From decay of correlations to locality and stability of the Gibbs state

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Joint work with A. Capel, S. Teufel and T. Wessel
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How to measure the locality of the Gibbs state?

Interacting quantum spin system defined on $\Lambda \subseteq \mathbb{Z}^\nu$, $\nu \in \mathbb{N}$.
Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x \,;\,, \qquad \mathcal{H}_x \equiv \mathbb{C}^N$.
Algebra of observables $\mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda) = \bigotimes_{x\in\Lambda} \mathcal{B}(\mathcal{H}_x)$.

$$\mathcal{A}_\Lambda = \overline{\bigcup_{\Lambda' \Subset \Lambda} \mathcal{A}_{\Lambda'}}^{\|\cdot\|}$$

Local Hamiltonian H_Λ: $\to$ *sum of local terms*
$\to$ An *interaction* on $\mathbb{Z}^\nu$ is a function

$$\Psi \colon \{X \Subset \mathbb{Z}^\nu\} \to \mathcal{A}_{\mathbb{Z}^\nu}, \quad X \mapsto \Psi(X) \in \mathcal{A}_X \text{ with } \Psi(X) = \Psi(X)^*.$$

For each $\Lambda \Subset \mathbb{Z}^\nu$ we define

$$H_\Lambda := \sum_{X \subset \Lambda} \Psi(X).$$

Gibbs state at inverse temperature $\beta > 0$, $\Lambda \subset \mathbb{Z}^\nu$

$$\rho_\beta^\Lambda[H_\Lambda] := \frac{e^{-\beta H_\Lambda}}{\mathrm{Tr}_\Lambda\left(e^{-\beta H_\Lambda}\right)},$$

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Problem

How the locality of the Hamiltonian translates to the Gibbs state?

Lieb-Robinson bound

H satisfies a Lieb-Robinson bound with decay
$\zeta_{LR} : \mathcal{P}_0(\Lambda) \times \mathcal{P}_0(\Lambda) \times [0,\infty) \times [0,\infty) \to [0,\infty)$ if

$$\left\| \left[\mathrm{e}^{-\mathrm{i}tH} A \mathrm{e}^{\mathrm{i}tH}, B\right] \right\| \leq \|A\| \|B\| \zeta_{\mathrm{LR}}(X, Y, |t|)$$

for all $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$ with $X, Y \subset \Lambda$ and $t \in \mathbb{R}$.

$\to$ We focus on three different ways to **measure** the locality of the Gibbs state.

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$$\left\| \left(1 - \mathbb{E}_{X_r}\right) \left(\mathrm{e}^{-\mathrm{i}tH}\, A\, \mathrm{e}^{\mathrm{i}tH}\right) \right\| \leq \|A\|\, \zeta_{LR}(X, \Lambda \setminus X_r, |t|)$$

$X_r := \{x \in \mathbb{Z}^\nu \,|\, d(x, X) \leq r\}$

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Short-range $\Rightarrow \zeta_{LR}(X, Y, |t|) = |X| e^{-b(d(X,Y) - v_b |t|)}$

$$\sup_{z \in \Lambda} \sum_{\substack{Z \in \Lambda: \\ z \in Z}} \|\Psi(Z)\| e^{a|Z| + b\,\mathrm{diam}(Z)} < +\infty$$

$\rightarrow$ We focus on three different ways to **measure** the locality of the Gibbs state.

(1) Decay of correlations

$$\mathrm{Cov}_\rho(A,B) := \mathrm{Tr}_\Lambda\left(\rho AB\right) - \mathrm{Tr}_\Lambda\left(\rho A\right)\mathrm{Tr}_\Lambda\left(\rho B\right)$$

$$\mathrm{Cov}_\rho(X;Y) = \sup_{\substack{A\in\mathcal{A}_X\,:\,\|A\|=1\,;\\ B\in\mathcal{A}_Y\,:\,\|B\|=1}} |\mathrm{Cov}_\rho(A,B)|.$$

$$|\mathrm{Cov}_\rho(A,B)| \approx \mathrm{e}^{-c d(\mathrm{supp}(A),\mathrm{supp}(B))}$$

(Uniform) decay of correlations

Let $\Lambda \subset \mathbb{Z}^\nu$ and ρ be a state on Λ. We say that ρ satisfies ***decay of correlations*** with respect to $\zeta_{DC}, f_{DC} : [0,+\infty) \to [0,+\infty)$ and $n \geq 0$ if and only if

$$\mathrm{Cov}_\rho(X;Y) \leq |X|^n f_{DC}(|Y|)\zeta_{DC}\left(\mathrm{d}(X,Y)\right)$$

for all $X, Y \subset \Lambda$.

H_Λ/Ψ satisfies ***uniform decay of correlations*** (at inverse temperature β) $\iff$ the Gibbs states $\rho_\beta^{\Lambda'}[H_{\Lambda'}]$ satisfy decay of correlations for the same functions and n for every $\Lambda' \subset \Lambda$.

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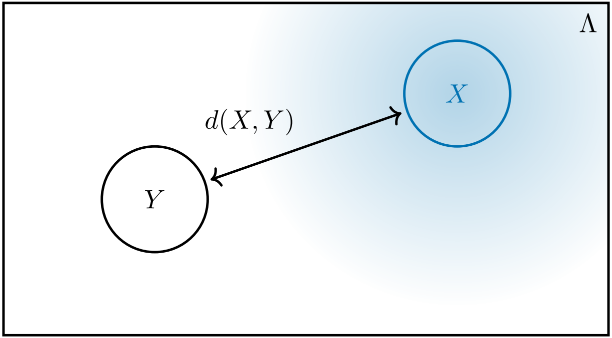
- $\nu \geq 1$, $\beta^{-1} = T > T^*$: Kliesch-Gogolin-Kastoryano-Riera-Eisert 2014 (finite range); Frölich-Ueltschi 2015 (short range);
- $\nu = 1$ and $T > 0$ + translation invariance: Araki 1969; Perez–Garcia-Perez–Hernandez 2023. For the infinite chain!
- $T = 0$ + spectral gap: Hastings-Koma 2006, Nachergaele-Sims 2006.

