Roma Tre Università degli Studi

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Adiabatic evolution of low-temperature many-body quantum systems

joint work with R. L. Greenblatt, M. Lange and M. Porta

Commun. Math. Phys. (2024)

Università degli studi di Palermo - *2nd international AMS-UMI joint meeting* - 25/07/2024

Outline

- *Introduction*: adiabatic theorems for quantum systems.

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- *Setting*: interacting fermionic lattice systems.
  Dynamics: initial state in thermal equilibrium then a weak and slowly-varying time-dependent perturbation is introduced.

- *Main result*: convergent expansion for expectation values of local observables, at small temperature. Corollary: adiabatic theorem.

paper QR code

Adiabatic Theorem (AT)

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  Let $s := \eta t$ be the slow scaled time.

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- Suppose that there exists a family of spectral projections $s \mapsto P(s)$ satisfying the *gap condition*.
  For simplicity, for every s suppose that $\mathcal{H}(s)$ has an eigenvalue $E(s)$ such that

$$\mathbf{dist}(E(s), \sigma(\mathcal{H}(s)) \setminus \{E(s)\}) = g > 0 \text{ for all } s \in [-1, 0]$$

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- AT: Any initial state $\boxed{\phi(-1)}$ in $\mathbf{Ran}\, P(-1)$ evolves under the Schrödinger evolution into a state $\boxed{\psi(s)}$ that is localised in $\mathbf{Ran}\, P(s)$ up to error of order η.

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- [Born–Fock 1928, Kato 1950] AT implies that there exists C_0 independent of η:

$$\left\|P(s)^{\perp}\psi(s)\right\| = \|(\mathbb{1} - P(s))\psi(s)\| \leq C_0\eta \tag{1}$$

for all $s \in [-1, 0]$, where C_0 depends linearly in $\|\dot{\mathcal{H}}(s)\| < \infty$.

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In a quantum spin or fermionic lattice system on $\Gamma_L = \mathbb{Z}^d/(L\mathbb{Z}^d)$, one gets:

$$\|\dot{\mathcal{H}}(s)\| = \left\|\sum_{X\subseteq\Gamma_L} \dot{\Phi}_X(s)\right\| \propto L^d \implies \text{(1) is useless for large } L \text{ at fixed } \eta.$$

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- [Bachmann–De Roeck–Fraas, CMP '18] In spin lattice system let O_X be a local operator, then there exists C_1 independent of η and the system size L:

$$|\langle \psi(s)|\, O_X\psi(s)\rangle - \langle \phi(s)|\, O_X\phi(s)\rangle| \leq C_1\eta \quad \text{for all } s \in [-1, 0].$$

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  For fermionic lattice system, similar result in [Monaco–Teufel RMP '19, Henheik, Teufel FMΣ '20] for finite/infinite volume. In the setting one-body (infinitely extended) continuum system an analogous result in [Elgart–Schlein CPAM '04, Marcelli LMP '22].

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- (Standard) Adiabatic theorem can be applied whenever one considers $T = 0$: Initial state is in the range of the projection on the ground state of many-body gapped Hamiltonian (or Fermi projection of one-body gapped Hamiltonian).

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- Consider $\mathcal{H}(\eta t)$. Let $\rho_{\beta,\mu,L} = \frac{\mathrm{e}^{-\beta(\mathcal{H}-\mu\mathcal{N})}}{\mathbf{Tr}(\mathrm{e}^{-\beta(\mathcal{H}-\mu\mathcal{N})})}$ be the equilibrium Gibbs state of $\mathcal{H} \equiv \mathcal{H}(-1)$ where $\beta = 1/T$.
  The state $\rho(t)$ of the system is determined by the Cauchy problem:

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In particular, $T \to 0^+$ after $L \to \infty$.

Setting

- We consider interacting fermions on $\Gamma_L := \mathbb{Z}^d/(L\mathbb{Z}^d)$, including $M \in \mathbb{N}$ internal degree of freedom: the total configuration space $\boxed{\Lambda_L := \Gamma_L \times \{1, \dots, M\}}$.

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- Let the number operator $\mathcal{N} := \sum_{x \in \Lambda_L} a_x^+ a_x^- \in \mathcal{A}_{\Lambda_L}$, $\qquad \mathcal{A}_X^{\mathcal{N}} := \{O \in \mathcal{A}_X \,|\, [O, \mathcal{N}] = 0\}$.

