Intrinsic geometry and the invariant trace field of hyperbolic 3–manifolds

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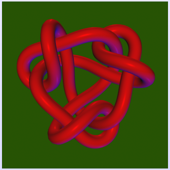
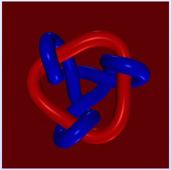
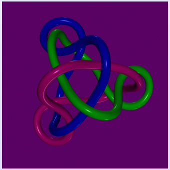
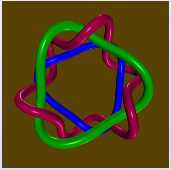
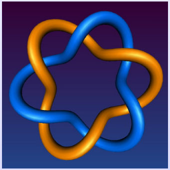
Joint with Walter Neumann, based on earlier joint work with Morwen Thistlethwaite

Cusped hyperbolic 3-manifolds

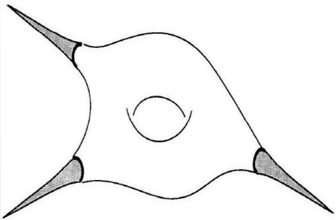
As a corollary of Thurston's Hyperbolization Theorem, many 3–manifolds have hyperbolic metric or can be decomposed into pieces with hyperbolic metric. In view of Mostow-Prasad rigidity, for a manifold with finite volume such metric is unique as long as it is complete. Additionally, Thurston showed that every link in S^3 is either a (generalized) **torus link**, a **satellite link** (*i.e.* contains it an incompressible, non-boundary parallel torus in its complement), or a **hyperbolic link**, and these three categories are mutually exclusive.

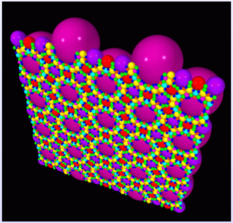
    

We may view a hyperbolic 3–manifold as the quotient of $\mathbb{H}^3$ by a discrete group of hyperbolic (fixed point free) isometries. This group of isometries is then isomorphic to the fundamental group of M. Let us restrict our attention to finite-volume cusped hyperbolic 3-manifolds in this talk. Example of such a manifold is a hyperbolic knot or link complement in the 3–sphere S^3.

Cusped hyperbolic 3-manifolds

Hyperbolic 3-manifolds are abundant. For example (Hoste-Thistlethwaite-Weeks):
Of the 14 knots up to 7 crossings, only 3 are non-hyperbolic.
Of the 1,701,935 knots up to 16 crossings, 32 are non-hyperbolic.
Of the 8,053,378 knots with 17 crossings, 30 are non-hyperbolic.

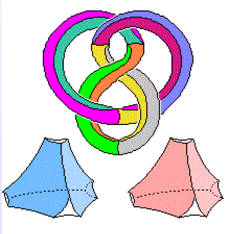


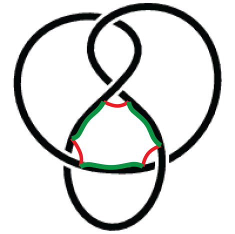


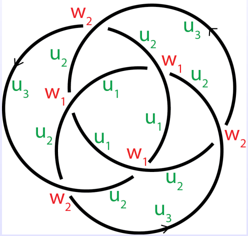
We will use the **upper half space model** for $\mathbb{H}^3$. The preimage of a cusp in $\mathbb{H}^3$ is a set of horoballs (on the right). There are horoballs of an arbitrarily small Euclidean diameter and one additional horoball, the plane $z = 1$. Cusps may be chosen so that the horoballs have disjoint interiors. Their points of tangency with the boundary of $\mathbb{H}^3$ form a dense subset of this boundary. The knot group acts transitively on the entire collection of horoballs, and for a link there is one orbit for every component.

Methods for computing hyperbolic structures

A well-known method for computing the structure of a hyperbolic 3-manifold by Thurston was implemented in the program SnapPea (Weeks). It is based on a decomposition of the manifold into ideal tetrahedra.







An alternative method (M. Thistlethwaite, A. T.) is based on ideal polygons bounding the regions of a diagram of the link. Given a link diagram, a complex number is associated to every crossing, and to each side of every edge. The numbers are defined so that they contain all the information about the horoball packing in $\mathbb{H}^3$. The polynomial system that allows one to find these numbers can be written by looking at the link diagram. We will utilize similar ideas to look at the structure of an arbitrary hyperbolic 3-manifold. First, we will do it in the context of connecting arithmetic invariants with the geometry, and then we will discuss the (more general) perspective this provides.

