

THE LOCAL GINZBURG-RALLIS MODEL FOR GENERIC REPRESENTATIONS

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ABSTRACT. We consider the local Ginzburg-Rallis model for generic representations. We prove that the summation of the multiplicities is always equal to 1 over every generic L-packet for the p-adic case and real case. This is a sequel work of [Wan15], [Wan16a] and [Wan16b] in which we considered the p-adic case and the real case for tempered representations, and considered the complex case for generic representations.

CONTENTS

1. Introduction and Main Result	2
2. Preliminary	6
2.1. Notations and conventions	6
2.2. The representations of GL_2 , and GL_4	7
2.3. The orbit method	8
2.4. The integrals over $B_2(F)\backslash GL_2(F)$ and $\bar{B}_2(F)\backslash GL_2(F)$	9
3. The relations between the Ginzburg-Rallis model and the middle model	10
3.1. The problem	11
3.2. The double cosets $P_{4,2}(F)\backslash G(F)/H(F)$	13
3.3. The orbits $P_{4,2}(F)w_iH(F)$ for $2 \leq i \leq 8$	14
3.4. The p-adic case	16
3.5. The archimedean case	18
3.6. The quaternion case	18
4. The relations between the middle model and the trilinear GL_2 model	19
4.1. The problem	19
4.2. The double cosets $P_{2,2}(F) \times GL_2(F)\backslash G_1(F)/H_1(F)$	20
4.3. The p-adic case	20
4.4. The archimedean case	21
4.5. The quaternion case	22
5. The proof of Theorem 1.2	22

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6.	The reduced models	24
6.1.	The reduced model for $\mathrm{GL}_5(F)$	24
6.2.	The reduced model for $\mathrm{GL}_4(F)$	26
6.3.	The reduced model for $\mathrm{GL}_3(F) \times \mathrm{GL}_3(F)$	26
6.4.	The reduced model for $\mathrm{GL}_3(F) \times \mathrm{GL}_2(F)$	28
7.	The proof of Theorem 1.6	30
7.1.	The problem	30
7.2.	The $\{5, 1\}$ case	30
7.3.	The $\{4, 1, 1\}$ case	35
7.4.	The $\{3, 3\}$ case	36
7.5.	The $\{3, 2, 1\}$ case	37
7.6.	The $\{3, 1, 1, 1\}$ case	38
Appendix A.	The middle model case	43
Appendix B.	Holomorphic continuation of the generalized Jacquet integrals	43
References		44

1. INTRODUCTION AND MAIN RESULT

This paper is a continuation of [Wan15], [Wan16a] and [Wan16b]. For an overview of the Ginzburg-Rallis model, see Section 1 of [Wan15]. We recall from there the definition of the Ginzburg-Rallis models and the conjectures.

Let F be a local field (p-adic or archimedean), D be the unique quaternion algebra over F if $F \neq \mathbb{C}$. Take $P = P_{2,2,2} = MU$ be the standard parabolic subgroup of $G = \mathrm{GL}_6$ whose Levi part M is isomorphic to $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$, and whose unipotent radical U consists of elements of the form

$$(1.1) \quad u = u(X, Y, Z) := \begin{pmatrix} I_2 & X & Z \\ 0 & I_2 & Y \\ 0 & 0 & I_2 \end{pmatrix}.$$

We define a character ξ on $U(F)$ by

$$(1.2) \quad \xi(u(X, Y, Z)) := \psi(\mathrm{tr}(X) + \mathrm{tr}(Y))$$

where ψ is a non-trivial additive character of F . It's clear that the stabilizer of ξ is the diagonal embedding of $\mathrm{GL}_2(F)$ into $M(F)$, which is denoted by $H_0(F)$. For a given character χ of F^\times , one induces a character ω of $H_0(F)$ given by $\omega(\mathrm{diag}(h, h, h)) := \chi(\det(h))$. Combining ξ and ω , we get a character $\omega \otimes \xi$ on $H(F) := H_0(F) \rtimes U(F)$. The pair (G, H) is the Ginzburg-Rallis model introduced by D. Ginzburg and S. Rallis in their paper [GR00]. Let π be an irreducible admissible representation of $G(F)$ with central character χ^2 , we are interested in the Hom space $\mathrm{Hom}_{H(F)}(\pi, \omega \otimes \xi)$, the dimension of which is denoted by $m(\pi)$ and is called the multiplicity.

On the other hand, if $F \neq \mathbb{C}$, define $G_D = \mathrm{GL}_3(D)$, similarly we can define U_D , $H_{0,D}$ and H_D . We also define the character $\omega_D \otimes \xi_D$ on $H_D(F)$

in the same way except that the trace in the definition of ξ is replaced by the reduced trace of the quaternion algebra D and the determinant in the definition of ω is replaced by the reduced norm of the quaternion algebra D . Then for an irreducible admissible representation π_D of $G_D(F)$ with central character χ^2 , we can also talk about the multiplicity $m(\pi_D) := \text{Hom}_{H_D(F)}(\pi_D, \omega_D \otimes \xi_D)$.

The purpose of this paper is to study the multiplicities $m(\pi)$ and $m(\pi_D)$. First, it was proved in [N06] and [JSZ11] that both multiplicities are less or equal to 1: $m(\pi), m(\pi_D) \leq 1$. In other words, the pairs (G, H) and (G_D, H_D) are Gelfand pairs. In this paper, we are interested in the relation between $m(\pi)$ and $m(\pi_D)$ under the local Jacquet-Langlands correspondence established in [DKV84]. The local conjecture has been expected since the work of [GR00], and was first discussed in details by Jiang in his paper [J08].

