Kubo's Formula for Disordered Systems

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Outline

1. Introduction
2. Local Structures of Wonderland
3. Locabatic Theorem and Kubo Formula

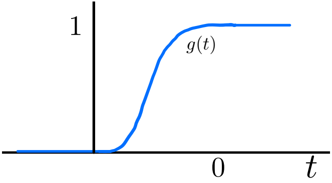
Setting

We consider Hamiltonian on $l^2(\mathbb{Z}^d)$,

$$H(t) = H + \beta W(t)$$

Main Example:

- $H = \Delta + V_\omega$ is magnetic Laplacian with disorder
- $W(t) = g(t)V$ is a time-dependent perturbation, $g(t)$ is a compactly supported switch function.



Linear Response Theory

Linear response theory aims to justify Ohm's law

$$\langle \mathsf{J} \rangle = \sigma \mathsf{V},$$

and to give a microscopic formula for the conductance σ.

Framework of Linear Response

- $W(t) = g(t)V(x)$ with $V(x)$ electric potential of unit voltage
- At $t = -\infty$, state ρ is equilibrium state of H
- Solve $\dot{\rho}_t = -i[H(t), \rho_t], \quad H(t) = H + \beta e^{\epsilon t} V(x)$
- Measure the current J at time $t = 0$

Then the measured conductance is

$$\sigma_m(\epsilon,\, \beta) = \beta^{-1} \mathrm{Tr}(\rho_0 J).$$

In experiments $\epsilon / \beta < 10^{-9}$.

Kubo's formula

Kubo's formula [Kubo '57] for conductance is

$$\sigma = \lim_{\epsilon\to 0}\lim_{\beta\to 0}\sigma_m(\epsilon,\,\beta) = \lim_{\epsilon\to 0} i\int_{-\infty}^{0} e^{\epsilon t}\mathrm{Tr}(\rho[e^{iHt}Je^{-iHt}, V])dt.$$

The problem of linear response [Simon '84]

Show that the joint limit

$$\lim_{\epsilon << \beta\to 0}\sigma_m(\epsilon,\,\beta)$$

exists and equal σ or provide an alternative explanation for the validity of Kubo's formula.

Kubo's formula

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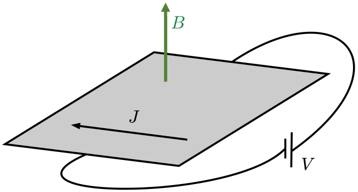
Question

Is there a microscopic proof of Kubo formula not related to quantum Hall effect?

Kubo's formula for Hall conductance

In $d = 2$, Hall conductance, σ_H, is a ratio of current J in direction x_1 to applied electric field in direction x_2.



Let Λ_n be the characteristic function of the set $\{x_n \geq 0\}$, $n = 1, 2$.

- $J = i[H, \Lambda_1]$,
- $V = \Lambda_2$,
- At zero temperature $\rho = P_F := \chi_{<E_F}(H)$, with Fermi energy E_F in the mobility gap.

The Kubo's formula is then given by [Aizenman-Graf 89']

$$\sigma_H = \mathrm{i}\mathrm{Tr}(P_F[[P_F, \Lambda_1], [P_F, \Lambda_2]]) \in \mathbb{Z}/(2\pi).$$

History

No disorder: If E_F belongs to a gap then limits $\epsilon \to 0$, $\beta \to 0$ commute, i.e.

$$\sigma_H = \lim_{\beta \to 0} \lim_{\epsilon \to 0} \sigma_m(\epsilon, \beta).$$

[Avron, Seiler, Yaffe, Bachmann, De Roeck, Fraas , Teufel, Marcelli, ...]

With disorder:

- For ϵ fixed and $\beta \to 0$, $\sigma_m \to \sigma_H$ [Bouclet, Germinet, Klein, Schenker '05]
- For $\beta = \epsilon$ and complete localization $\sigma_m \to \sigma_H = 0$.[Nakano, Kaminaga '99]

How to prove Kubo with a gap

1. By adiabatic theorem, solving $\dot{\rho}_t = -i[H(t), \rho_t]$, gives

$$\rho_t = P_F(t) + O(\epsilon), \quad P_F(t) := \chi_{<E_F}(H(t)).$$

2. By first order perturbation theory, $H(t) = H + \beta g(t)\Lambda_2$,

$$P_F(0) = P_F + \beta \mathrm{Ad_H}^{-1}(P_F\Lambda_2(1-P_F) + (1-P_F)\Lambda_2 P_F) + O(\beta^2).$$

3. By a bit of algebra,

$$\mathrm{Tr}(\rho_0 i[H, \Lambda_1]) = \beta \mathrm{Tr}(P_F[[P_F, \Lambda_1], [P_F, \Lambda_2]]) + O(\beta^2) + O(\epsilon).$$

It remains to be able to bound the errors.