(2) Local stability

V perturbation supported on $X \subset \Lambda$.
$H'_\Lambda := H_\Lambda + V$.
$\rho^\Lambda_\beta[H_\Lambda + V]$ the Gibbs state associated to H'_Λ.
B supported on a set Y *far* from X

$$\mathrm{Tr}_\Lambda\left((\rho^\Lambda_\beta[H_\Lambda]\,B\right) \approx \mathrm{Tr}_\Lambda\left((\rho^\Lambda_\beta[H_\Lambda + V]\,B\right)$$



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We say that H satisfies ***LPPL*** (at inverse temperature β) with respect to f_{lppl}, $g_{\text{lppl}}, \zeta_{\text{lppl}} : [0,\infty) \to [0,\infty)$ and $n \geq 0$, if and only if

$$\left|\mathrm{Tr}\left(\rho_\beta^\Lambda[H]B\right) - \mathrm{Tr}\left(\rho_\beta^\Lambda[H+V]B\right)\right| \leq \|B\||X|^n f_{\text{lppl}}(|Y|) g_{\text{lppl}}(\|V\|) \zeta_{\text{lppl}}(d(X,Y))$$

for all $X, Y \subset \Lambda, V \in \mathcal{A}_X$ self-adjoint and $B \in \mathcal{A}_Y$.
H_Λ/Ψ satisfies ***uniform L P P L*** (at inverse temperature β) $\iff$ $H_{\Lambda'}$ satisfies LPPL for every $\Lambda' \subset \Lambda$.

- $\nu \geq 1, T > T^*$: Kliesch-Gogolin-Kastoryano-Riera-Eisert 2014 (finite range).
- $T = 0$ + spectral gap + V *small*: Yarotski 2005; Bachmann-Michalakis-Nachtergaele-Sims 2011; Henheik-Teufel-Wessel 2022; Bachmann-de Roeck-Donvil-Fraas 2022.

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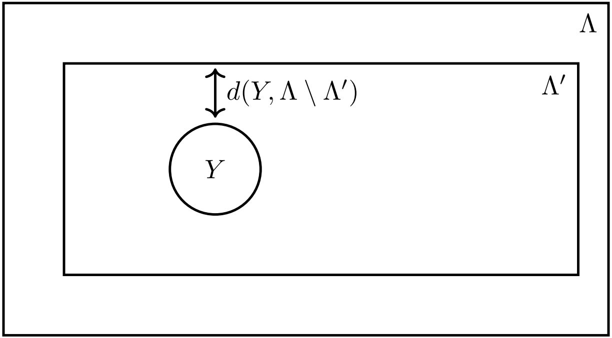
(3) Local indistinguishability

$\Lambda' \subseteq \Lambda$. Local Gibbs state $\rho_\beta^{\Lambda'}$.
B observable supported in an inner region $X \subset \Lambda' \subset \Lambda$.

$$\mathrm{Tr}_\Lambda \left(\rho_\beta^\Lambda B\right) \approx \mathrm{Tr}_{\Lambda'} \left(\rho_\beta^{\Lambda'} B\right).$$



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Let $\Lambda \subset \mathbb{Z}^\nu$. We say that Ψ satisfies ***local indistinguishability*** (at inverse temperature β) on Λ with respect to f_{LI} and $\zeta_{LI} : [0,\infty) \to [0,\infty)$ if and only if

$$\left| \mathrm{Tr}\left(\rho_\beta^\Lambda \left[H_\Lambda\right] B\right) - \mathrm{Tr}\left(\rho_\beta^{\Lambda'} \left[H_{\Lambda'}\right] B\right) \right| \leq \|B\| f_{\mathrm{LI}}(|Y|) \zeta_{\mathrm{LI}}\left(d\left(Y, \Lambda \backslash \Lambda'\right)\right)$$

for all $Y \subset \Lambda' \subset \Lambda$ and $B \in \mathcal{A}_Y$.

- $\nu \geq 1$, $T > T^*$: Kliesch et al. 2014 (finite range Hamiltonians); Brandão-Kastoryano 2019 (finite range Hamiltonians).
- $T > 0$, $\nu = 1$, finite range and translation invariant: Bluhm-Capel-Perez–Hernanzed 2022.
- $T = 0 \rightsquigarrow$ Local Topological Quantum Order for gapped ground states.

How to connect (1), (2) and (3) ?

Spoiler!