Conjecture 1.1 (Jiang, [J08]). *For any irreducible generic representation π of $GL_6(F)$ with central character χ^2 , let π_D be the local Jacquet-Langlands correspondence of π to $GL_3(D)$ if it exists, and zero otherwise. In particular, π_D is always 0 if $F = \mathbb{C}$. Then we have*

$$(1.3) \quad m(\pi) + m(\pi_D) = 1.$$

The assertion in Conjecture 1.1 can be formulated in terms of Vogan L-packets. For details, see Section 1 of the previous paper [Wan15].

In the previous papers [Wan15] and [Wan16a], we proved Conjecture 1.1 for the case when F is a p -adic local field or \mathbb{R} , and π is an irreducible tempered representation of $GL_6(F)$. In [Wan16b], we proved Conjecture 1.1 for a majority of the generic representations when $F = \mathbb{C}$, and we also have some partial results for the generic representations when $F = \mathbb{R}$. In this paper, we considered the case when F is a p -adic local field or \mathbb{R} , and π is a generic representation. For the rest part of this paper, F will be either a p -adic local field or \mathbb{R} . Our first result is the following theorem.

Theorem 1.2. *With the notations above, let π be an irreducible generic representation of $G(F)$ with central character χ^2 . Assume that π_D exists. Then Conjecture 1.1 holds (i.e. $m(\pi) + m(\pi_D) = 1$).*

Remark 1.3. (1) *The theorem above implies that for all irreducible generic representations π of $G(F)$ with central character χ^2 , we have $m(\pi) + m(\pi_D) \leq 1$.*

(2) *By the theory of the local Jacquet-Langlands correspondence, we know that π_D exists if and only if π is an essentially discrete series, or the parabolic induction of some essential discrete series of $GL_4(F) \times GL_2(F)$ or $GL_2(F) \times GL_2(F) \times GL_2(F)$. Here we say a representation of $GL_n(F)$ is an essential discrete series if it can be written as a discrete series of $GL_n(F)$ twist by some (not necessarily unitary) character of F^\times .*

The idea for proving Theorem 1.2 is to show that the multiplicity $m(\pi)$ is invariant under the parabolic induction by applying the orbit method. To

be specific, if π is the parabolic induction of some essential discrete series τ of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$ or $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$, we let $m(\tau)$ be the multiplicity of the reduced model. Here if τ is an essential discrete series of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$, the reduced model is the trilinear GL_2 model; if τ is an essential discrete series of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$, the reduced model is the middle model defined in Appendix B.4 of [Wan15], which can be viewed as the model between the Ginzburg-Rallis model and the trilinear GL_2 model (we will recall the definition of these reduced models in later sections). Then we will show that $m(\pi) = m(\tau)$. Similarly, we can also show that the multiplicity $m(\pi_D)$ is invariant under the parabolic induction. Then since the multiplicities for the Ginzburg-Rallis model, the middle model, and the trilinear GL_2 model are invariant under the unramified twist, we can reduce to the case when π is tempered which has already been proved in the previous papers [Wan15] and [Wan16a].

Then we will consider the case when $\pi_D = 0$. In this case, Conjecture 1.1 is equivalent to say that $m(\pi) = 1$. When $F = \mathbb{R}$, this has been proved in the previous paper [Wan16b] under the assumption that certain generalized Jacquet integrals have holomorphic continuation.

Theorem 1.4 (Theorem 1.5 of [Wan16b]). *$F = \mathbb{R}$, assume that the generalized Jacquet integrals in Section 7 of [Wan16b] have holomorphic continuation. Let π be an irreducible generic representation of $\mathrm{GL}_6(F)$ with central character χ^2 . Assume that $\pi_D = 0$. Then $m(\pi) = 1$.*

Remark 1.5. *In Appendix B, we will briefly recall the assumption on the generalized Jacquet integrals. As we discussed in the previous paper [Wan16b], this assumption allows us to apply the open orbit method to show that if π is induced from a generic representation (not necessarily essential discrete series) τ of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$ or $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$ with the multiplicity $m(\tau)$ of the reduced model being nonzero, then we have $m(\pi) \neq 0$. In general, if we don't make this assumption, then we can only prove $m(\tau) \neq 0 \Rightarrow m(\pi) \neq 0$ when the exponent of π satisfies certain conditions. And this will only allow us to prove Theorem 1.4 when the exponent of π satisfies these conditions. We refer the readers to [Wan16b] for more details.*

The second main result of this paper is to extend Theorem 1.4 to the p -adic case.

Theorem 1.6. *Let F be a p -adic local field. We still assume that the generalized Jacquet integrals have holomorphic continuation. Let π be an irreducible generic representation of $\mathrm{GL}_6(F)$ with central character χ^2 . Assume that $\pi_D = 0$. Then $m(\pi) = 1$.*

It is worth to mention that the proof of Theorem 1.6 is much more complicated than the real case in [Wan16b]. To be specific, if $F = \mathbb{R}$, since only $\mathrm{GL}_1(F)$ and $\mathrm{GL}_2(F)$ have discrete series, we can always find a generic representation τ of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$ such that π is the parabolic

induction of τ . Then by applying the open orbit method, we only need to show that the multiplicity $m(\tau)$ for the trilinear GL_2 model is nonzero, which follows from the result in [P90]. Now when F is a p -adic field, the group $\mathrm{GL}_n(F)$ has discrete series (and even supercuspidal representations) for all n . As a result, we can not always reduce to the trilinear GL_2 model case. Instead, we need to consider all the reduced models of the Ginzburg-Rallis model. In Section 6, by applying the Rankin-Selberg integrals of $\mathrm{GL}_3 \times \mathrm{GL}_3$ and $\mathrm{GL}_3 \times \mathrm{GL}_2$ in [JPSS], and the Rankin-Selberg integrals for the exterior square L-functions of GL_4 and GL_5 in [K11], we will show that all these reduced models have nonzero multiplicity. Another difficulty for the p -adic case is that for all the reduced models other than the middle model and the trilinear GL_2 model, when we applying the open orbit method, we are not just integrating certain unipotent subgroups. Hence we need to make sense of these integrals and show that they are nonzero before we apply the open orbit method. For details, see Section 7.