Assumptions and technicalities

For $\Theta \subset \mathbb{Z}^d$, H^Θ is the restriction of H to Θ. Assumptions:

- H, V finite range, $g(t)$ smooth compactly supported. H^Θ, H^Φ independent if $\operatorname{dist}(\Theta, \Phi) > \mathit{range}$;
- Fractional moment for an interval J_{loc} of spectrum. There exists $q > 0$, such that for all Θ and $x, y \in \Theta$,

$$\sup_{E \in J_{loc}} \mathbb{E}\left[|(H^\Theta - E - i\eta)^{-1}(x,y)|^q \right] \le C_q e^{-c|x-y|_\Theta}.$$

Redefine conductance by averaging:

$$\sigma_m(\beta, \epsilon) := \beta^{-1} \epsilon \int_0^{1/\epsilon} \operatorname{Tr}\left(J(\rho_t - \rho) \right) dt.$$

The result

Theorem (De Roeck, Elgart, Fraas 23')

Suppose that $E_F \in J_{loc}$. Then there exist $p > 0$ such that for all β small enough,

$$\mathbb{E}|\sigma_H - \sigma_m| \leq e^{-\beta^{-p/2}},$$

provided $\epsilon = e^{-\beta^{-p}}$.

Naive Idea

Let U_t be the solution of $i\partial_t U_t = H(\epsilon t)U_t$. If

$$U_t P_{J_{loc}} U_t^* - P_{J_{loc}}(t) \approx 0,$$

then

$$\tilde{H}(\epsilon t) = (1 - P_{J_{loc}}(t))H(\epsilon t)(1 - P_{J_{loc}}(t))$$

is gapped and we can proceed as before.

Part 2: Local Structures of Wonderland

Welcome to Wonderland

Wonderland Theorem [del Rio, Makarov, Simon '94, Gordon '94]:

The rank one perturbation family

$$H(\beta) = H + \beta \chi_{\{0\}},$$

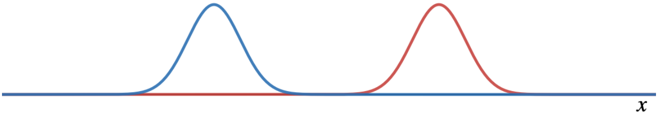
exhibits almost sure singular continuous spectrum for a G_δ-dense set of β's.

Remarks:

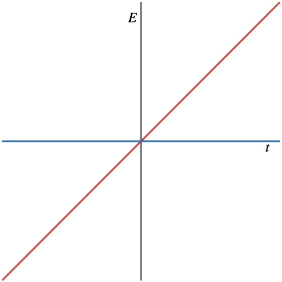
- The spectral transitions are due to resonant hybridization,
- Dynamical localization breaks, propagation is logarithmic [del Rio, Jitomirskaya, Last, Simon 94'],
- This picture is expected to be generic, beyond rank one perturbation.

Eigenstate Hybridization

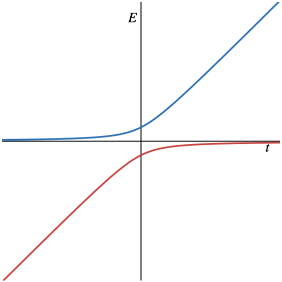
Two far away eigenstates of H close in energy



Hybridize as we add the perturbation tV supported on right:



without overlap



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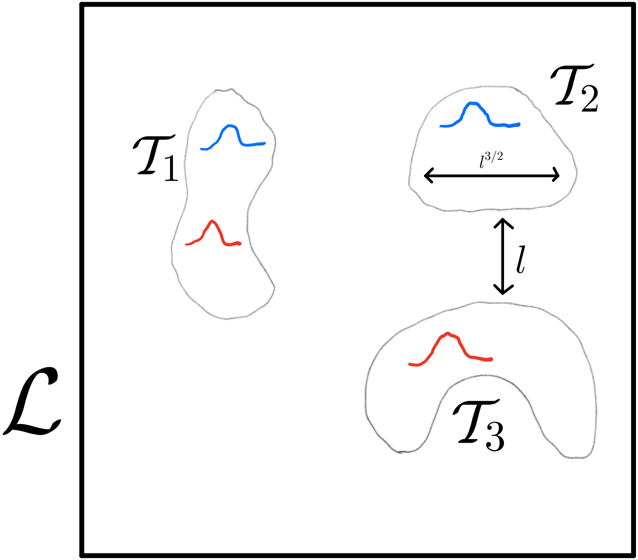
Local Structures of Wonderland