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- We say that $\mathcal{O} \in \mathcal{A}_{\Lambda_L}$ is a *finite-range* operator if there exist
  - (i) $R > 0$ independent of L such that $\mathcal{O}_X = 0$ if $\mathrm{diam}(X) > R$
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- Prototypical example:

$$\mathcal{H} = \sum_{x,y \in \Lambda_L} a_x^+ H(x,y) a_y^- + \lambda \sum_{x,y \in \Lambda_L} a_x^+ a_y^+ v(x,y) a_y^- a_x^-$$

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- Let $O_X \in \mathcal{A}_X$. Grand-canonical Gibbs state:

$$\langle O_X \rangle_{\beta,\mu,L} := \mathbf{Tr}_{\mathcal{F}_L} \left(O_X \, \rho_{\beta,\mu,L} \right) \text{ with } \rho_{\beta,\mu,L} := \frac{\mathbf{e}^{-\beta(\mathcal{H}-\mu\mathcal{N})}}{\mathbf{Tr}_{\mathcal{F}_L}(\mathbf{e}^{-\beta(\mathcal{H}-\mu\mathcal{N})})}.$$

Driving the system out of equilibrium

- We introduce for $t \leq 0$

$$\mathcal{H}(\eta t) = \mathcal{H} + \varepsilon\, g(\eta t)\, \mathcal{P} \quad \text{with } \varepsilon \in \mathbb{R} \text{ and } \eta > 0\,,$$

-$\mathcal{P}$ is finite-range and self-adjoint operator in $\mathcal{A}^{\mathcal{N}}_{\Lambda_L}$,

-for simplicity $g(t) = \mathbf{e}^t$ (the function g can be chosen in a suitable switch function class).

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- Evolution of the state: the density matrix of the system is determined by

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$$\mathbf{Tr}\left(O_X\, \rho(t)\right) \equiv \mathbf{Tr}\left(O_X\, \rho_{\varepsilon,\eta,\beta,\mu,L}(t)\right).$$

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Order of limits: the adiabatic regime $\eta \to 0^+$ and ε small uniformly in η, uniformly in the system size L. Comparison with the instantaneous Gibbs state of $\mathcal{H}(\eta t)$.

Perturbation theory for quantum dynamics

- Let the Heisenberg evolution be:

$$\mathbb{R} \ni t \mapsto \tau_t(O) := \mathbf{e}^{\mathrm{i}\mathcal{H}t} O \, \mathbf{e}^{-\mathrm{i}\mathcal{H}t} \quad \text{for any } O \in \mathcal{A}_{\Lambda_L}.$$

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- The Duhamel expansion for quantum dynamics via interaction picture:

$$\mathbf{Tr}\, O_X \rho(t) = \mathbf{Tr}\, O_X \rho_{\beta,\mu,L} + \sum_{n\geq 1} (-\mathrm{i}\varepsilon)^n \int_{\text{symplex}} d\underline{s}\, \mathbf{e}^{\eta(s_1+\cdots+s_n)} \langle [\cdots[\tau_t(O_X), \tau_{s_1}(\mathcal{P})] \cdots \tau_{s_n}(\mathcal{P})] \rangle_{\beta,\mu,L}$$

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where symplex means $-\infty \leq s_n \leq \cdots \leq s_1 \leq t \leq 0$.

- Using the unitarity of the dynamics a very rough bound for the n-th term is:

$$C^n \frac{|\varepsilon|^n}{\eta^n} \frac{|\Lambda_L|^n}{n!}.$$

Thus, for $L < \infty$ and $\eta > 0$ the series is absolutely convergent. But it is useless for large L and small η.

Perturbation theory for equilibrium states

- Let the Euclidean evolution be:

$$\mathbb{R} \ni t \mapsto \gamma_t(O) := \mathbf{e}^{t(\mathcal{H}-\mu\mathcal{N})}\, O\, \mathbf{e}^{-t(\mathcal{H}-\mu\mathcal{N})} \quad \text{for any } O \in \mathcal{A}_{\Lambda_L}.$$

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- The n-th order cumulant/connected correlation function is defined as:

$$\begin{aligned}&\langle O_1; O_2; \ldots; O_n\rangle_{\beta,\mu,L}\\ &= \frac{\partial^n}{\partial\lambda_1\partial\lambda_2\cdots\lambda_n} \log\left\{1 + \sum_{\emptyset\neq I\subseteq\{1,2,\ldots,n\}} \Pi_{i\in I}\lambda_i \langle \Pi_{i\in I} O_i\rangle_{\beta,\mu,L}\right\}\Bigg|_{\lambda_i=0}.\end{aligned}$$