Invariant trace field and the geometry of a hyperbolic 3-manifold

For two hyperbolic 3-manifolds, commensurability is equivalent to the existence of a common finite-sheeted cover. One of the ideas motivating a study of hyperbolic 3-manifolds from the number-theoretical point of view is to classify hyperbolic 3-manifolds up to commensurability. To approach questions about commensurability, the invariant trace field is often used.

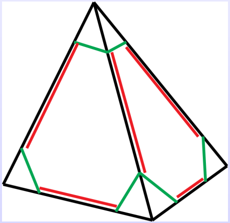
Consider an orientable cusped hyperbolic 3-manifold M with finite volume. Recall: M can be viewed as $\mathbb{H}^3/\Gamma$, where Γ is a discrete group of hyperbolic (fixed-point-free) isometries. One can think of elements of Γ as of 2×2 matrices that represent Möbius transformations (*i.e.* elements of $\mathrm{PSL}(2, \mathbb{C})$). Let $\Gamma^2 =< \gamma^2 | \gamma$ is in $\Gamma >$. The invariant trace field is a field generated by the traces of Γ^2 over $\mathbb{Q}$.

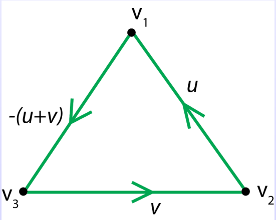
The field is a topological and commensurability invariant of a manifold (A. Reid). But how to efficiently compute it? And what is a relation of this field with the intrinsic geometry of a hyperbolic 3-manifold?

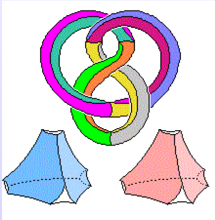
The first computations used the definition. A number of results that provided shortcuts were obtained (Coulson, Goodman, Hodgson, Neumann, Reid).

Invariant trace field and the geometry of a hyperbolic 3-manifold

Consider a Euclidean **cusp cross-section** of an ideal hyperbolic tetrahedron. Suppose that the (complex) translations corresponding to the sides of the cross-section are $u, v, -(u+v)$. Any of $-\frac{v}{u}$, $\frac{u}{u+v}$, $\frac{u+v}{v}$ can be taken as a shape of the tetrahedron (terahedron parameter). Once a manifold is decomposed into tetrahedra, the shapes together with the combinatorial picture of the tetrahedra describe the hyperbolic structure (Thurston's method, implemented in SnapPea).





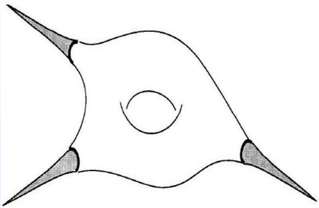


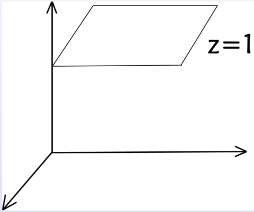
Theorem (Neumann, Reid). The invariant trace field equals the field generated by the shapes of the ideal tetrahedra of any ideal triangulation of the manifold.

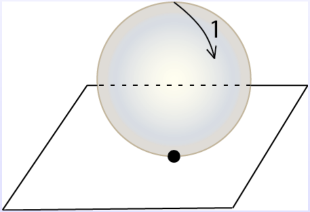
This is used in the program Snap (Coulson, Goodman, Hodgson, Neumann).

Horoball structures: preliminaries

We will see more: the invariant trace is closely connected with the geometric picture provided by the horoball structure, and can be generated by a few (often, just one) selected geometric parameters. But first we need some preliminaries.





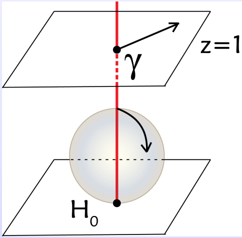


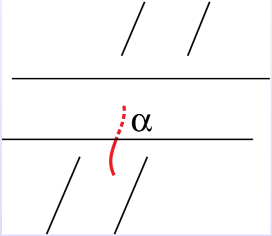
Take horospherical cross-sections of the cusps of $S^3 - L$, so that length of a meridian on each cross-section is 1. Choose the coordinates in $\mathbb{H}^3$ so that a component of the preimage of some particular cross-section is the Euclidean plane $z = 1$. Parameterize Euclidean translations on each horosphere by complex numbers so that the meridional translation corresponds to 1.