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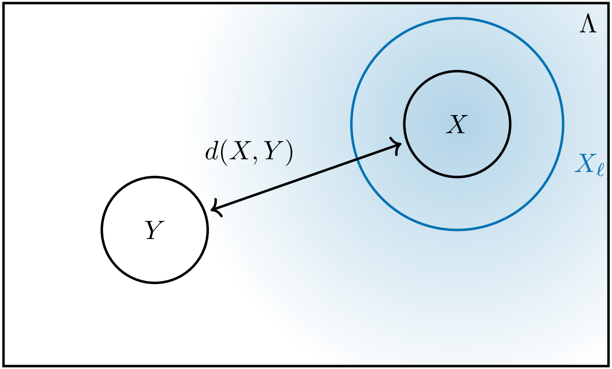
Key tool: quantum belief propagation (QBP)

V perturbation supported on $X \subset \Lambda$.
$H'_\Lambda := H_\Lambda + V$.
$\tilde{\rho}^\Lambda_\beta$ the Gibbs state associated to H'_Λ.
$X_\ell := \{x \in \mathbb{Z}^\nu \,|\, d(x, X) \leq \ell\}$. $\tilde{\eta}_\ell \in \mathcal{A}_{X_\ell}$.

$$\tilde{\rho}^\Lambda_\beta \approx \tilde{\eta}_\ell \rho^\Lambda_\beta \tilde{\eta}^*_\ell$$



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Quantum Belief Propagation (Hastings 2007,+ *)

$\mathcal{H}$ finite dimensional Hilbert space. H, V self-adjoint operators on $\mathcal{H}$.
$H(s) := H + sV$. Then,

- $$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{-\beta H(s)} = -\frac{\beta}{2}\left\{\mathrm{e}^{-\beta H(s)}, \Phi_\beta^{H(s)}(V)\right\},$$

  where $\Phi_\beta^{H(s)}(V) := \int_{-\infty}^{\infty} \mathrm{d}t f_\beta(t) \mathrm{e}^{-\mathrm{i}tH(s)} V \mathrm{e}^{\mathrm{i}tH(s)}$ with f_β an exponentially decaying L^1-function

  $$f_\beta(t) = \frac{2}{\beta\pi}\log\left(\frac{\mathrm{e}^{\pi|t|/\beta}+1}{\mathrm{e}^{\pi|t|/\beta}-1}\right)$$

- $\exists\ s \mapsto \eta(s) := \mathcal{T}\exp\left(-\frac{\beta}{2}\int_0^s \Phi_\beta^{H(\sigma)}(V)\mathrm{d}\sigma\right)$
- $s \mapsto \rho_\beta(s) := \rho_\beta^{\mathcal{H}}[H(s)]$

  $$\frac{\mathrm{d}}{\mathrm{d}s}\rho_\beta(s) = -\frac{\beta}{2}\left\{\rho_\beta(s), \Phi_\beta^{H(s)}\left(V - \langle V\rangle_{\rho_\beta(s)}\right)\right\}.$$

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$$\mathrm{e}^{-\beta H(s)} = \eta(s)\mathrm{e}^{-\beta H(0)}\eta(s)^* \qquad \|\eta(s)\| \leq \mathrm{e}^{\frac{\beta}{2}s\|V\|}.$$

- $s \mapsto \rho_\beta(s) := \rho_\beta^{\mathcal{H}}[H(s)]$

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- $\exists\ s \mapsto \tilde{\eta}(s) = \exp\left(-\frac{\beta}{2}\int_0^s \langle V\rangle_{\rho_\beta(\sigma)}\mathrm{d}\sigma\right)\eta(s)$ such that

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$$\|\rho_\beta(0) - \rho_\beta(s)\|_1 \leq \mathrm{e}^{2\beta s\|V\|} - 1.$$

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Quantum Belief Propagation (Hastings 2007,+ *)

Path of Hamiltonians $H(s) = H + s\,V$, $V \in \mathcal{A}_X$, that satisfy **Lieb-Robinson bound** with a uniform ζ_{LR}. Then, $s \mapsto \rho_\beta(s) := \rho_\beta[H(s)]$ satisfies the differential equation

$$\frac{\mathrm{d}\rho_\beta(s)}{\mathrm{d}s} = -\frac{\beta}{2}\left\{\rho_\beta(s), \Phi_\beta^{H(s)}\left(V - \langle V\rangle_{\rho_\beta(s)}\right)\right\}.$$

Moreover, $\exists\, \zeta_{QBP}$, which only depends on ζ_{LR} and β, such that:

- $\Phi_\beta^{H(s)}(V)$ can be **approximated by local operators** $\Phi_{\beta,\ell}^{H(s)}(V) \in \mathcal{A}_{X_\ell}$ supported on X_ℓ, such that $\left\|\Phi_{\beta,\ell}^{H(s)}(W)\right\| \leq \left\|\Phi_\beta^{H(s)}(W)\right\|$ and

$$\left\|\Phi_\beta^{H(s)}(W) - \Phi_{\beta,\ell}^{H(s)}(W)\right\| \leq \|W\|\zeta_{\mathrm{QBP}}(X, \ell).$$

- $\eta(s)$ and $\tilde{\eta}(s)$ can be **approximated by local operators** $\eta_\ell(s), \tilde{\eta}_\ell(s) \in \mathcal{A}_{X_\ell}$ supported on X_ℓ, such that $\|\tilde{\eta}_\ell(s)\| \leq \|\tilde{\eta}(s)\|$ and

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$$\|\tilde{\eta}(s) - \tilde{\eta}_\ell(s)\| \leq \beta s \|V\| \mathrm{e}^{\beta s \|V\|}\zeta_{\mathrm{QBP}}(X,\ell).$$

$*$: Kim 2012; Kato-Brandão '19; Harrow-Mehraban-Soleimanifar '20; Anshu-Arunachalam-Kuwahara-Soleimanifar '21; Alhambra 2022.

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$*$: Kim 2012; Kato-Brandão '19; Harrow-Mehraban-Soleimanifar '20; Anshu-Arunachalam-Kuwahara-Soleimanifar '21; Alhambra 2022.