Finally, by combining Theorem 1.2, 1.4 and 1.6, we have proved Conjecture 1.1 for all generic representations.

Theorem 1.7. *F is p -adic or \mathbb{R} . Assume that the generalized Jacquet integrals have holomorphic continuation. Then Conjecture 1.1 holds.*

Remark 1.8. *By the same arguments as in [Wan16a] and [Wan16b], we can also prove the epsilon dichotomy conjecture for the generic representations. To be specific, when $F = \mathbb{R}$, for all irreducible generic representations π of $\mathrm{GL}_6(F)$ with central character χ^2 , we have*

$$(1.4) \quad m(\pi) = 1 \iff \epsilon\left(\frac{1}{2}, \pi, \wedge^3 \otimes \chi^{-1}\right) = 1, \quad m(\pi) = 0 \iff \epsilon\left(\frac{1}{2}, \pi, \wedge^3 \otimes \chi^{-1}\right) = -1.$$

Here $\epsilon(s, \pi, \wedge^3 \otimes \chi^{-1})$ is the epsilon factor of $(\wedge^3 \phi_\pi) \otimes \chi^{-1}$ (NOT $\wedge^3(\phi_\pi \otimes \chi^{-1})$) where ϕ_π is the Langlands parameter of π . When F is a p -adic field, we can prove (1.4) when π is not an essential discrete series or the parabolic induction of an essential discrete series of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$. Since the proof is the same as the cases in the previous papers, we will skip it here.

Remark 1.9. *The arguments in this paper can also be applied to the middle model case. For details, see Appendix A.*

The paper is organized as following: In Section 2, we introduce basic notations and conventions of this paper. We will also give a brief overview of the orbit method. From Section 3 to 5, we deal with the case when $\pi_D \neq 0$. In Section 3, we study the relations between the Ginzburg-Rallis model and the middle model. Then in Section 4, we study the relations between the middle model and the trilinear GL_2 model. Finally in Section 5, we will prove Theorem 1.2. Start from Section 6, we focus on the case when F is p -adic and $\pi_D = 0$. In Section 6, we study several reduced models of the Ginzburg-Rallis model, the goal is to show that the the multiplicities for those reduced models are always nonzero. Finally in Section 7, we will

prove Theorem 1.6 by applying the open orbit method together with the results in Section 6.

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2. PRELIMINARY

2.1. Notations and conventions. Throughout this paper, F will be either a p-adic local field or \mathbb{R} . When F is a p-adic field, let \mathcal{O}_F be the ring of integers of F , and we fix an uniformizer ϖ_F .

We will use the permutation group S_n to label the Weyl group W_n of GL_n . For a partition $s = (i_1 i_2 \cdots i_n) \in S_n$, we define the correspondent Weyl element w_s to be the matrix with entries 1 in the (i_k, k) positions and

0 elsewhere. For example, if $s = (231)$, then $w_s = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Let $|\cdot| = |\cdot|_F$ be the absolute value on F . We define the character ν on F^\times to be $\nu(x) = |x|$. Then any character α of F^\times is of the form $\alpha_0 \nu^s$ where α_0 is some unitary character of F^\times and $s \in \mathbb{R}$. We say α is nonnegative (resp. not positive) if $s \geq 0$ (resp. $s \leq 0$). Moreover, if $F = \mathbb{R}$, then any character α of F^\times is of the form $\alpha_0 \nu^s$ with $s \in \mathbb{C}$ and α_0 is either the trivial character or the sign character sgn .

Since the multiplicity for the Ginzburg-Rallis model is invariant under the unramified twisted, up to twist π by some unramified characters, we will assume that χ is a unitary character. When $F = \mathbb{R}$, we can even assume that $\chi = 1$ or sgn .

Let $P = MN$ be a parabolic subgroup of G . Let π be a representation of $G(F)$ and let τ be the representation of $M(F)$. We will use $I_P^G(\tau)$ to denote the normalized parabolic induction. In other words, $I_P^G(\tau) = \mathrm{Ind}_P^G(\delta_P^{1/2} \tau)$ where $\mathrm{Ind}_P^G(\cdot)$ is the usual induced representation and δ_P is the modular character. We will also use $J_N(\pi)$ to denote the Jacquet module of π with respect to the unipotent subgroup N .

For $n > 0$, let $B_n = T_n N_n$ (resp. $\bar{B}_n = T_n \bar{N}_n$) be the upper triangular (resp. lower triangular) Borel subgroup of GL_n . When $n = n_1 + n_2$, we will use $P_{n_1, n_2} = M_{n_1, n_2} N_{n_1, n_2}$ (resp. $\bar{P}_{n_1, n_2} = M_{n_1, n_2} \bar{N}_{n_1, n_2}$) to denote the parabolic subgroup of GL_n of type (n_1, n_2) such that $B_n \subset P_{n_1, n_2}$ (resp. $\bar{B}_n \subset \bar{P}_{n_1, n_2}$). For example, $P_{2,2}(F)$ will be the parabolic subgroup $\left\{ \begin{pmatrix} A & X \\ 0 & B \end{pmatrix} \mid A, B \in \mathrm{GL}_2(F), X \in \mathrm{Mat}_{2 \times 2}(F) \right\}$ of $\mathrm{GL}_4(F)$. When F is p-adic, let $K_n = \mathrm{GL}_n(\mathcal{O}_F)$ be the maximal open compact subgroup of $\mathrm{GL}_n(F)$.

Finally, if π is a representation of $\mathrm{GL}_n(F)$ and $s \in \mathbb{C}$, we use $\pi\nu^s$ to denote the representation of $\mathrm{GL}_n(F)$ given by $g \mapsto \nu^s(\det(g))\pi(g)$.