Our choices guarantee that the horoballs in $\mathbb{H}^3$ have disjoint interiors (C. Adams).

Intercusp arcs and associated parameters

For two horospheres H_0, H_1, define an *intercusp parameter* $w(H_0, H_1)$ that captures geometric information about the geodesic traveling between their centers. Let $|w| = e^{-d}$, where d is the hyperbolic intercusp distance along the geodesic, and let the argument of w be the dihedral angle between two half-planes, each determined by the geodesic and one of the meridional translations.

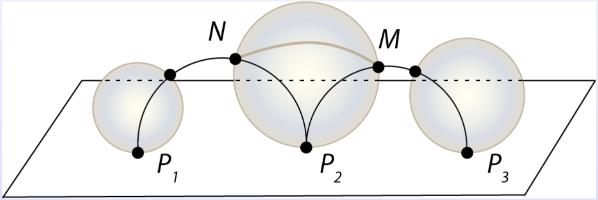


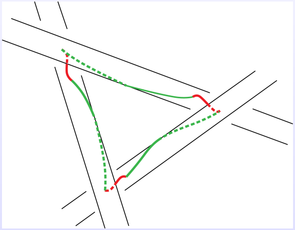


This parameter appeared before in the alternative method for computing hyperbolic structures of links (M. Thistlethwaite, A.T.). Such a label was assigned to every crossing in a link diagram (“crossing label"), and decoded information about the intercusp arc at a crossing. Independently, W. Neumann and A. Reid introduced $\log w$ as “the complex length" of the geodesic between two horospheres.

Translation parameters

For distinct horospheres H_1, H_2, H_3 with centers P_1, P_2, P_3 a *translation parameter* $u(H_1, H_2, H_3)$ is a complex number determining a translation on a horosphere mapping N to M.



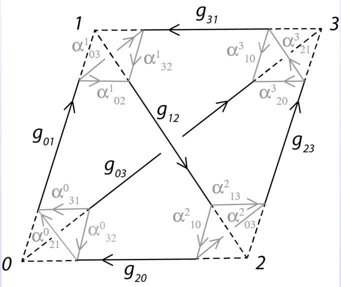


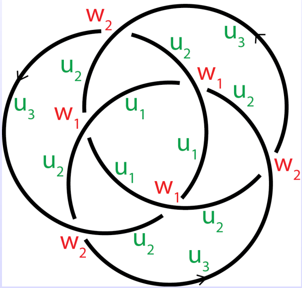
This was first introduced for parameterizing translations on the boundary torus of a link between intercusp crossing arcs, in the alternative method for computing hyperbolic structures of links. The complex number was called an edge label.

Theorem 1 (W. Neumann, A. T.). For a hyperbolic 3-manifold, all intercusp and translation parameters lie in the invariant trace field.

Invariant trace field and the geometry of a hyperbolic 3-manifold

Theorem 2 (W. Neumann, A. T.). Suppose X is a union of cusp arcs and intercusp arcs of M, and the preimage of X in $\mathbb{H}^3$ is connected (equivalently, $\pi_1(X) \to \pi_1(M)$ is surjective). Then the intercusp and translation parameters corresponding to these arcs generate the invariant trace field.



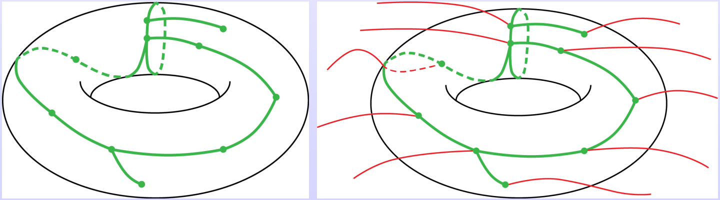


There are two "generic" ways to choose such X. One way is to construct a geometric triangulation of the fundamental region for the manifold, and take the labels that correspond to its long and short edges (as in C. Zickert). Another way, for links, is to take parameters for all edges and crossings (*i.e.* edge and crossing labels) of a link diagram that satisfies a few mild restrictions.