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- Relies only on the locality of the system through *Lieb-Robinson* bounds.
- **Exponential** Lieb-Robinson bound $\Rightarrow$ **Exponential decay** in ζ_{QBP}

$$\zeta_{\mathrm{QBP}}(X,\ell) \le C\,|X|\,\mathrm{e}^{\frac{-b\,\ell}{1+bv_b\beta/\pi}}$$

- Who is the **approximated generator** $\Phi^{H(s)}_{\beta,\ell}$?

$$\Phi^{H(s)}_{\beta,\ell} = \int_{-\infty}^{\infty} \mathrm{d}t\, f_\beta(t)\, \mathbb{E}_{X_r}\left(\mathrm{e}^{-\mathrm{i}tH(s)}\,V\,\mathrm{e}^{\mathrm{i}tH(s)}\right)$$

- ζ_{QBP} comes from a **balance** between the **decay of** $f_\beta(t)$ and the **spreading of the support of** V:

$$\left\|\Phi^{H(s)}_{\beta}(W) - \Phi^{H(s)}_{\beta,\ell}(W)\right\| \le \left\|\int_{-\infty}^{\infty} \mathrm{d}t\, f_\beta(t)\,\left(1-\mathbb{E}_{X_r}\right)\left(\mathrm{e}^{-\mathrm{i}tH(s)}\,V\,\mathrm{e}^{\mathrm{i}tH(s)}\right)\right\|$$

$$\le \|V\| \int_{|t|\le T} f_\beta(t)\,\zeta_{\mathrm{LR}}(X,\Lambda\setminus X_r,t)\mathrm{d}t + 2\,\|V\| \int_{|t|\ge T} f_\beta(t)\mathrm{d}t$$

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Main results: from DC to LPPL

Theorem [A. Capel, M.M., S. Teufel, T. Wessel]

$X \subset \Lambda$, $V \in \mathcal{A}_X$. Path of Hamiltonians $H_\Lambda(s) := H_\Lambda + sV$, that satisfy a Lieb-Robinson bound with a uniform ζ_{LR}.
Then, for all $B \in \mathcal{A}_Y$, with $Y \subset \Lambda$ and all $r \in \mathbb{N}$, we have

- DC in the unperturbed state $\Rightarrow$ LPPL

$$\left|\mathrm{Tr}\left(\rho_\beta^\Lambda(0)B\right) - \mathrm{Tr}\left(\rho_\beta^\Lambda(1)B\right)\right| \leq \mathrm{e}^{2\beta\|V\|}\|B\|\left(\mathrm{Cov}_{\rho_\beta^\Lambda(0)}(X_r;Y) + 4\beta\|V\|\zeta_{\mathrm{QBP}}(X,r)\right).$$

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- **No restriction on the range** of the Hamiltonian
- Need only **decay of the covariance** (Kliesch et al. results requires *generalized covariance*: $\left|\mathrm{Tr}(\rho^\tau A \rho^{1-\tau} B) - \mathrm{Tr}(\rho A)\,\mathrm{Tr}(\rho B)\right|$)

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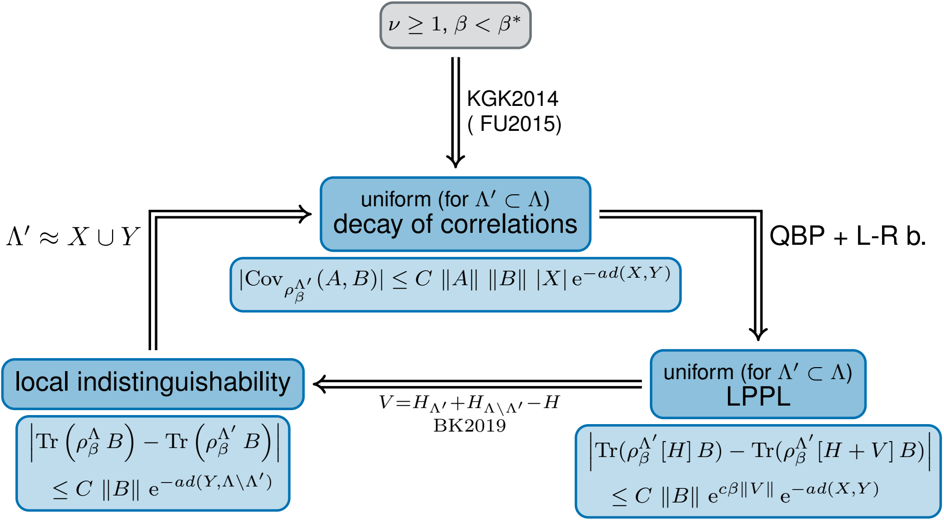
$\nu \geq 1,\ \beta < \beta^*$
KGK2014
(FU2015)
uniform (for $\Lambda' \subset \Lambda$)
decay of correlations
$|\mathrm{Cov}_{\rho_\beta^{\Lambda'}}(A,B)| \leq C\ \|A\|\ \|B\|\ |X|\, \mathrm{e}^{-a d(X,Y)}$
$\Lambda' \approx X \cup Y$
QBP + L-R b.
local indistinguishability
$\left|\mathrm{Tr}\left(\rho_\beta^{\Lambda} B\right) - \mathrm{Tr}\left(\rho_\beta^{\Lambda'} B\right)\right| \leq C\ \|B\|\ \mathrm{e}^{-a d(Y, \Lambda\setminus\Lambda')}$
$V = H_{\Lambda'} + H_{\Lambda\setminus\Lambda'} - H$
BK2019
uniform (for $\Lambda' \subset \Lambda$)
LPPL
$\left|\mathrm{Tr}(\rho_\beta^{\Lambda'}[H]\,B) - \mathrm{Tr}(\rho_\beta^{\Lambda'}[H+V]\,B)\right| \leq C\ \|B\|\ \mathrm{e}^{c\beta\|V\|}\ \mathrm{e}^{-a d(X,Y)}$

