$$\frac{1}{2}\sum_{\substack{p,q,r\in\Lambda^*:\\ p,q,p+r,q-r\in P_{\mathrm{L}}}} \langle \varphi_{p+r}\otimes\varphi_{q-r}, V_{\mathrm{ren}}\varphi_p\otimes\varphi_q\rangle a^*_{p+r}a^*_{q-r}a_pa_q$$
$$\geq 4\pi\mathfrak{a}N^{1+\kappa} - CN^{\kappa}\mathcal{N}_{>N^\alpha} - CN^{\kappa+3\alpha}$$

- on the other hand, recalling that $\mathcal{K} = \sum_{r\in\Lambda_+^*}|r|^2a_r^*a_r$, we lower bound

$$\sum_{r\in\Lambda_+^*}|r|^2c_r^*c_r \geq \sum_{|r|\leq N^\alpha} c_r^*c_r \geq \sum_{r\in\Lambda_+^*:|r|\leq N^\alpha} a_r^*a_r - CN^{\kappa-\alpha-1}\mathcal{K}\mathcal{N}_{>N^\alpha},$$

- controlling $\mathcal{K}\mathcal{N}_{>N^\alpha}$ as in [Adhikari-B.-Schlein '21] essentially by

$$\mathcal{K}\mathcal{N}_{>N^\alpha} = O\big(N^{1+\kappa}\big)\times O\big(N^{1+\kappa-2\alpha}\big)$$

and combining this with similar bounds on R_N implies for SMALL κ that

$$H_N \geq 4\pi\mathfrak{a}N^{1+\kappa} + c\mathcal{N}_+ + o(N)$$

- finally, let us point out that

$$c_r = a_r + \sum_{(p,q)\in P_{\mathrm{L}}^2} \langle \varphi_{p+q-r}\otimes\varphi_r, \eta\,\varphi_p\otimes\varphi_q\rangle a^*_{p+q-r}a_pa_q$$
$$= a_r + \frac{1}{2}\sum_{u,p,q\in\Lambda^*} \langle \varphi_{p+u}\otimes\varphi_{q-u}, \eta\,\varphi_p\otimes\varphi_q\rangle \big[a_r, a^*_{p+u}a^*_{q-u}a_pa_q\big],$$

so that the identification $\eta \approx (1-f)(N^{1-\kappa}(x_1-x_2))$ in $L^2_s(\Lambda^2)$ suggests

$$e^{B_\eta}c_re^{-B_\eta} \approx c_r - [a_r, B_\eta] \approx a_r + [a_r, B_\eta] - [a_r, B_\eta] = a_r$$

- in particular, we expect that

$$\sum_{r\in\Lambda^*_+} |r|^2 e^{B_\eta}c^*_rc_re^{-B_\eta} \approx \sum_{r\in\Lambda^*_+} |r|^2 a^*_ra_r$$

and one can derive similarly the emergence of V_{ren}, connecting the key effects of the conjugation $e^{B_\eta}(\cdot)e^{-B_\eta}$ to the Schur complement formula
- these structural observations can also be used to give a simple proof e.g. of the dynamical stability of BEC in the GP regime [B.-Kroschinsky '24]

Notation

- the creation and annihilation operators a_p^*, a_q for $p, q \in \Lambda^*$ satisfy

$$[a_p, a_q^*] = \delta_{pq}, \qquad [a_p, a_q] = [a_p^*, a_q^*] = 0$$

  and they act on a suitable, dense subspace of $\mathcal{F} = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L_s^2(\Lambda^n)$
- a_p is the adjoint of a_p^*, whichs acts on $\xi \in L_s^2(\Lambda^{n-1})$ as

$$\begin{aligned}(a_p^*\xi)(x_1, \dots, x_n) &= (\varphi_p \otimes_s \xi)(x_1, \dots, x_n) \\ &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \varphi_p(x_j)\xi(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in L_s^2(\Lambda^n)\end{aligned}$$

- in particular, we count the number of particles with momenta $p \in \Lambda^*$ via

$$N = \langle \psi_N, \sum_{i=1}^{N} \mathbf{1}_{x_i} \psi_N \rangle = N \langle \psi_N, \sum_{p \in \Lambda^*} |\varphi_p\rangle\langle\varphi_p\big|_{x_1} \psi_N \rangle = \sum_{p \in \Lambda^*} \langle \psi_N, a_p^* a_p \psi_N \rangle$$

- similarly, one can express H_N through the a_p, a_q^* in form sense as

$$H_N = \sum_{p\in\Lambda^*} |p|^2 a_p^* a_p + \frac{N^\kappa}{2N} \sum_{p,q,r\in\Lambda^*} \widehat{V}(r/N^{1-\kappa}) a_{p+r}^* a_{q-r}^* a_p a_q$$