Local Structures of Wonderland theorem

Theorem (Local Structures)

Pick a large enough scale l*, and let* $\mathbb{T}$ *be a torus of size* $\mathcal{L} = e^{\sqrt{l}}$*. Suppose that* $\beta \leq l^{-p_1}$*. Then with probability* $> 1 - e^{-\sqrt{l}}$*,* $H^{\mathbb{T}}(s)$ *has a local structure* $\mathcal{T}_\gamma$ *for energies in* $J = (E - l^{-d/q}, E + l^{-d/q})$*.*

Moreover, each $H^{\mathcal{T}_\gamma}$ *has interval* $J_\gamma \subset J$ *of energies of size* $\delta = cl^{-d/q}$ *separated by a gap of size* $\Delta = l^{-d-1/2}l^{-d/q}$ *in* J*.*

Part 3: Locabatic Theorem and Kubo Formula

Theorem (Local adiabatic theorem)

Fix $N \in \mathbb{N}$. Suppose $\epsilon \geq e^{-\sqrt{l}}$ and $\beta \leq l^{-p_1}$. Then for ℓ large enough, there exists a smooth family of orthogonal projections $\mathcal{Q}(s)$ with the following properties:

1. $\|[\mathcal{Q}(s), H^{\mathbb{T}}(s)]\| \leq C_N \left(\epsilon\Delta^{-1} + e^{-c\sqrt{\ell}}\right)$;
2. $\|P_{<E-6\delta}(H^{\mathbb{T}}(s))\bar{\mathcal{Q}}(s)\| + \|\mathcal{Q}(s)P_{>E+6\delta}(H^{\mathbb{T}}(s))\| \leq C_N \left(\epsilon\Delta^{-1} + e^{-c\sqrt{\ell}}\right)$;
3. *Let $\mathcal{Q}_\epsilon(s)$ the solution of $i\epsilon\dot{\mathcal{Q}}_\epsilon(s) = [\mathcal{Q}_\epsilon(s), H^{\mathbb{T}}(s)]$, $\mathcal{Q}_\epsilon(0) = \mathcal{Q}(0)$, we have*

$$\|\mathcal{Q}_\epsilon(s) - \mathcal{Q}(s)\| \leq C_N \left(\epsilon^N \left(\frac{1}{\Delta^N} + \frac{1}{\delta^{2N+1}}\right) + e^{-c\sqrt{\ell}}\right).$$

Furthermore, for $s = 0$ and $s = 1$, the inequalities in (i) and (ii) hold without the terms proportional to ϵ.

Sketch of Proof

- $Q_\gamma(s) = \chi_{\hat{\mathcal{T}}_\gamma} P_{J_\gamma}(H^{\mathcal{T}_\gamma}(s))\chi_{\hat{\mathcal{T}}_\gamma}$ evolves adiabatically within it's local structure $\mathcal{T}_\gamma$.
- $Q(s) = \sum_\gamma Q_\gamma(s)$ evolves adiabatically within $\cup_\gamma \mathcal{T}_\gamma$. Let Q_s be the superadiabatic projection of order N.
- Then Hamiltonian $\bar{H}(s) = \bar{Q}_s H(s) \bar{Q}_s$ has a gap of order δ at E_F and we take
$$\mathcal{Q}(t) := \chi_{<E_F}(\bar{H}(t)).$$

Proof of Kubo formula

1. By the adiabatic theorem

$$\rho_t = \mathcal{Q}(t) + O(e^{-c\sqrt{\ell}}).$$

   and $P_F(0) = \mathcal{Q}(0) + R$.

2. Show that R does not contribute.
3. Show that for $\mathcal{Q} = \mathcal{Q}(-1)$,

$$\mathrm{Tr}(P_F[[P_F, \Lambda_1], [P_F, \Lambda_2]]) = \mathrm{Tr}(\mathcal{Q}[[\mathcal{Q}, \Lambda_1], [\mathcal{Q}, \Lambda_2]]) + O(e^{-c\sqrt{\ell}})$$

Thank you