One-dimensional quantum spin chains

$\nu = 1$, $\beta < \beta^*$,
translation invariant Hamiltonian

$\Downarrow$ Araki1969, PP2023

infinite chain
decay of correlations

$$\left|\mathrm{Cov}_{\rho_\beta}(A,B)\right| \le C\,\|A\|\,\|B\|\,\mathrm{e}^{-a\,\mathrm{dist}(X,Y)}$$

$\Downarrow \quad \Uparrow$

finite chain
decay of correlations

$$\left|\mathrm{Cov}_{\rho^I_\beta}(A,B)\right| \le C\,\|A\|\,\|B\|\,\mathrm{e}^{-a\,\mathrm{dist}(X,Y)}$$

local indistinguishability

$$\left|\mathrm{Tr}\left(\rho^I_\beta\, B\right) - \mathrm{Tr}\left(\rho^{I'}_\beta\, B\right)\right| \le C\,\|B\|\,\mathrm{e}^{-a\,\mathrm{dist}(X,Y)}$$

$\Leftarrow$

local perturbations perturb locally

$$\left|\mathrm{Tr}\left(\rho^I_\beta[H]\, B\right) - \mathrm{Tr}\left(\rho^I_\beta[H+V]\, B\right)\right| \le C\,\|B\|\,\mathrm{e}^{c\beta\|V\|}\,\mathrm{e}^{-a\,\mathrm{dist}(X,Y)}$$

Summary and open questions

- Decay of correlations implies locality and stability of the Gibbs state
- (Uniform) Decay of correlations $\iff$ (Uniform) LPPL $\iff$ Local Indistinguishability

Open questions:

- how the "circle" looks like at $T = 0$?
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Thank you for your attention!

Capel. A., M. M., Teufel, S., Wessel, T.: From decay of correlations to locality and stability of the Gibbs state ArXiv: 2310.09182 (2023).

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Idea of the proof: DC along the path $\Rightarrow$ LPPL

Integrate the differential equation

$$\begin{aligned}
&\mathrm{Tr}(\rho(1)B) - \mathrm{Tr}(\rho(0)B) \\
&\quad = -\frac{\beta}{2}\int_0^1 \mathrm{Tr}\left(\left\{\rho(s), \Phi_\beta^{H(s)}(V)\right\} - 2\,\mathrm{Tr}\left(\rho(s)\Phi_\beta^{H(s)}(V)\right)\rho(s)B\right)\mathrm{d}s \\
&\quad = -\frac{\beta}{2}\int_0^1 \mathrm{Cov}_{\rho(s)}\left(\Phi_\beta^{H(s)}(V), B\right) + \mathrm{Cov}_{\rho(s)}\left(B, \Phi_\beta^{H(s)}(V)\right)\mathrm{d}s.
\end{aligned}$$

$\Rightarrow$ exploit **locality of the generator**!

$$\begin{aligned}
&\left|\mathrm{Cov}_{\rho(s)}\left(B, \Phi_\beta^{H(s)}(V)\right) - \mathrm{Cov}_{\rho(s)}\left(B, \Phi_{\beta,r}^{H(s)}(V)\right)\right| \\
&\leq 2\|B\|\|\Phi_\beta^{H(s)}(V) - \Phi_{\beta,r}^{H(s)}(V)\| \leq 2\|B\|\|V\|\zeta_{\mathrm{QBP}}(X,r)
\end{aligned}$$

Therefore

$$|\,\mathrm{Tr}(\rho(1)B) - \mathrm{Tr}(\rho(0)B)| \leq \beta\|V\|\|B\|\left(\int_0^1 \mathrm{Cov}_{\rho(s)}(X_r;Y)\,\mathrm{d}s + \zeta_{\mathrm{QBP}}(X,r)\right)$$

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Idea of the proof: DC in the unperturbed state $\Rightarrow$ LPPL

Using $[B, \tilde{\eta}_\ell] = 0$

$$\begin{aligned}&\mathrm{Tr}(\rho(s)\,B) - \mathrm{Tr}(\rho\,B) = \mathrm{Tr}(\tilde{\eta}\,\rho\,\tilde{\eta}^*\,B) - \mathrm{Tr}(\rho\,B)\\ &= \mathrm{Tr}((\tilde{\eta} - \tilde{\eta}_\ell)\,\rho\,\tilde{\eta}^*\,B) + \mathrm{Tr}(\tilde{\eta}_\ell\,\rho\,(\tilde{\eta}^* - \tilde{\eta}_\ell^*)\,B) + \mathrm{Tr}(\rho\,\tilde{\eta}_\ell^*\,\tilde{\eta}_\ell\,B) - \mathrm{Tr}(\rho\,B)\end{aligned}$$