- denote by $\mathcal{N}_+$ the NUMBER OF EXCITATIONS operator

$$\mathcal{N}_+ = \sum_{p\in\Lambda_+^*} a_p^* a_p \qquad \text{for} \qquad \Lambda_+^* = \Lambda^* \setminus \{0\},$$

- then, we observe that proving complete BEC into φ_0 is equivalent to

$$\lim_{N\to\infty} \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle = \lim_{N\to\infty} \frac{1}{N} \langle \psi_N, a_0^* a_0 \psi_N \rangle = 1 \leftrightarrow \lim_{N\to\infty} \frac{1}{N} \langle \psi_N, \mathcal{N}_+ \psi_N \rangle = 0$$

- a natural strategy to prove BEC consists of showing a COERCIVITY BOUND

$$H_N \geq 4\pi \mathfrak{a} N^{1+\kappa} + c\mathcal{N}_+ + o(N) \qquad (2)$$

  and combining this with a matching upper bound on $E_N = \inf \sigma(H_N)$
- NOTE: since[1] $E_N = 4\pi\mathfrak{a}N^{1+\kappa} + c_{LHY} N^{\frac{5}{2}\kappa} + o(N^{\frac{5}{2}\kappa})$, such a strategy and suitably adjusted variants thereof become harder the larger $\kappa \geq 0$

[1] [Yau-Yin '09, Basti-Cenatiempo-Schlein '21] (upper bound) and [Fournais-Solovej '20 '23, Haberberger-Hainzl-Nam-Seiringer-Triay '23] (lower bound)

BEC in GP for Small Interactions:

- based on a simple argument introduced in [Brietzke-Fournais-Solovej '20], BEC can be proved easily in the GP regime if $\mathfrak{a}$ is sufficiently small
- we follow [Nam-Napiórkowski-Ricaud-Triay '22] and recall that $f_N = f(N.)$ $(\kappa = 0)$ is the solution of the zero energy scattering eq. for $V_N = N^2 V(N.)$

$$(-2\Delta + V_N) f_N = 0, \qquad f_N(x) \stackrel{|x|\to\infty}{\to} 1$$

- setting $\mathrm{P}_0 = |\varphi_0\rangle\langle\varphi_0|$, the positivity of V_N implies the operator bound

$$\big(1 - \mathrm{P}_0 \otimes \mathrm{P}_0 f_N(x_1 - x_2)\big) V_N(x_1 - x_2) \big(1 - f_N(x_1 - x_2) \mathrm{P}_0 \otimes \mathrm{P}_0\big) \geq 0$$

- plugging this into the many body Hamiltonian, one finds that

$$\begin{aligned} H_N \geq \sum_{p\in\Lambda_+^*} \Big(|p|^2 a_p^* a_p + \frac{1}{2} \widehat{Vf}(p/N) b_p^* b_{-p}^* + \frac{1}{2} \widehat{Vf}(p/N) b_p b_{-p} \Big) \\ + \frac{1}{2} \int_\Lambda (2f_N - f_N^2) V_N \, a_0^* a_0^* a_0 a_0, \end{aligned}$$

where $b_p^* = N^{-\frac{1}{2}} a_p^* a_0$, for $p \in \Lambda^*$, denote modified creation operators

- using $\widehat{Vf}(0) = 8\pi\mathfrak{a}$, $a_p^* a_p \geq b_p^* b_p$ and the general lower bound

$$A_p\big(b_p^* b_p + b_{-p}^* b_{-p}\big) + B_p\big(b_p^* b_{-p}^* + b_p b_{-p}\big) \geq -\big(A - \sqrt{A^2 - B^2}\big)\,\frac{[b_p, b_p^*] + [b_{-p}, b_{-p}^*]}{2},$$

  we shift H_N by $\mu\mathcal{N}_+$ for some $0 < \mu < 4\pi^2 - 8\pi\mathfrak{a}$, ASSUMING $\mathfrak{a}$ TO BE SMALL ENOUGH, and basic manipulations then imply that

$$H_N \geq 4\pi\mathfrak{a}N + \big(\mu - 16\pi\mathfrak{a}\big)\mathcal{N}_+ + O(1)$$

- combining this with the upper bound $E_N \leq 4\pi\mathfrak{a}N + O(1)$, we conclude that the ground state ψ_N exhibits complete BEC into φ_0:

$$4\pi\mathfrak{a}N + O(1) \geq \langle\psi_N, H_N\psi_N\rangle \geq 4\pi\mathfrak{a}N + c\langle\psi_N, \mathcal{N}_+\psi_N\rangle + O(1)$$

- NOTE: for $\kappa > 0$, the potential energy has size $\mathcal{O}(N^{1+\kappa}) \gg \mathcal{O}(N)$ which prohibits the application of similar perturbative arguments