# *Correlation energy for the low density Fermi gas*


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Based on


1. M. Falconi, E.L. G., C. Hainzl, M. Porta, *The dilute Fermi gas via Bogoliubov theory*, Ann. Henri Poincaré (2021).
2. E.L. G., *An optimal upper bound for the dilute Fermi gas in three dimensions*, Journal of Functional Analysis (2023).
3. E.L. G., C. Hainzl, P.T. Nam, R. Seiringer, *The Huang-Yang formula for the low-density Fermi gas: an upper bound*, in preparation.
4. E.L. G., *An optimal lower bound for the low density Fermi gas*, in preparation

# The Setting

- $N$ interacting spin 1/2 fermions in a box $\Lambda_L := [-L/2, L/2]^3$, with **periodic** boundary conditions.

$$H_N = -\sum_{i=1}^{N} \Delta_{x_i} + \sum_{i<j=1}^{N} V(x_i - x_j), \quad \mathfrak{h} := \wedge^N L^2(\Lambda_L; \mathbb{C}^2)$$

  - $N_\sigma = \#$ particles with spin $\sigma \in \{\uparrow, \downarrow\}$, $N = N_\uparrow + N_\downarrow$.
  - $V$ is the 'periodization' on $\Lambda_L$ of a potential $V_\infty$ on $\mathbb{R}^3$ positive, radial, compactly supported and in $L^2$.

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- The **ground state energy density** is ($\rho_\sigma = N_\sigma / L^3$, $\rho = \rho_\uparrow + \rho_\downarrow$)

$$e_L(\rho_\uparrow, \rho_\downarrow) = \frac{E_L(N_\uparrow, N_\downarrow)}{L^3} = \inf_{\psi \in \mathfrak{h}(N_\uparrow, N_\downarrow)} \frac{\langle \psi, H_N \psi \rangle}{\langle \psi, \psi \rangle}$$

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- We are interested in the **thermodynamic limit**:

$$e(\rho_\uparrow, \rho_\downarrow) = \lim_{\substack{N, L \to \infty \\ \rho = N/L^3 = \text{const.}}} e_L(\rho_\uparrow, \rho_\downarrow)$$

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- We focus on the **dilute regime** ($\rho a^3 \to 0$)

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In physical units such that $\hbar = 1$, $m = 1/2$, if $\rho_\uparrow = \rho_\downarrow = \rho/2$, then as $\rho \to 0$

$$e(\rho_\uparrow, \rho_\downarrow) = \frac{3}{5}(3\pi^2)^{\frac{2}{3}}\rho^{\frac{5}{3}} + 2\pi a \rho^2 + \frac{4(11 - 2\log 2)}{35\pi^2}\left(\frac{3}{4\pi}\right)^{4/3} a^2 \rho^{7/3} + o(\rho^{7/3}).$$

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- The first term $\propto \rho^{\frac{5}{3}}$ is a purely kinetic term (Free Fermi Gas)

$$\Phi_{\mathrm{FFG}} = \frac{1}{\sqrt{N!}}\det\big(f_{k_i}^{\sigma_i}(x_j)\big)_{1\le i,j\le N}, \qquad f_{k_i}^{\sigma_i}(x) = \frac{e^{ik_i\cdot x}}{\sqrt{L^3}}, \qquad |k_i| \le k_F^{\sigma_i} \sim \rho_{\sigma_i}^{\frac{1}{3}}$$

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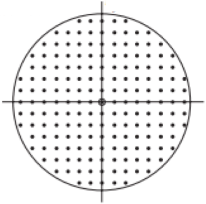

$$\mathcal{B}_F^{\sigma_i} = \{|k| \leq k_F^{\sigma_i} \sim \rho_{\sigma_i}^{\frac{1}{3}}\}$$

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- The 2nd and the 3rd corrections depend on the *scattering length* $a$, which describes the effective range of the interaction:

$$8\pi a = \int \mathrm{d}x\, V(x) f(x), \qquad -\Delta f + \frac{1}{2} V f = 0, \qquad \lim_{|x|\to\infty} f(x) = 1$$

$\rightsquigarrow$ correlations play a role.

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- Main challenge: to study the **correlation energy**

$$\boxed{E_{\mathrm{corr}} = E_N - E_N^{HF}}$$

# Main Results

**Theorem 1 [E.L. G., Journal of Functional Analysis (2023)]**

*Suppose the interaction potential $V$ is* ***positive***, ***radial***, ***compactly supported*** *and* ***smooth***, *then as $\rho \to 0$,*

$$e(\rho_\uparrow, \rho_\downarrow) \leq \frac{3}{5}(6\pi^2)^{\frac{2}{3}}(\rho_\uparrow^{\frac{5}{3}} + \rho_\downarrow^{\frac{5}{3}}) + 8\pi a \rho_\uparrow \rho_\downarrow + C\rho^{\frac{7}{3}},$$

*where $a$ is the scattering length of the potential $V$. Moreover, as $\rho \to 0$*

$$e(\rho_\uparrow, \rho_\downarrow) \geq \frac{3}{5}(6\pi^2)^{\frac{2}{3}}(\rho_\uparrow^{\frac{5}{3}} + \rho_\downarrow^{\frac{5}{3}}) + 8\pi a \rho_\uparrow \rho_\downarrow - C\rho^{2+\frac{1}{5}}.$$

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- Positive temperature case by **Seiringer (Comm. Math. Phys. 2006)**
- Recently **Lauritsen (Ann. Henri Poincaré (2024))** obtained an almost optimal upper bound ($\mathcal{O}(\rho^{7/3-\varepsilon})$) for hard-core interactions via Cluster expansion methods.