Minimal subcollection of parameters

More generally, a collection of intercusp arcs is a tunnel collection if the result of removing open horoball neighborhoods of the cusps and tubular neighborhoods of the intercusp arcs is a handlebody. The edge and crossing arcs constitute a tunnel collection for a link complement already, but usually a small subset of it is enough to generate the field.

Theorem 3 (W. Neumann, A.T.). If a cusped hyperbolic 3-manifold has a tunnel collection consisting of k intercusp arcs, then the invariant trace field can be generated by the k intercusp parameters of these arcs together with 2k translation parameters.



Cusp arcs are depicted in green (lying on a cusp torus), intercusp arcs are depicted in red.

Minimal subcollection of parameters

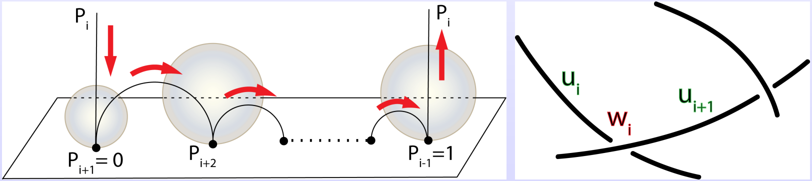
Empirically, for a random knot chosen from existing knot tables it is rare that the field is not generated by **a single one of the parameters**. It is unusual that the invariant trace field has a proper subfield of degree > 1.

Recall: all the parameters lie in the invariant trace field, as well as all tetrahedra shapes. Therefore, not only the field can be computed from just a few geometric parameters (very often, just one parameter!) describing the intercusp arcs, but also all the hyperbolic structure (the other parameters are just rational functions of the specified few parameters), and other invariants.

But how to compute complex values of the parameters that we discussed? Let us turn our attention to the alternative method for computing hyperbolic structures of links again.

How to compute the parameters

Consider a cyclic sequence of horospheres $H_1, \ldots, H_k$ with centers $P_1, \ldots, P_k$ in the preimage of the manifold in $\mathbb{H}^3$. It determines the ideal polygon with the vertices $P_1, \ldots, P_k$ in the boundary of $\mathbb{H}^3$. For links, one can choose polygons that correspond to regions of a link diagram, i.e. to a union of arcs on the boundary torus and arcs at crossings (as in alternative method for computing hyperbolic structure). To every geodesic in the polygon, assign the parameter $\xi_i = \frac{\pm w_i}{u_i u_{i+1}}$. Geometrically, ξ_i assigned to the geodesic $P_i P_{i+1}$, is the cross-ratio $\frac{(P_{i-1}-P_i)(P_{i+1}-P_{i+2})}{(P_{i-1}-P_{i+1})(P_i-P_{i+2})}$.



Perform an isometry to place the vertices P_{i-1}, P_i, P_{i+1} of the the polygon at $1, \infty, 0$ respectively. Then the the Möbius transformation $\rho_i : z \rightarrow \frac{-\xi_i}{z-1}$ determines an isometry of $\mathbb{H}^3$ which maps P_{i-1}, P_i, P_{i+1} to P_i, P_{i+1}, P_{i+2} respectively, rotating the truncated polygon forward one notch. Since every k-sided polygon closes up, the composite of k such isometries is identity.

An alternative method: region relations

If we represent the Möbius transformations by 2×2 matrices, we see that the product

$$\begin{pmatrix} 0 & -\xi_k \\ 1 & -1 \end{pmatrix} \dots \begin{pmatrix} 0 & -\xi_1 \\ 1 & -1 \end{pmatrix}$$

is a scalar multiple of the identity matrix. From the matrix entries we read off three independent polynomial relations for every region in the parameters.

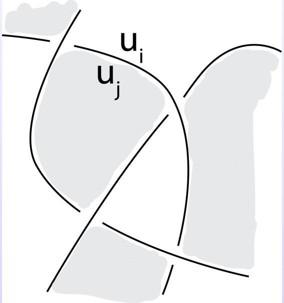
One can write a general combinatorial formula for the relations, that depends only on the number of sides in a region. E.g.,

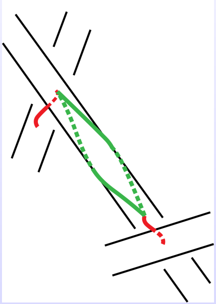
for a 3–sided region, $\xi_i = 1$;
for a 4-sided region, $\xi_i + \xi_{i+1} = 1$;
for a 5–sided region, $1 - \xi_i - \xi_{i+1} - \xi_{i+2} + \xi_i \xi_{i+2} = 0$, $1 \le i \le 3$.