2.2. The representations of GL_2 , and GL_4 . In this subsection, we discuss some basic representation theory of $\mathrm{GL}_2(F)$ and $\mathrm{GL}_4(F)$. This will be used in Section 3 and 4. Our main reference is [Z80]. We start with $\mathrm{GL}_2(F)$. We want to study the essential discrete series of $\mathrm{GL}_2(F)$ and its Jacquet module.

When F is p -adic, we know that an essential discrete series π of $\mathrm{GL}_2(F)$ is either a supercuspidal representation or the Steinburg representation $St_2(\eta)$ where $St_2(\eta)$ is the unique subrepresentation of $I_{B_2}^{\mathrm{GL}_2}(\eta\nu^{1/2} \otimes \eta\nu^{-1/2})$ and η is some character of F^\times . If π is a supercuspidal representation, the Jacquet module $J_{N_2}(\pi)$ is zero. If π is the Steinburg representation $St_2(\eta)$, then $J_{N_2}(\pi) = \eta\nu \otimes \eta\nu^{-1}$.

When $F = \mathbb{R}$, the essential discrete series π of $\mathrm{GL}_2(F)$ is of the form $St(\eta, \alpha, n)$ where η is some character of F^\times , α is either the trivial character or the sign character, n is a positive integer such that n is odd when $\alpha = 1$ and n is even when $\alpha = \mathrm{sgn}$ (i.e. $\alpha = (\mathrm{sgn})^{n+1}$). Then $St(\eta, \alpha, n)$ is the unique subrepresentation of $I_{B_2}^{\mathrm{GL}_2}(\eta\alpha\nu^{n/2} \otimes \eta\nu^{-n/2})$. In this case, the Jacquet module $J_{N_2}(\pi)$ is equal to $\eta\alpha\nu^{\frac{n+1}{2}} \otimes \eta\nu^{-\frac{n+1}{2}}$.

Then we study the representations of $\mathrm{GL}_4(F)$. If F is p -adic, we know that an essential discrete series π of $\mathrm{GL}_4(F)$ is either a supercuspidal representation, or the Steinburg representation $St_4(\tau)$ where τ is a character of F^\times or a supercuspidal representation of $\mathrm{GL}_2(F)$. If τ is a character of F^\times , $St_4(\tau)$ is defined to be the unique subrepresentation of $I_{B_4}^{\mathrm{GL}_4}(\tau\nu^{3/2} \otimes \tau\nu^{1/2} \otimes \tau\nu^{-1/2} \otimes \tau\nu^{-3/2})$. If τ is a supercuspidal representation of $\mathrm{GL}_2(F)$, then $St_4(\tau)$ is defined to be the unique subrepresentation of $I_{P_{2,2}}^{\mathrm{GL}_4}(\tau\nu^{1/2} \otimes \tau\nu^{-1/2})$.

We are interested in the Jacquet module $J_{N_{3,1}}(\pi)$, which is a representations of $M_{3,1}(F) = \mathrm{GL}_3(F) \times \mathrm{GL}_1(F)$. If π is a supercuspidal representation or the Steinburg representation $St_4(\tau)$ with τ being a supercuspidal representation of $\mathrm{GL}_2(F)$, then $J_{N_{3,1}}(\pi) = 0$. If $\pi = St_4(\tau)$ with τ be a character of F^\times , then $J_{N_{3,1}}(\pi) = St_3(\tau\nu) \otimes \tau\nu^{-3}$ where $St_3(\tau\nu)$ is an essential discrete series of $\mathrm{GL}_3(F)$ which is defined to be the unique subrepresentation of $I_{B_3}^{\mathrm{GL}_3}(\tau\nu^2 \otimes \tau\nu^1 \otimes \tau)$. Finally, if $\pi = I_{P_{2,2}}^{\mathrm{GL}_4}(\tau_1 \otimes \tau_2)$ is the parabolic induction of some essential discrete series of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$, there are four cases. If τ_1 and τ_2 are supercuspidal representations, then $J_{N_{3,1}}(\pi) = 0$. If τ_1 is supercuspidal and $\tau_2 = St_2(\eta_2)$ is a Steinberg representation, then $J_{N_{3,1}}(\pi) = I_{P_{2,1}}^{\mathrm{GL}_3}(\tau_1\nu^{1/2} \otimes \eta_2\nu) \otimes \eta_2\nu^{-2}$. If τ_2 is supercuspidal and $\tau_1 = St_2(\eta_1)$ is a Steinberg representation, then $J_{N_{3,1}}(\pi) = I_{P_{1,2}}^{\mathrm{GL}_3}(\eta_1\nu \otimes \tau_2\nu^{1/2}) \otimes \eta_1\nu^{-2}$. If $\tau_1 \otimes \tau_2 = St_2(\eta_1) \otimes St_2(\eta_2)$ is a Steinberg representation, then $J_{N_{3,1}}(\pi) = (I_{P_{2,1}}^{\mathrm{GL}_3}(St_2(\eta_1\nu^{1/2}) \otimes \eta_2\nu) \otimes \eta_2\nu^{-2}) \oplus (I_{P_{1,2}}^{\mathrm{GL}_3}(\eta_1\nu \otimes St_2(\eta_2\nu^{1/2})) \otimes \eta_1\nu^{-2})$.