- First two terms:
  $|\mathrm{Tr}((\tilde{\eta} - \tilde{\eta}_\ell)\,\rho\,\tilde{\eta}^*\,B) + \mathrm{Tr}(\tilde{\eta}_\ell\,\rho\,(\tilde{\eta}^* - \tilde{\eta}_\ell^*)\,B)| \le \|\tilde{\eta} - \tilde{\eta}_\ell\|\,(\|\tilde{\eta}\| + \|\tilde{\eta}_\ell\|)\,\|B\|$
  $$(B = \mathbb{I}) \Rightarrow |\mathrm{Tr}(\rho\tilde{\eta}_\ell^*\tilde{\eta}_\ell) - 1| \le \|\tilde{\eta} - \tilde{\eta}_\ell\|\,(\|\tilde{\eta}\| + \|\tilde{\eta}_\ell\|)$$
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  $|\mathrm{Tr}((\tilde{\eta} - \tilde{\eta}_\ell)\, \rho\, \tilde{\eta}^*\, B) + \mathrm{Tr}(\tilde{\eta}_\ell\, \rho\, (\tilde{\eta}^* - \tilde{\eta}_\ell^*)\, B)| \leq \|\tilde{\eta} - \tilde{\eta}_\ell\|\, (\|\tilde{\eta}\| + \|\tilde{\eta}_\ell\|)\, \|B\|$
  $$(B = \mathbb{I}) \Rightarrow |\mathrm{Tr}(\rho \tilde{\eta}_\ell^* \tilde{\eta}_\ell) - 1| \leq \|\tilde{\eta} - \tilde{\eta}_\ell\|\, (\|\tilde{\eta}\| + \|\tilde{\eta}_\ell\|)$$
- The others: $|\mathrm{Tr}(\rho\, \tilde{\eta}_\ell^*\, \tilde{\eta}_\ell\, B) - \mathrm{Tr}(\rho\, B)| =$
  $|\mathrm{Tr}(\rho\, \tilde{\eta}_\ell^*\, \tilde{\eta}_\ell\, B) - \mathrm{Tr}(\rho\, B)\, \mathrm{Tr}(\rho \tilde{\eta}_\ell^* \tilde{\eta}_\ell) + (\mathrm{Tr}(\rho \tilde{\eta}_\ell^* \tilde{\eta}_\ell) - 1)\, \mathrm{Tr}(\rho\, B)| \leq$
  $|\mathrm{Cov}_\rho(\tilde{\eta}_\ell^* \tilde{\eta}_\ell, B)| + \|\tilde{\eta} - \tilde{\eta}_\ell\|\, (\|\tilde{\eta}\| + \|\tilde{\eta}_\ell\|)$

Therefore we obtain

$$|\, \mathrm{Tr}(\rho(s) B) - \mathrm{Tr}(\rho B)|$$
$$\leq 2\, \|\tilde{\eta} - \tilde{\eta}_\ell\|\, (\|\tilde{\eta}\| + \|\tilde{\eta}_\ell\|)\, \|B\| + |\mathrm{Cov}_\rho\, (\tilde{\eta}_\ell^* \tilde{\eta}_\ell, B)|$$
$$\leq 4 \|B\| \mathrm{e}^{2\beta s \|V\|} \zeta_{\mathrm{QBP}}(X, \ell) + \|B\| \mathrm{e}^{2\beta s \|V\|}\, \mathrm{Cov}_\rho\, (X_\ell; Y)$$

Idea of the proof: DC in the unperturbed state $\Rightarrow$ LPPL

Using $[B, \tilde{\eta}_\ell] = 0$

$$\mathrm{Tr}(\rho(s)\, B) - \mathrm{Tr}(\rho\, B) = \mathrm{Tr}(\tilde{\eta}\, \rho\, \tilde{\eta}^*\, B) - \mathrm{Tr}(\rho\, B)$$
$$= \mathrm{Tr}((\tilde{\eta} - \tilde{\eta}_\ell)\, \rho\, \tilde{\eta}^*\, B) + \mathrm{Tr}(\tilde{\eta}_\ell\, \rho\, (\tilde{\eta}^* - \tilde{\eta}_\ell^*)\, B) + \mathrm{Tr}(\rho\, \tilde{\eta}_\ell^*\, \tilde{\eta}_\ell\, B) - \mathrm{Tr}(\rho\, B)$$

- First two terms: $|\mathrm{Tr}((\tilde{\eta} - \tilde{\eta}_\ell)\, \rho\, \tilde{\eta}^*\, B) + \mathrm{Tr}(\tilde{\eta}_\ell\, \rho\, (\tilde{\eta}^* - \tilde{\eta}_\ell^*)\, B)| \leq \|\tilde{\eta} - \tilde{\eta}_\ell\|\, (\|\tilde{\eta}\| + \|\tilde{\eta}_\ell\|)\, \|B\|$
$$(B = \mathbb{I}) \Rightarrow |\mathrm{Tr}(\rho \tilde{\eta}_\ell^* \tilde{\eta}_\ell) - 1| \leq \|\tilde{\eta} - \tilde{\eta}_\ell\|\, (\|\tilde{\eta}\| + \|\tilde{\eta}_\ell\|)$$

- The others: $|\mathrm{Tr}(\rho\, \tilde{\eta}_\ell^*\, \tilde{\eta}_\ell\, B) - \mathrm{Tr}(\rho\, B)| =$ $|\mathrm{Tr}(\rho\, \tilde{\eta}_\ell^*\, \tilde{\eta}_\ell\, B) - \mathrm{Tr}(\rho\, B)\, \mathrm{Tr}(\rho \tilde{\eta}_\ell^* \tilde{\eta}_\ell) + (\mathrm{Tr}(\rho \tilde{\eta}_\ell^* \tilde{\eta}_\ell) - 1)\, \mathrm{Tr}(\rho\, B)| \leq$ $|\mathrm{Cov}_\rho(\tilde{\eta}_\ell^* \tilde{\eta}_\ell, B)| + \|\tilde{\eta} - \tilde{\eta}_\ell\|\, (\|\tilde{\eta}\| + \|\tilde{\eta}_\ell\|)$

Therefore we obtain

$$|\,\mathrm{Tr}(\rho(s)B) - \mathrm{Tr}(\rho B)|$$
$$\leq 2\, \|\tilde{\eta} - \tilde{\eta}_\ell\|\, (\|\tilde{\eta}\| + \|\tilde{\eta}_\ell\|)\, \|B\| + |\mathrm{Cov}_\rho\, (\tilde{\eta}_\ell^* \tilde{\eta}_\ell, B)|$$
$$\leq 4 \|B\| \mathrm{e}^{2\beta s \|V\|} \zeta_{\mathrm{QBP}}(X, \ell) + \|B\| \mathrm{e}^{2\beta s \|V\|}\, \mathrm{Cov}_\rho\, (X_\ell; Y)$$