The recursive formula can be written as well:
$f_1 \equiv f_2 \equiv 0$, $f_3 \equiv 1 - \zeta_2$, $f_4 \equiv 1 - \zeta_2 - \zeta_3$, $f_n \equiv f_{n-1} - \zeta_n f_{n-2}$ $(n \ge 5)$,
where f_k is the relation for a k-sided region.

An alternative method: additional relations

To obtain more relations, we may also look at translation parameters on the cusp torus that form a meridian. E.g., for a link, color the regions of the link diagram in black and white as a checkerboard. Each edge of a region gives rise to two arcs: on the boundary of the black region, and on the boundary of the white one.

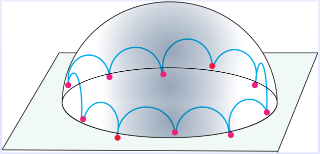


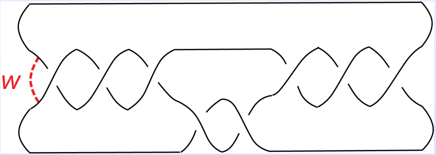


On the boundary torus, traveling along one arc and returning along another corresponds to a meridian, which is 1. In an alternating diagram, this is always the case, and $u_i - u_j = 1$ holds for every edge. In a non-alternating diagram this difference may be 1, -1 or 0.

Minimal subcollection of parameters

The system of equations in the parameters has several solutions. The solution with the greatest volume is the geometric one (S. Francaviglia). Empirically, for links the geometric solution is the one for which the ideal polygons that correspond to regions of the diagram are closest to being regular. In many cases (empirically, in all cases we have seen), the set of solutions is 0-dimensional, and it is easy to choose the geometric one.





This process can be simplified even further.

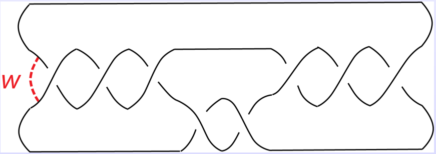
Example 1: 2-bridge links. Recall: if a cusped hyperbolic 3-manifold has a tunnel collection consisting of k intercusp arcs, then the invariant trace field can be generated by the k intercusp parameters of these arcs together with 2k translation parameters.

Since these are tunnel number one links, we need no more than three parameters to generate the field, to obtain hyperbolic structure and invariants. We will show that just one intercusp parameter associated with the tunnel arc is enough.

2-bridge links: reducing the system of equations to a polynomial

Proceeding region by region from left to right and using the alternative relations, we obtained the recursive formulas for all labels from w (the formulas can be found on ArXiv; one needs to assign labels to the link diagram, and then plug them in together with number of crossings and twists). The formulas also allow one to construct the polynomial from a reduced alternating diagram. The intercusp (crossing) label w is a root; other labels can be written as rational functions of it.

$$-1 + 7w - 18w^2 + 16w^3 + 9w^9 - 19w^5 - 4w^6 + 10w^7 + 4w^8 = 0.$$



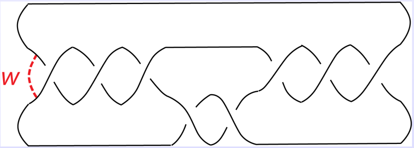
Closed formulas can often be obtained as well. E.g., for a twist knot with $k+2$ crossings, the polynomial is

$$\sum_{j=0}^{n} \binom{2n-j}{j} w^{2j} + \sum_{j=0}^{n-1} \binom{2n-j-1}{k} w^{2j+1} = 0, \text{ where } n = k/2.$$

(O. Dasbach, LSU VIGRE student group, A. T.)