# Main Results – In Preparation

## Theorem 2 [E.L. G., C. Hainzl, P.T. Nam, R. Seiringer]

*Suppose the interaction potential $V$ is* ***positive*****,** ***radial*****,** ***compactly supported*** *and $L^2$, then if $\rho_\uparrow = \rho_\downarrow = \rho/2$, we have as $\rho \to 0$*

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- Theorem 2 holds in a more general setting, i.e., $\rho_\uparrow \neq \rho_\downarrow$, with

$$\frac{4(11 - 2\log 2)}{35\pi^2}\left(\frac{3}{4\pi}\right)^{4/3} a^2\rho^{7/3} \rightsquigarrow a^2\rho_\uparrow^{\frac{7}{3}} F(\rho_\downarrow/\rho_\uparrow) = a^2\rho_\downarrow^{\frac{7}{3}} F(\rho_\uparrow/\rho_\downarrow)$$

# Main Ideas of the Strategy of the Proofs

→ Minimize the energy over anty-symmetric product states:

$$E_{\mathrm{FFG}} = \langle \Phi_{\mathrm{FFG}}, H_N \Phi_{\mathrm{FFG}} \rangle = \tfrac{3}{5}(6\pi^2)^{\frac{2}{3}}(\rho_\uparrow^{\frac{5}{3}} + \rho_\downarrow^{\frac{5}{3}})L^3 + \underbrace{\widehat{V}(0)}_{>8\pi a}\rho_\uparrow\rho_\downarrow L^3 + \mathcal{O}(L^3\rho^{\frac{8}{3}})$$

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→ Describe the effective correlation energy ($E_{\mathrm{corr}} = E_N - E_{\mathrm{FFG}}$) via quasi-bosonic creation/annihilation operators:

$$b_{p,\sigma} := \sum_r b_{p,r,\sigma}, \quad b_{p,r,\sigma} = \hat{a}_{p+r,\sigma}\hat{a}_{r,\sigma} \quad |r| < k_F^\sigma < |p+r|, \qquad b^*_{p,\sigma} = (b_{p,\sigma})^*$$

$$\mathcal{H}^{\mathrm{eff}}_{\mathrm{corr}} \sim \frac{1}{L^3}\sum_\sigma\sum_k A_k b^*_{k,\sigma} b_{k,\sigma} + \frac{1}{L^3}\sum_k B_k(b_{k,\uparrow}b_{-k,\downarrow} + b^*_{-k\downarrow}b^*_{k,\uparrow}),$$

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→ Conjugate $\mathcal{H}^{\mathrm{eff}}_{\mathrm{corr}}$ via a quasi-bosonic Bogoliubov transformation

$$T \sim \exp\left(\frac{1}{2L^3}\sum_{p,r,r'}\hat{\varphi}_{r,r'}(p)b_{p,r,\uparrow}b_{-p,r',\downarrow} - \mathrm{h.c.}\right).$$

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⇝ through the conjugation we diagonalize the $\mathcal{H}^{\mathrm{eff}}_{\mathrm{corr}}$ and re-normalize the interaction potential ($V \rightsquigarrow V_\varphi$)

# Main Ideas of the Strategy of the Proofs

- $T_1 = \exp\left(\frac{1}{L^3}\sum_p \hat{\varphi}(p) b_{p,\uparrow} b_{-p,\downarrow} - \text{h.c.}\right)$, for $\rho^{\frac{1}{3}-\gamma} \lesssim |p|$
- $T_2 = \exp\left(\frac{1}{L^3}\sum_{p,r,r'} \hat{\eta}_{r,r'}(p) b_{p,r\uparrow} b_{-p,r'\downarrow} - \text{h.c.}\right)$, for $|p| \lesssim \rho^{\frac{1}{3}-\gamma}$

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$\rightsquigarrow$ After $T_1 T_2$, we have

$$e(\rho_\uparrow, \rho_\downarrow) \sim \frac{3}{5}(6\pi^2)^{\frac{2}{3}}(\rho_\uparrow^{\frac{5}{3}} + \rho_\downarrow^{\frac{5}{3}}) + 8\pi a \rho_\uparrow \rho_\downarrow - \frac{(8\pi a)^2}{2L^3} \sum_{\substack{r\in\mathcal{B}_F^\uparrow, r'\in\mathcal{B}_F^\downarrow \\ r+p\notin\mathcal{B}_F^\uparrow, r'-p\notin\mathcal{B}_F^\downarrow}} \left(\frac{2}{|r+p|^2 - |r|^2 + |r'-p|^2 - |r'|^2} - \frac{1}{|p|^2}\right)$$

Thank you for the attention!