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- The instantaneous Gibbs state of the perturbed Hamiltonian $\mathcal{H}(\eta t)$:

$$\langle O_X\rangle_{\eta t} = \frac{\mathbf{Tr}\, O_X e^{-\beta(\mathcal{H}(\eta t)-\mu\mathcal{N})}}{\mathbf{Tr}\, e^{-\beta(\mathcal{H}(\eta t)-\mu\mathcal{N})}}\,.$$

Perturbation theory for equilibrium states

- Cumulant expansion around the Gibbs state of the unperturbed Hamiltonian $\mathscr{H}$:

$$\langle O_X\rangle_{\eta t} = \mathbf{Tr}\, O_X \rho_{\beta,\mu,L} + \sum_{n\geq 1} \frac{\left(-\varepsilon \mathrm{e}^{\eta t}\right)^n}{n!} \int_{[0,\beta]^n} d\underline{s}\, \langle \mathbf{T} \gamma_{s_1}(\mathcal{P})\,; \gamma_{s_2}(\mathcal{P})\,; \cdots\,; \gamma_{s_n}(\mathcal{P})\,; O_X\rangle_{\beta,\mu,L}.$$

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$$\langle O_X \rangle_{\eta t} = \mathbf{Tr}\, O_X \rho_{\beta,\mu,L} + \sum_{n \geq 1} \frac{\left(-\varepsilon \mathbf{e}^{\eta t}\right)^n}{n!} \int_{[0,\beta]^n} d\underline{s}\, \langle \mathbf{T} \gamma_{s_1}(\mathcal{P})\,;\gamma_{s_2}(\mathcal{P})\,;\cdots\,;\gamma_{s_n}(\mathcal{P})\,;O_X \rangle_{\beta,\mu,L}.$$

- For a relevant class of models:

$$\int_{[0,\beta]^n} d\underline{s} \left| \langle \mathbf{T} \gamma_{s_1}(\mathcal{P})\,;\gamma_{s_2}(\mathcal{P})\,;\cdots\,;\gamma_{s_n}(\mathcal{P})\,;O_X \rangle_{\beta,\mu,L} \right| \leq \mathfrak{c}^n n!$$

for a constant $\mathfrak{c}$ that might depend on β but is uniform in L.

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for a constant $\mathfrak{c}$ that might depend on β but is uniform in L.

Is there a relation between these two PTs?

Assumptions

- Let $\mathcal{H}(\eta t) = \mathcal{H} + \varepsilon \mathbf{e}^{\eta t}\mathcal{P}$ with $\mathcal{H}, \mathcal{P}$ finite-range, self-adjoint in $\mathcal{A}^{\mathcal{N}}_{\Lambda_L}$.
- Assume the Integrability of time-ordered cumulants:
  For $\mathcal{O}^{(i)}$ finite-range operators,

$$\int_{[0,\beta]^n} d\underline{t}(1 + |\underline{t}|_\beta) \sum_{X_i \subseteq \Lambda_L} \left|\langle \mathbf{T}\gamma_{t_1}(\mathcal{O}^{(1)}_{X_1}); \cdots; \gamma_{t_n}(\mathcal{O}^{(n)}_{X_n}); \mathcal{O}^{(n+1)}_X \rangle_{\beta,\mu,L}\right| \leq \mathfrak{c}^n n!$$

with $|\underline{t}|_\beta = \sum_{i=1}^n \mathbf{min}_{m\in\mathbb{Z}} |t_i - m\beta|$ and $\mathfrak{c} = \mathfrak{c}(\beta)$.

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Remark: This assumption holds true for many-body perturbations of quadratic Hamiltonians by cluster expansion [Brydges–Battle–Federbush formula & Gawedzki–Kupiainen–Lesniewski bound], as for example:

$$\mathcal{H} = \sum_{x,y\in\Lambda_L} a^+_x H(x,y) a^-_y + \lambda \sum_{x,y\in\Lambda_L} a^+_x a^+_y v(x,y) a^-_y a^-_x$$

with $|\lambda| < \lambda_0$ small independent of L, both H and v finite-range. If H is gapped, and μ is in the gap, the constants λ_0, $\mathfrak{c}$ are independent of β.