Exact computations

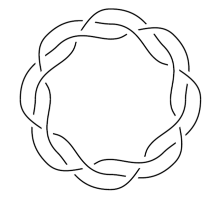
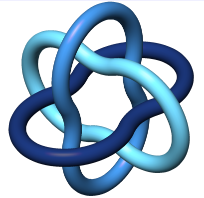
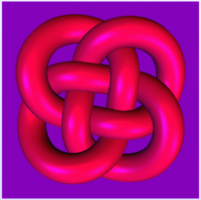
We first obtained these formulas in the context of computing the exact hyperbolic volume. After the polynomial is constructed, the volume can be computed from the link diagram as an analytic function of w. Note: Thurston's method for computing the volume, implemented in SnapPea by Weeks, produces a decimal approximation rather than the exact value. Sakuma and Weeks earlier reduced Thurston's gluing equations for hyperbolic 2-bridge links to a polynomial, but by working with triangulations and tetrahedra shapes rather than the diagram.



Now the above theorems guarantee that the invariant trace field can be computed exactly as well, as $\mathbb{Q}(w)$. Compare this with the methods of the program Snap by Goodman, Hodgson, Neumann. Snap approximates the hyperbolic structure to a high precision using Thurston's and Weeks' methods, and then makes an intelligent guess of the corresponding algebraic numbers using LLL algorithm (from which the invariant trace field can be computed).

As mentioned earlier, it is unusual that the invariant trace field has a proper subfield of degree > 1. Therefore the phenomenon of one parameter that determines hyperbolic structure and hyperbolic invariants is abundant.

Example 2: The closure of the braid $(\sigma_1\sigma_2^{-1})^n$, $n \geq 3$.
For $n = 3$, this is the Borromean rings; for $n = 4$ this is the Turk's head knot.

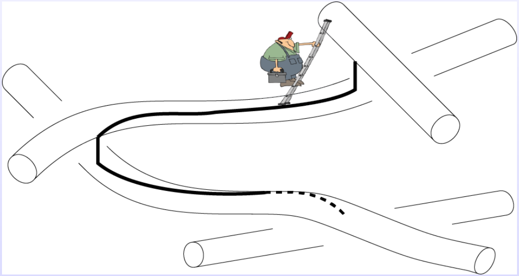




Due to the symmetry of the diagram, the central region corresponds to a regular ideal polygon, and the polynomial for one of the parameters can be easily constructed. The invariant trace field is then generated over $\mathbb{Q}$ by $\cos(\pi/n)$ and $\sqrt{-3 - 4\cos(\pi/n) + 4\cos^2(\pi/n)}$.

E.g., for the Borromean rings this is $\mathbb{Q}(i)$, and the intecusp parameter associated to a crossing has the argument $\pi/2$. Indeed, from the symmetries of the link it transpires that strands at crossings intersect at right angles.

The holonomy representation of the fundamental group

The parameters determine the holonomy representation of the fundamental group into $\mathrm{PSL}(2,\mathbb{C})$. To compute it for a link complement in S^3, associate to every edge label u the matrix $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ and to every crossing label w the matrix $\begin{pmatrix} 0 & -w \\ 1 & 0 \end{pmatrix}$.



Imagine walking from a basepoint to one of the crossings along a loop that is an element of the fundamental group. Walk on knotted tubes and the ladders joining underpasses with overpasses. By keeping track of the labels that we passed on the way, we obtain matrices for Wirtinger generators that are conjugates of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ by products of the above matrices.

Proof of the "invariant trace field and the intrinsic geometry" Thm.

Recall the statement of the theorem: Suppose X is a union of cusp arcs and intercusp arcs of M, and the preimage of X in $\mathbb{H}^3$ is connected (equivalently, $\pi_1(X) \to \pi_1(M)$ is surjective). Then the intercusp and translation parameters corresponding to these arcs generate the invariant trace field.

Proof. Denote the invariant trace field of the manifold M by $k(M)$.

1) Neumann, Reid: the shapes of tetrahedra in an ideal geometric triangulation of M lie in $k(M)$. Use hyperbolic geometry to find intercusp and translation parameters from the shapes, and deduce that the parameters lie in $k(M)$ as well.

2) If k is a field then the square of an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(k)$ equals $\frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 \in \mathrm{PSL}_2(k)$ and hence has PSL trace in k.

3) From the previous slide: each covering transformation h in Γ of the covering map $\mathbb{H}^3 \to M$ is a product of matrices with the intercusp and translation parameters as their elements. Therefore, h is in $\mathrm{PGL}_2(k(M))$, and its square has PSL trace in $k(M)$. But the traces of squares of elements of Γ generate $k(M)$.

Questions

A knot and a preimage of one of its intercusp arcs in $\mathbb{H}^3$.