If $F = \mathbb{R}$, there is no essential discrete series of $\mathrm{GL}_4(F)$. Let $\pi = I_{P_{2,2}}^{\mathrm{GL}_4}(\pi_1 \otimes \pi_2)$ with $\pi_i = \mathrm{St}(\eta_i, \alpha_i, n_i)$ are some Steinburg representations of $\mathrm{GL}_2(F)$. Then we know that $J_{N_{3,1}}(\pi) = (I_{P_{2,1}}^{\mathrm{GL}_3}(\mathrm{St}(\eta_1 \nu^{1/2}, \alpha_1, n_1) \otimes \eta_2 \alpha_2 \nu^{\frac{n_2+1}{2}}) \otimes \eta_2 \nu^{-\frac{n_2+3}{2}}) \oplus (I_{P_{1,2}}^{\mathrm{GL}_3}(\eta_1 \alpha_1 \nu^{\frac{n_1+1}{2}} \otimes \mathrm{St}(\eta_2 \nu^{1/2}, \alpha_2, n_2)) \otimes \eta_1 \nu^{-\frac{n_1+3}{2}})$.

2.3. The orbit method. In this subsection, we give a brief overview of the orbit method. The purpose of this method is to study the distinction of induced representations, it is an application of the geometric lemma due to Bernstein and Zelevinsky. Let G be a connected reductive group defined over F , and $H \subset G$ be a closed subgroup such that $X = H \backslash G$ is a spherical variety of G (i.e. the Borel subgroup has an open orbit). For simplicity, we assume that H is unimodular. Let $P = MU$ be a parabolic subgroup of G and (τ, V_τ) be an irreducible admissible representation of $M(F)$. We want to study the Hom space $\mathrm{Hom}_{H(F)}(I_P^G(\tau), \chi)$ where χ is some character of $H(F)$. We say that $(\pi, V_\pi) = (I_P^G(\tau), I_P^G(V_\tau))$ is (H, χ) -distinguished (or just H-distinguished if χ is trivial) if the Hom space is nonzero. For simplicity, we assume that χ is trivial.

The geometric lemma (Bernstein-Zelevinsky, [BZ77]) There is an ordering $\{P(F)y_iH(F)\}_{i=1}^N$ on the double cosets $P(F) \backslash G(F) / H(F)$ such that

$$Y_i = \cup_{j=1}^i P(F)y_jH(F)$$

is an open subset of $G(F)$ for all $1 \leq i \leq N$.

With the filtration above, for $1 \leq i \leq N$, define

$$V_i = \{f \in I_P^G(V_\tau) \mid \mathrm{supp}(f) \subset Y_i\}.$$

Then we have $V_1 \subset V_2 \subset \dots \subset V_N = V_\pi$ and V_i is $H(F)$ -invariant for all i . In particular, this implies that if $I_P^G(\tau)$ is H-distinguished, there exists i such that $\mathrm{Hom}_{H(F)}(V_i/V_{i-1}, 1) \neq 0$ (here $V_0 = \{0\}$). Moreover, for any $1 \leq i \leq N$, it is easy to see that the map

$$f \in V_i \mapsto \phi_f(h) := f(y_i h)$$

is an isomorphism between V_i/V_{i-1} and $\mathrm{ind}_{H_i}^H(\delta_P^{1/2} \tau^{y_i}|_{H_i})$ ($\mathrm{ind}_{H_i}^H$ is the compact induction). Here $H_i = H(F) \cap y_i^{-1} P(F) y_i = y_i^{-1} P_i(F) y_i$ with $P_i(F) = P(F) \cap y_i H(F) y_i^{-1}$. By applying the reciprocity law, we have a necessary condition for $I_P^G(\tau)$ to be H-distinguished.

Proposition 2.1. *If $I_P^G(\tau)$ is H-distinguished, there exists i such that τ is $(P_i, \delta_{P_i} \delta_P^{-1/2})$ -distinguished. Here we view τ as a representation of $P(F)$ by making it trivial on $U(F)$.*

In many cases including the Ginzburg-Rallis model case in this paper, one wants to show that $I_P^G(\tau)$ is H-distinguished if and only if the open orbit is distinguished (i.e. τ is $(P_1, \delta_{P_1} \delta_P^{-1/2})$ -distinguished). In order to show this, one needs to prove two statements:

- (1) If τ is $(P_1, \delta_{P_1} \delta_P^{-1/2})$ -distinguished, then $I_P^G(\tau)$ is H-distinguished.
- (2) For all $i > 1$, τ is not $(P_i, \delta_{P_i} \delta_P^{-1/2})$ -distinguished.

The first statement is usually proved by the open orbit method. For the Ginzburg-Rallis model case, when $\pi_D \neq 0$, this has been proved in the previous paper [Wan16b]. Note that although the previous paper is for the case when F is archimedean, but the same argument can also be applied to the p-adic case. Hence in this paper, when $\pi_D \neq 0$, we only need to focus on the second statement (Section 3-4). In other words, we need to show that all the non-open orbits are not distinguished. On the other hand, when $\pi_D = 0$, we first need to prove that the open orbit is always distinguished (Section 6), and then prove the first statement by the open orbit method (Section 7).

2.4. The integrals over $B_2(F) \backslash \mathrm{GL}_2(F)$ and $\bar{B}_2(F) \backslash \mathrm{GL}_2(F)$. Let F be a p-adic field and α be a character of F^\times . The goal of this section is to prove the following proposition, which will allow us to apply the open orbit method in Section 7 for the case when $\pi_D = 0$.

Proposition 2.2. *$F : \mathrm{GL}_2(F) \rightarrow \mathbb{C}$ be a locally constant function with*

$$F\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \left|\frac{a}{b}\right| \alpha(ab) F(g), \quad \forall g \in \mathrm{GL}_2(F), \quad a, b \in F^\times, x \in F.$$

Then for all $g \in \mathrm{GL}_2(F)$, we have

$$(2.1) \quad \int_{K_2} F(kg) \alpha^{-1}(\det(k)) dk = \alpha(\det(g)) \int_{K_2} F(k) \alpha^{-1}(\det(k)) dk.$$

Before we prove the proposition, we first prove a lemma.