Main result

Theorem (R. L. Greenblatt, M. Lange, G. M., M. Porta)
Under the previous assumptions, there exists $\varepsilon_0 \equiv \varepsilon_0(\mathfrak{c})$ such that for every $|\varepsilon| < \varepsilon_0$:

1. We have that

$$\mathbf{Tr}\,(O_X \rho(t)) = \langle O_X \rangle_{\beta,\mu,L} + \sum_{n\geq 1} \frac{(-\varepsilon)^n}{n!} I^{(n)}_{\beta,\mu,L}(\eta, t) + R_{\beta,\mu,L}(\varepsilon, \eta, t)\ ,$$

where

$$I^{(n)}_{\beta,\mu,L}(\eta, t) = \int_{[0,\beta]^n} d\underline{s} \Big[\prod_{j=1}^n \mathrm{e}^{\eta_\beta (t - i s_j)} \Big] \langle \mathbf{T} \gamma_{s_1}(\mathcal{P}); \gamma_{s_2}(\mathcal{P}); \cdots; \gamma_{s_n}(\mathcal{P}); O_X \rangle_{\beta,\mu,L}$$

with $\eta_\beta \in \frac{2\pi}{\beta}\mathbb{N}$, $0 < \eta_\beta - \eta \leq \frac{2\pi}{\beta}$,

$$\Big| I^{(n)}_{\beta,\mu,L}(\eta, t) \Big| \leq \mathfrak{c}^n n! \quad \text{and} \quad \Big| R_{\beta,\mu,L}(\varepsilon, \eta, t) \Big| \leq K \frac{|\varepsilon|}{\eta^{d+2}\beta}.$$

Main result

2. Let $\langle \cdot \rangle_{\eta t}$ be the instantaneous Gibbs state of $\mathcal{H}(\eta t)$. Then

$$\left| \mathbf{Tr}\left(O_X \rho(t) \right) - \langle O_X \rangle_{\eta t} \right| \leq \frac{K|\varepsilon|}{\eta^{d+2}\beta} + C_1 |\varepsilon| \left(\eta + \frac{1}{\beta} \right) + \frac{C_2 |\varepsilon|}{\beta \eta} \, .$$

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Remark:

-if $\mathcal{H}$ is a many-body perturbation of a non-interacting gapped Hamiltonian (of the types considered before) then $\mathfrak{c}$ is actually independent of β. Hence, for this latter class of models ε_0 is independent of L, η and β.

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-To make sure that the remainder term $R_{\beta,\mu,L}(\varepsilon, \eta, t)$ is small, one needs to choose β large enough so that

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Thank you very much!

Sketch of the proof

- For simplicity consider $g(\eta t) = \mathbf{e}^{\eta t}$, thus let $g_{\beta,\eta}(t) = \mathbf{e}^{\eta_\beta t}$ with $\eta_\beta \in \frac{2\pi}{\beta}\mathbb{N}$.

Sketch of the proof

- For simplicity consider $g(\eta t) = \mathbf{e}^{\eta t}$, thus let $g_{\beta,\eta}(t) = \mathbf{e}^{\eta_\beta t}$ with $\eta_\beta \in \frac{2\pi}{\beta}\mathbb{N}$.
- We replace the perturbed dynamics generated by $\mathcal{H}(\eta t) = \mathcal{H} + \varepsilon g(\eta t)\mathcal{P}$ with the one generated by $\widetilde{\mathcal{H}}_{\beta,\eta}(t) = \mathcal{H} + \varepsilon g_{\beta,\eta}(t)\mathcal{P}$, by making an error $\mathcal{O}\left(\frac{|\varepsilon|}{\eta^{d+2}\beta}\right)$.

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- We write down the Duhamel expansion:

$$
\begin{aligned}
&\mathbf{Tr}_{\mathcal{F}_L} O_X \tilde{\rho}(t) = \mathbf{Tr}_{\mathcal{F}_L} O_X \rho_{\beta,\mu,L} \\
&+ \sum_{n\geq 1} (-\mathbf{i}\varepsilon)^n \int_{\text{symplex}} d\underline{s} \left[\prod_{j=1}^{n} g_{\beta,\eta}(s_j)\right] \langle [\cdots [\tau_t(O), \tau_{s_1}(\mathcal{P})] \cdots \tau_{s_n}(\mathcal{P})] \rangle_{\beta,\mu,L}.
\end{aligned}
$$

Sketch of the proof

- Each term in the Duhamel expansion is "Wick rotated" (complex deformation):

$$\int_{-\infty \le s_n \le \ldots \le s_1 \le t} d\underline{s} \left[\prod_{j=1}^{n} g_{\beta,\eta}(s_j) \right] \langle [\cdots [\tau_t(O), \tau_{s_1}(\mathcal{P})], \cdots, \tau_{s_n}(\mathcal{P})] \rangle_{\beta,\mu,L}$$

$$= \frac{(-\mathbf{i})^n}{n!} \int_{[0,\beta]^n} d\underline{s} \left[\prod_{j=1}^{n} g_{\beta,\eta}(t - i s_j) \right] \langle \mathbf{T} \gamma_{s_1}(\mathcal{P}); \cdots; \gamma_{s_n}(\mathcal{P}); O \rangle_{\beta,\mu,L}.$$

