Interacting many-particle systems in the random Kac–Luttinger model and proof of Bose–Einstein condensation

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Introduction I: BEC in Random Potentials

- Non-interacting Bose gas without random potential:
  BEC occurs only in dimension 3 and higher.
- Non-interacting Bose gas with certain random potentials:
  BEC possible also in dimension 1 and 2.
- However, more realistic to consider repulsive interaction between the particles.
- Does BEC still occur?

The model

- $N \in \mathbb{N}$ bosons "at zero temperature" in dimension 2 or higher
- confined in the box $\Lambda_N := (-L_N/2, L_N/2)^d \subset \mathbb{R}^d$, $d \geq 2$, $L_N = (\rho^{-1} N)^{1/d}$, $\rho > 0$
- $H_N = \sum_{i=1}^{N} (-\Delta_i + V(\omega, x_i)) + \sum_{1 \leq i < j \leq N} v_N(x_i - x_j)$ on $L^2_{\mathrm{sym}}(\Lambda_N^N)$
- hard Poissonian obstacles: $V(\omega, x) = \sum_j W(x - \hat{x}_j)$ where $W(x) = \infty \cdot \mathbb{1}_{\|x\| \leq r}$ and $\{\hat{x}_j\}_j$ distributed according to a PPP on $\mathbb{R}^d$ with arbitrary, fixed intensity
- interparticle interaction: $v_N \in (L^1 \cap L^\infty)(\mathbb{R}^d)$ nonnegative, even, positive-definite function s.t. $\widehat{v}_N \in L^1(\mathbb{R}^d)$ $\forall N \in \mathbb{N}$

Main Result

1. $\forall \epsilon > 0\ \exists \kappa > 0$: If $\|v_N\|_1 \leq \kappa N^{-1}(\ln N)^{-2/d}$ for all but finitely many $N \in \mathbb{N}$ and $v_N(0) \ll (\ln N)^{-(1+2/d)}$, then $\forall \zeta > 0$
$$\liminf_{N\to\infty} \mathbb{P}\left(\left|\frac{n_N^{1,\omega}}{N} - 1\right| < \zeta\right) \geq 1 - \epsilon$$
i.e., complete BEC with probability almost one
2. If $\|v_N\|_1 \ll N^{-1}(\ln N)^{-2/d}$ and $v_N(0) \ll (\ln N)^{-(1+2/d)}$, then $\forall \zeta > 0$
$$\lim_{N\to\infty} \mathbb{P}\left(\left|\frac{n_N^{1,\omega}}{N} - 1\right| < \zeta\right) = 1$$
i.e., there is complete BEC in probability.

Note: $n_N^{1,\omega} = N\,\mathrm{tr}(\rho^{(1)}|u_N^{\tilde{k},\omega}\rangle\langle u_N^{\tilde{k},\omega}|)$ where $u_N^{\tilde{k},\omega}$ is the minimizer of

$$\mathcal{E}_N^{k,\omega}[\psi] = \int_{\Lambda_N^{\tilde{k},\omega}} |\nabla\psi(x)|^2\,\mathrm{d}x + \frac{N-1}{2}\int_{\Lambda_N^{\tilde{k},\omega}}\int_{\Lambda_N^{\tilde{k},\omega}} v_N(x-y)|\psi(x)|^2|\psi(y)|^2\,\mathrm{d}x\mathrm{d}y$$

Main Result: Example

1. $\forall \epsilon > 0\ \exists \kappa > 0$: If $\|v_N\|_1 \leq \kappa N^{-1}(\ln N)^{-2/d}$ for all but finitely many $N \in \mathbb{N}$ and $v_N(0) \ll (\ln N)^{-(1+2/d)}$, then $\forall \zeta > 0$

$$\liminf_{N\to\infty} \mathbb{P}\left(\left| \frac{n_N^{1,\omega}}{N} - 1 \right| < \zeta \right) \geq 1 - \epsilon$$

   i.e., complete BEC with probability almost one

2. If $\|v_N\|_1 \ll N^{-1}(\ln N)^{-2/d}$ and $v_N(0) \ll (\ln N)^{-(1+2/d)}$, then $\forall \zeta > 0$

$$\lim_{N\to\infty} \mathbb{P}\left(\left| \frac{n_N^{1,\omega}}{N} - 1 \right| < \zeta \right) = 1$$

   i.e., there is complete BEC in probability.

For example,

$$v_N(x) = \frac{\kappa V(x)}{N(\ln N)^{2/d}} \ ,$$

which leads to a potential energy per particle comparable in size to the spectral gap of the Dirichlet Laplacian.

Brief remark regarding the proof

- A.-S. Sznitman, *On the spectral gap in the Kac–Luttinger model and Bose–Einstein condensation*, Stoch. Process. Their Appl. (2023)

$$\lim_{\sigma\to 0}\liminf_{N\to\infty}\mathbb{P}\left(e_N^{1,\omega}-e_N^{2,\omega}\geq\sigma(\ln N)^{-(1+2/d)}\right)=1$$

  regarding the gap between the two lowest eigenvalues of the Dirichlet Laplacian in a Poissonian cloud of hard spherical obstacles
- For any $\xi\in L^1(\mathbb{R}^d)$ we have

$$\sum_{1\leq i<j\leq N} v_N(x_i-x_j)\geq\sum_{j=1}^{N}(\xi * v_N)(x_j)$$
$$-\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v_N(x-y)\xi(x)\xi(y)\,\mathrm{d}x\mathrm{d}y-N\frac{v_N(0)}{2}\ ,$$

  see M. Lewin, *Mean-field limit of Bose systems: rigorous results*, Proceedings of the International Congress of Mathematical Physics, Santiago de Chile (2015).