Idea of the proof: QBP

$\{\psi_a(s)\}_a$ be the eigenbasis of $H(s)$ such that $H(s) = \sum_a E_a(s) \left|\psi_a(s)\right\rangle \left\langle\psi_a(s)\right|$.

$$\Rightarrow V = \sum_{a,b} V_{a,b}(s) \left|\psi_a(s)\right\rangle \left\langle\psi_b(s)\right| .$$

Using Duhamel's formula $\Rightarrow$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{-\beta H(s)} &= -\beta \int_0^1 \mathrm{e}^{-\beta\tau H(s)} V \mathrm{e}^{-\beta(1-\tau)H(s)} \mathrm{d}\tau \\
&= -\beta \sum_{a,b} \int_0^1 \mathrm{e}^{-\beta\tau H(s)} V_{a,b}(s) \left|\psi_a(s)\right\rangle \left\langle\psi_b(s)\right| \mathrm{e}^{\beta\tau H(s)} \mathrm{d}\tau \mathrm{e}^{-\beta H(s)} \\
&= -\beta \sum_{a,b} V_{a,b}(s) \int_0^1 \mathrm{e}^{\beta\tau \Delta E_{a,b}(s)} \mathrm{d}\tau \left|\psi_a(s)\right\rangle \left\langle\psi_b(s)\right| \mathrm{e}^{-\beta H(s)} \\
&= -\beta \sum_{a,b} V_{a,b}(s) \left(1 + \mathrm{e}^{\beta \Delta E_{a,b}(s)}\right)^{-1} .
\end{aligned}$$

$$\int_0^1 \mathrm{e}^{\beta\tau \Delta E_{a,b}(s)} \mathrm{d}\tau \left\{ \mathrm{e}^{-\beta H(s)}, \left|\psi_a(s)\right\rangle \left\langle\psi_b(s)\right| \right\}$$

Idea of the proof: QBP

$\{\psi_a(s)\}_a$ be the eigenbasis of $H(s)$ such that $H(s) = \sum_a E_a(s) |\psi_a(s)\rangle \langle\psi_a(s)|$.

$$\Rightarrow V = \sum_{a,b} V_{a,b}(s) |\psi_a(s)\rangle \langle\psi_b(s)| \, .$$

Using Duhamel's formula $\Rightarrow$

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{-\beta H(s)} = -\beta \int_0^1 \mathrm{e}^{-\beta\tau H(s)} V \mathrm{e}^{-\beta(1-\tau)H(s)} \mathrm{d}\tau$$

$$= -\beta \sum_{a,b} \int_0^1 \mathrm{e}^{-\beta\tau H(s)} V_{a,b}(s) |\psi_a(s)\rangle \langle\psi_b(s)| \, \mathrm{e}^{\beta\tau H(s)} \mathrm{d}\tau \mathrm{e}^{-\beta H(s)}$$

$$= -\beta \sum_{a,b} V_{a,b}(s) \int_0^1 \mathrm{e}^{\beta\tau\Delta E_{a,b}(s)} \mathrm{d}\tau \, |\psi_a(s)\rangle \langle\psi_b(s)| \, \mathrm{e}^{-\beta H(s)}$$

$$= -\beta \sum_{a,b} V_{a,b}(s) \left(1 + \mathrm{e}^{\beta\Delta E_{a,b}(s)}\right)^{-1} .$$

$$\int_0^1 \mathrm{e}^{\beta\tau\Delta E_{a,b}(s)} \mathrm{d}\tau \left\{\mathrm{e}^{-\beta H(s)}, |\psi_a(s)\rangle \langle\psi_b(s)|\right\}$$

$$= -\frac{\beta}{\;} \left\{\mathrm{e}^{-\beta H(s)}, \Phi^{H(s)}(V)\right\}$$

Idea of the proof: QBP

$\{\psi_a(s)\}_a$ be the eigenbasis of $H(s)$ such that $H(s) = \sum_a E_a(s) \left|\psi_a(s)\right\rangle \left\langle\psi_a(s)\right|$.

$$\Rightarrow V = \sum_{a,b} V_{a,b}(s) \left|\psi_a(s)\right\rangle \left\langle\psi_b(s)\right| .$$

Using Duhamel's formula $\Rightarrow$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{-\beta H(s)} &= -\beta \int_0^1 \mathrm{e}^{-\beta\tau H(s)} V \mathrm{e}^{-\beta(1-\tau)H(s)} \mathrm{d}\tau \\
&= -\beta \sum_{a,b} \int_0^1 \mathrm{e}^{-\beta\tau H(s)} V_{a,b}(s) \left|\psi_a(s)\right\rangle \left\langle\psi_b(s)\right| \mathrm{e}^{\beta\tau H(s)} \mathrm{d}\tau \mathrm{e}^{-\beta H(s)} \\
&= -\beta \sum_{a,b} V_{a,b}(s) \int_0^1 \mathrm{e}^{\beta\tau \Delta E_{a,b}(s)} \mathrm{d}\tau \left|\psi_a(s)\right\rangle \left\langle\psi_b(s)\right| \mathrm{e}^{-\beta H(s)} \\
&= -\beta \sum_{a,b} V_{a,b}(s) \left(1 + \mathrm{e}^{\beta \Delta E_{a,b}(s)}\right)^{-1} .
\end{aligned}$$