Sketch of the proof

- Let us show this equality at first order:

$$\lim_{T\to\infty}\int_{-T}^{t} ds\, g_{\beta,\eta}(s)\langle[\tau_t(O),\tau_s(\mathcal{P})]\rangle_{\beta,\mu,L} =$$

$$\lim_{T\to\infty}\int_{-T}^{t} ds\, g_{\beta,\eta}(s)\Big\{\langle\tau_{s-\mathrm{i}\beta}(\mathcal{P})\tau_t(O)\rangle_{\beta,\mu,L} - \langle\tau_s(\mathcal{P})\tau_t(O)\rangle_{\beta,\mu,L}\Big\} =$$

$$\lim_{T\to\infty}\Big\{\int_{-T}^{t} ds\, g_{\beta,\eta}(s-\mathrm{i}\beta)\langle\tau_{s-\mathrm{i}\beta}(\mathcal{P})\tau_t(O)\rangle_{\beta,\mu,L}$$

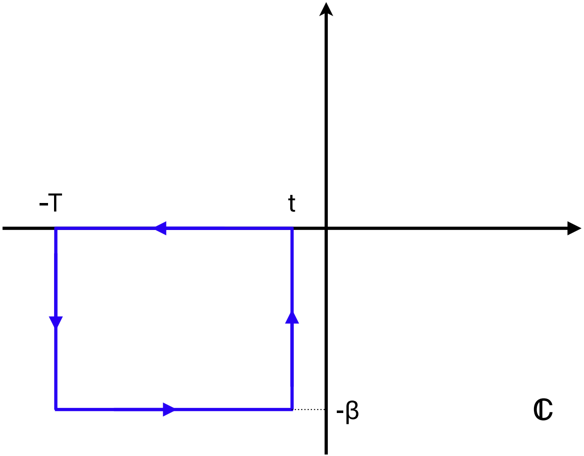
$$-\int_{-T}^{t} ds\, g_{\beta,\eta}(s)\langle\tau_s(\mathcal{P})\tau_t(O)\rangle_{\beta,\mu,L}\Big\} =$$

$$-\mathrm{i}\int_0^{\beta} ds\, g_{\beta,\eta}(t-\mathrm{i}s)\langle\tau_{t-\mathrm{i}s}(\mathcal{P})\tau_t(O)\rangle_{\beta,\mu,L} =$$

$$-\mathrm{i}\int_0^{\beta} ds\, g_{\beta,\eta}(t-\mathrm{i}s)\langle\gamma_s(\mathcal{P})O\rangle_{\beta,\mu,L} = -\mathrm{i}\int_0^{\beta} ds\, g_{\beta,\eta}(t-\mathrm{i}s)\langle\mathbf{T}\gamma_s(\mathcal{P});O\rangle_{\beta,\mu,L}.$$

Sketch of the proof

Let $f(z) := g_{\beta,\eta}(z)\langle\tau_z(\mathcal{P})\tau_t(O)\rangle_{\beta,\mu,L} = \mathrm{e}^{\eta_\beta z}\langle\tau_z(\mathcal{P})\tau_t(O)\rangle_{\beta,\mu,L}$.



Decay of time-ordered cumulants

- If $\mathcal{H}_0$ and Ψ_{X_i} are quadratic (interaction parameter $\lambda = 0$), the bound follows from Wick's rule and the decay of non-interacting two-point function

$$g_2(t, x; s, y) := \langle \mathsf{T} \gamma_t(a_x^-) \gamma_s(a_y^+) \rangle_{\beta,\mu,L}^{\lambda=0}.$$

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In particular, if H is gapped and μ is in the gap, then $\mathfrak{c}$ is independent of β.

- For finite-range $\Psi_{X_i} \in \mathcal{A}_{X_i}^{\mathcal{N}}$, $\mathcal{H} = \mathcal{H}_0 + \lambda\mathcal{V}$ with $\mathcal{H}_0$ quadratic and $\mathcal{V}$ quartic or higher even power, the bound $(**)$ follows from cluster expansion [Brydges–Battle–Federbush formula & Gawedzki–Kupiainen–Lesniewski bound].