Lemma 2.3. *Let $\varphi : \mathrm{GL}_2(F) \rightarrow \mathbb{C}$ be a locally constant compactly supported function, and let*

$$\varphi_0(g) := \int_{F^\times} \int_{F^\times} \int_F \varphi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \alpha^{-1}(ab) dx dadb.$$

Then φ_0 satisfies

$$(2.2) \quad \varphi_0\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \left|\frac{a}{b}\right| \alpha(ab) \varphi_0(g), \quad \forall g \in \mathrm{GL}_2(F), \quad a, b \in F^\times, x \in F.$$

Moreover, every locally constant function $F : \mathrm{GL}_2(F) \rightarrow \mathbb{C}$ which satisfies (2.2) is equal to φ_0 for some φ .

Proof. The proof of (2.2) is trivial. For the second part, given $F : \mathrm{GL}_2(F) \rightarrow \mathbb{C}$ satisfies (2.2), define a locally constant compactly supported function $\varphi : \mathrm{GL}_2(F) \rightarrow \mathbb{C}$ to be

$$\varphi(g) = \begin{cases} F(g), & \text{if } g \in K_2; \\ 0, & \text{else.} \end{cases}$$

Then for all $k \in K_2$, we have

$$\begin{aligned}
\varphi_0(k) &= \int_{F^\times} \int_{F^\times} \int_F \varphi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k\right) \alpha^{-1}(ab) dx dadb \\
&= \int_{\mathcal{O}_F^\times} \int_{\mathcal{O}_F^\times} \int_{\mathcal{O}_F} \varphi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k\right) \alpha^{-1}(ab) dx dadb \\
&= \int_{\mathcal{O}_F^\times} \int_{\mathcal{O}_F^\times} \int_{\mathcal{O}_F} \varphi(k) dx dadb = \varphi(k) = F(k).
\end{aligned}$$

Hence $\varphi_0|_{K_2} = F|_{K_2}$. Since both φ_0 and F satisfies (2.2), by the Iwasawa decomposition, we have $\varphi_0 = F$. This proves the lemma. \square

Now we are ready to prove Proposition 2.2. Given the function F as in the proposition, by the lemma above, we may assume that $F = \varphi_0$ for some φ . Then for all $g \in \mathrm{GL}_2(F)$, we have

$$\begin{aligned}
&\int_{K_2} F(kg) \alpha^{-1}(\det(k)) dk \\
&= \int_{K_2} \int_{F^\times} \int_{F^\times} \int_F \varphi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} kg\right) \alpha^{-1}(ab) \alpha^{-1}(\det(k)) dx dadb \\
&= \int_{\mathrm{GL}_2(F)} \varphi(hg) \alpha^{-1}(\det(h)) dh = \alpha(\det(g)) \int_{\mathrm{GL}_2(F)} \varphi(h) \alpha^{-1}(\det(h)) dh \\
&= \alpha(\det(g)) \int_{K_2} \int_{F^\times} \int_{F^\times} \int_F \varphi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k\right) \alpha^{-1}(ab) \alpha^{-1}(\det(k)) dx dadb \\
&= \alpha(\det(g)) \int_{K_2} F(k) \alpha^{-1}(\det(k)) dk.
\end{aligned}$$

This finishes the proof of Proposition 2.2. Similarly, we can also prove the following proposition.

Proposition 2.4. *Let $F : \mathrm{GL}_2(F) \rightarrow \mathbb{C}$ be a locally constant function such that*

$$F\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} g\right) = \left|\frac{b}{a}\right| \alpha(ab) F(g), \quad \forall g \in \mathrm{GL}_2(F), \quad a, b \in F^\times, x \in F.$$

Then for all $g \in \mathrm{GL}_2(F)$, we have

$$\int_{K_2} F(kg) \alpha^{-1}(\det(k)) dk = \alpha(\det(g)) \int_{K_2} F(k) \alpha^{-1}(\det(k)) dk.$$

3. THE RELATIONS BETWEEN THE GINZBURG-RALLIS MODEL AND THE MIDDLE MODEL

In this section, we study the relations between the Ginzburg-Rallis model and the middle model. The goal is to show that if some induced representation is distinguished by the Ginzburg-Rallis model, then the representation of the Levi subgroup is distinguished by the middle model. In Section 3.1,

we recall the definition of the middle model and state the problem. In Section 3.2, we study the double cosets $P_{4,2}(F)\backslash G(F)/H(F)$ where $P_{4,2}(F)$ is the standard parabolic subgroup of $\mathrm{GL}_6(F)$ of type $(4, 2)$. There are nine elements in the double cosets, and the model associated to the open orbit is the middle model. Hence by the orbit method, we would need to show that the rest 8 nonopen orbits are not distinguished. In Section 3.3, we will study 7 nonopen orbits. The last one will be studied in Section 3.4 for the p -adic case, and in Section 3.5 for the archimedean case. Finally in Section 3.6, we will consider the quaternion case.

3.1. The problem. We first recall the definition of the middle model. Let $G_1(F) = \mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$, and let $H_1(F) = H_0(F) \times U_1(F)$ be a subgroup of $G_1(F)$ with

$$H_0(F) = \left\{ \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \times h \mid h \in \mathrm{GL}_2(F) \right\}, \quad U_1(F) = \left\{ \begin{pmatrix} I_2 & X \\ 0 & I_2 \end{pmatrix} \times I_2 \mid X \in \mathrm{Mat}_{2 \times 2}(F) \right\}.$$

As in the Ginzburg-Rallis model case, we can define a character $\omega \otimes \xi_1$ of $H_1(F)$ by mapping $\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} I_2 & X \\ 0 & I_2 \end{pmatrix} \times h$ to $\psi(\mathrm{tr}(X))\chi(\det(h))$. The model (G_1, H_1) is called the middle model which can be viewed as the model between the Ginzburg-Rallis model and the trilinear GL_2 model. Let $\tau = \tau_1 \otimes \tau_2$ be an irreducible generic representation of $G_1(F)$ whose central character equals χ^2 on $Z_{H_0}(F)$, we define the multiplicity of the middle model to be

$$m(\tau) = \dim(\mathrm{Hom}_{H_1(F)}(\tau, \omega \otimes \xi_1)).$$

Similarly, we can also define the middle model $(G_{1,D}, H_{1,D})$ for the quaternion case with $G_{1,D}(F) = \mathrm{GL}_2(D) \times \mathrm{GL}_1(D)$.