$$\int_0^1 \mathrm{e}^{\beta\tau\Delta E_{a,b}(s)} \mathrm{d}\tau \left\{ \mathrm{e}^{-\beta H(s)}, \left|\psi_a(s)\right\rangle \left\langle\psi_b(s)\right| \right\}$$

$$= -\beta \left\{ \mathrm{e}^{-\beta H(s)}, \Phi^{H(s)}(V) \right\}$$

Idea of the proof: QBP

Using Duhamel's formula ⇒

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{-\beta H(s)} &= -\beta \int_0^1 \mathrm{e}^{-\beta\tau H(s)} V \mathrm{e}^{-\beta(1-\tau)H(s)} \mathrm{d}\tau \\
&= -\beta \sum_{a,b} \int_0^1 \mathrm{e}^{-\beta\tau H(s)} V_{a,b}(s) \left|\psi_a(s)\right\rangle \left\langle\psi_b(s)\right| \mathrm{e}^{\beta\tau H(s)} \mathrm{d}\tau \mathrm{e}^{-\beta H(s)} \\
&= -\beta \sum_{a,b} V_{a,b}(s) \int_0^1 \mathrm{e}^{\beta\tau \Delta E_{a,b}(s)} \mathrm{d}\tau \left|\psi_a(s)\right\rangle \left\langle\psi_b(s)\right| \mathrm{e}^{-\beta H(s)} \\
&= -\beta \sum_{a,b} V_{a,b}(s) \left(1 + \mathrm{e}^{\beta \Delta E_{a,b}(s)}\right)^{-1} \cdot \\
&\qquad \int_0^1 \mathrm{e}^{\beta\tau \Delta E_{a,b}(s)} \mathrm{d}\tau \left\{\mathrm{e}^{-\beta H(s)}, \left|\psi_a(s)\right\rangle \left\langle\psi_b(s)\right|\right\} \\
&= -\frac{\beta}{2} \left\{\mathrm{e}^{-\beta H(s)}, \Phi_\beta^{H(s)}(V)\right\},
\end{aligned}
$$

$\Delta E_{a,b}(s) := E_b(s) - E_a(s)$

Idea of the proof: QBP

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{-\beta H(s)} = -\frac{\beta}{2}\left\{\mathrm{e}^{-\beta H(s)}, \Phi_\beta^{H(s)}(V)\right\}$$

$$\Phi_\beta^{H(s)}(V) := \sum_{a,b} \hat{f}_\beta\left(\Delta E_{a,b}(s)\right) V_{a,b}(s) \left|\psi_a(s)\right\rangle \left\langle\psi_b(s)\right|$$

$$= \int_{-\infty}^{\infty} f_\beta(t)\mathrm{e}^{-\mathrm{i}tH(s)} V \mathrm{e}^{\mathrm{i}tH(s)}\mathrm{d}t,$$

where

$$\hat{f}_\beta(\omega) := 2\left(1+\mathrm{e}^{\beta\omega}\right)^{-1}\int_0^1 \mathrm{e}^{\beta\tau\omega}\mathrm{d}\tau = \begin{cases} \frac{2}{\beta\omega}\frac{\mathrm{e}^{\beta\omega}-1}{\mathrm{e}^{\beta\omega}+1} & \omega \neq 0 \\ 1 & \omega = 0 \end{cases} = \frac{\tanh\frac{\beta\omega}{2}}{\frac{\beta\omega}{2}}$$

and $f_\beta(t)$ is its inverse Fourier transform

$$f_\beta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t}\hat{f}_\beta(\omega)\mathrm{d}\omega = \frac{2}{\beta\pi}\log\left(\frac{\mathrm{e}^{\pi|t|/\beta}+1}{\mathrm{e}^{\pi|t|/\beta}-1}\right).$$

Main results: locality and stability above T*

Finite range interaction:
$\Psi : X \Subset \mathbb{Z}^\nu \mapsto \mathcal{A}_{\mathbb{Z}^\nu}$ s.t. $\Psi(X) = 0$ if diam$(X) > R$.

Theorem [Kliesch et al. 2014]

For $\beta < \beta^*$ there exist positive constants C, a and b such that

$$\mathrm{Cov}_{\rho_\beta^\Lambda}(X;Y) \leq C\mathrm{e}^{a|Y|}\mathrm{e}^{-b\mathrm{d}(X,Y)}$$

Main results: locality and stability above T*

- (LPPL) There exists constants $C, c > 0$ such that for all $\Lambda \subset \mathbb{Z}^\nu, X \subset \Lambda, V \in \mathcal{A}_X, Y \subset \Lambda$, $B \in \mathcal{A}_Y$ and $\beta < \beta^*$

$$\left|\mathrm{Tr}\left(\rho_\beta^\Lambda(0)B\right) - \mathrm{Tr}\left(\rho_\beta^\Lambda(1)B\right)\right| \leq C\beta\|V\|\|B\|\left(|X| + \mathrm{e}^{a|Y|}\right)\mathrm{e}^{-cd(X,Y)}.$$

- (Local Indistinguishability) There exists constants $C, c > 0$ such that for all $Y \subset \Lambda' \subset \Lambda \subset \mathbb{Z}^\nu, B \in \mathcal{A}_Y$ and $\beta < \beta^*$

$$\left|\mathrm{Tr}\left(\rho_\beta^\Lambda B\right) - \mathrm{Tr}\left(\rho_\beta^{\Lambda'} B\right)\right| \leq C\|B\|\mathrm{e}^{a|Y|}\mathrm{e}^{-cd\left(Y,\Lambda\setminus\Lambda'\right)}.$$