Let π be an irreducible generic representation of $\mathrm{GL}_6(F)$ with central character χ^2 . We assume that π is either the parabolic induction of some essential discrete series $\tau = \tau_1 \otimes \tau_2$ of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$, or the parabolic induction of some essential discrete series $\sigma = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$ of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$. We need to make some assumptions on the central characters, which will be used in Section 3.4 and 3.5.

- (1) If F is p -adic and π is the parabolic induction of some essential discrete series $\sigma = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$ of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$. Let ω_i be the central character of σ_i for $i = 1, 2, 3$. By switching the order of σ_i , we may assume that the characters $\omega_3\omega_1^{-1}$ and $\omega_3\omega_2^{-1}$ are not positive. Note that since $\pi = I_{P_{2,2,2}}^{\mathrm{GL}_6}(\sigma)$ is irreducible, switching the order of σ_i will not change π .
- (2) If $F = \mathbb{R}$ and π is the parabolic induction of some essential discrete series $\sigma = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$ of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$. Let $\sigma_i = \mathrm{St}(\eta_i, \alpha_i, n_i)$ for $i = 1, 2, 3$. The following lemma tells us that by switching the order of σ_i , we may assume that

$$(3.1) \quad \eta_2\eta_3^2\alpha_3\nu^{-\frac{n_2+3}{2}} \neq \nu^{-1}\chi, \quad \eta_1\eta_3^2\alpha_3\nu^{-\frac{n_1+3}{2}} \neq \nu^{-1}\chi.$$

Lemma 3.1. *When $F = \mathbb{R}$, with the notations above, by switching the order of σ_i , we can assume that the inequalities in (3.1) hold.*

Proof. For $i = 1, 2, 3$, fix $\sigma_i = St(\eta_i, \alpha_i, n_i)$ as above. Then the central character ω_i of σ_i is equal to $\eta_i^2 \alpha_i$. Hence the central character of π is equal to $(\eta_1 \eta_2 \eta_3)^2 \alpha_1 \alpha_2 \alpha_3$. Since $F = \mathbb{R}$, by our assumption on Section 2.1, the character χ is either 1 or sgn . Hence the central character of π is equal to $\chi^2 = 1$. Therefore we have

$$(3.2) \quad (\eta_1 \eta_2 \eta_3)^2 \alpha_1 \alpha_2 \alpha_3 = 1.$$

Since α_i is either 1 or sgn , (3.2) implies that

$$(3.3) \quad (\eta_1 \eta_2 \eta_3)^2 = \alpha_1 \alpha_2 \alpha_3 = 1.$$

If the inequalities in (3.1) already hold, then we are done. If not, without loss of generality, we may assume that

$$(3.4) \quad \eta_2 \eta_3^2 \alpha_3 \nu^{-\frac{n_2+3}{2}} = \nu^{-1} \chi.$$

Then if $\eta_2 \eta_1^2 \alpha_1 \nu^{-\frac{n_2+3}{2}} = \nu^{-1} \chi$, combining with (3.4), we have

$$(\eta_1 \eta_2 \eta_3)^2 \alpha_1 \alpha_3 = \nu^{n_2+1} \chi^2 = \nu^{n_2+1}.$$

This is a contradiction since $(\eta_1 \eta_2 \eta_3)^2 \alpha_1 \alpha_3 = \alpha_1 \alpha_3$ is a unitary character and $n_2 > 0$. Hence we have

$$(3.5) \quad \eta_2 \eta_1^2 \alpha_1 \nu^{-\frac{n_2+3}{2}} \neq \nu^{-1} \chi.$$

Now if $\eta_3 \eta_1^2 \alpha_1 \nu^{-\frac{n_3+3}{2}} \neq \nu^{-1} \chi$, we just need to change the order of σ_1 and σ_3 to make the inequalities in (3.1) hold and this proves the lemma. So it remains to consider the case when

$$(3.6) \quad \eta_3 \eta_1^2 \alpha_1 \nu^{-\frac{n_3+3}{2}} = \nu^{-1} \chi.$$

If $\eta_3 \eta_2^2 \alpha_2 \nu^{-\frac{n_3+3}{2}} = \nu^{-1} \chi$, combining with (3.6), we have

$$(\eta_1 \eta_2 \eta_3)^2 \alpha_1 \alpha_2 = \nu^{n_3+1} \chi^2 = \nu^{n_3+1}.$$

This is a contradiction since $(\eta_1 \eta_2 \eta_3)^2 \alpha_1 \alpha_2 = \alpha_1 \alpha_2$ is a unitary character and $n_3 > 0$. Hence we have

$$(3.7) \quad \eta_3 \eta_2^2 \alpha_2 \nu^{-\frac{n_3+3}{2}} \neq \nu^{-1} \chi.$$

On the mean time, if $\eta_1 \eta_2^2 \alpha_2 \nu^{-\frac{n_1+3}{2}} = \nu^{-1} \chi$, combining with (3.4) and (3.6), we have

$$(\eta_1 \eta_2 \eta_3)^3 \alpha_1 \alpha_2 \alpha_3 = \chi^3 \nu^{\frac{n_1+n_2+n_3+3}{2}} = \chi \nu^{\frac{n_1+n_2+n_3+3}{2}}.$$

By squaring both sides, we get

$$(\eta_1 \eta_2 \eta_3)^6 (\alpha_1 \alpha_2 \alpha_3)^2 = \chi^2 \nu^{n_1+n_2+n_3-6} = \nu^{n_1+n_2+n_3+3}.$$

Combining with (3.3), we have $n_1 + n_2 + n_3 = -3$ which is a contradiction since $n_i > 0$ for $i = 1, 2, 3$. Therefore we have

$$(3.8) \quad \eta_1 \eta_2^2 \alpha_2 \nu^{-\frac{n_1+3}{2}} \neq \nu^{-1} \chi.$$

By (3.7) and (3.8), we only need to change the order of σ_2 and σ_3 to make the inequalities in (3.1) hold. This finishes the proof of the lemma. \square

Now if π is the parabolic induction of some essential discrete series $\tau = \tau_1 \otimes \tau_2$ of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$, we let $m(\tau)$ be the multiplicity of the middle model. If π is the parabolic induction of some essential discrete series $\sigma = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$ of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$, let $\tau_1 = I_{P_{2,2}}^{\mathrm{GL}_4}(\sigma_1 \otimes \sigma_2)$ and let $\tau_2 = \sigma_3$. Then $\tau = \tau_1 \otimes \tau_2$ is an irreducible generic representation of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$ with $\pi = I_{P_{4,2}}^{\mathrm{GL}_6}(\tau)$. We still let $m(\tau)$ be the multiplicity of the middle model. Finally, let $m(\pi)$ be the multiplicity of the Ginzburg-Rallis model as usual. The goal of this section is to prove the following proposition.

Proposition 3.2. *With the notations above, we have*

$$m(\pi) \neq 0 \Rightarrow m(\tau) \neq 0.$$

3.2. The double cosets $P_{4,2}(F) \backslash G(F) / H(F)$. In this subsection, we will write down all the elements in the double cosets $P_{4,2}(F) \backslash G(F) / H(F)$. We start with the double cosets $P_{4,2}(F) \backslash G(F) / P(F)$ where $P = P_{2,2,2} = MU$ is the standard parabolic subgroup of type $(2, 2, 2)$. By the Bruhat decomposition, this is equivalent to the double cosets $S_4 \times S_2 \backslash S_6 / S_2 \times S_2 \times S_2$. Then an easy computation shows that the double cosets $P_{4,2}(F) \backslash G(F) / P(F)$ contain six elements $\{P_{4,2}(F)v_iP(F), 1 \leq i \leq 6\}$ with $v_1 = w_{(561234)}$, $v_2 = w_{(123456)}$, $v_3 = w_{(125634)}$, $v_4 = w_{(123546)}$, $v_5 = w_{(152346)}$, and $v_6 = w_{(152634)}$.

For $1 \leq i \leq 3$, it is easy to see that $M(F) \subset v_i^{-1}P_{4,2}(F)v_i$. Together with the fact that $P(F) = M(F)H(F)$, we know that $P_{4,2}(F)v_iP(F) = P_{4,2}(F)v_iH(F)$ for $1 \leq i \leq 3$. For $i = 4$, we have

$$M(F) \cap v_i^{-1}P_{4,2}(F)v_i = \{\mathrm{diag}(g, b_1, b_2) \mid g \in \mathrm{GL}_2(F), b_1, b_2 \in B_2(F)\}$$

Since the double cosets

$$\mathrm{GL}_2(F) \times B_2(F) \times B_2(F) \backslash \mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F) / \mathrm{GL}_2(F)^{\mathrm{diag}}$$

contain two elements which are represented by the elements $I_2 \times I_2 \times I_2$ and $I_2 \times w_{(21)} \times I_2$, we know that $P_{42}(F)v_4P(F)$ is the disjoint union of the double cosets $P_{42}(F)v_{41}H(F)$ and $P_{42}(F)v_{42}H(F)$ where

$$v_{41} = v_4 = w_{(123546)}, \quad v_{42} = v_4w_{(124356)} = w_{(125346)}.$$

Similarly, we can show that $P_{42}(F)v_5P(F)$ (resp. $P_{42}(F)v_6P(F)$) is the disjoint union of the double cosets $P_{42}(F)v_{51}H(F)$ and $P_{42}(F)v_{52}H(F)$ (resp. $P_{42}(F)v_{61}H(F)$ and $P_{62}(F)v_{62}H(F)$) where

$$v_{51} = v_5 = w_{(152346)}, \quad v_{52} = v_5w_{(123465)} = w_{(152364)};$$

$$v_{61} = v_6 = w_{(152634)}, \quad v_{62} = v_6w_{(124356)} = w_{(156234)}.$$

To summarize, the double cosets $P_{4,2}(F) \backslash G(F) / H(F)$ contain nine elements $\{P_{4,2}(F)v_iH(F), 1 \leq i \leq 9\}$ where

$$w_1 = w_{(561234)}, \quad w_2 = w_{(123456)}, \quad w_3 = w_{(125634)}, \quad w_4 = w_{(123546)}, \quad w_5 = w_{(125346)},$$

$$w_6 = w_{(152346)}, w_7 = w_{(152364)}, w_8 = w_{(156234)}, w_9 = w_{(152634)}.$$

Here the open orbit is $P_{4,2}(F)w_1H(F)$.

For $1 \leq i \leq 9$, let $P_i(F) = P_{4,2}(F) \cap w_iH(F)w_i^{-1}$. It is easy to see that the model associated to the open orbit $P_{4,2}(F)w_1H(F)$ is the middle model. Hence by the orbit method in Section 2.3, in order to prove Proposition 3.2, it is enough to show that for $2 \leq i \leq 9$, the representation τ of $P_{4,2}(F)$ is not $(P_i, \delta_P^{-1/2}\delta_{P_i}^{w_i}(\omega \otimes \xi))$ -distinguished. Here the character $w_i(\omega \otimes \xi)$ is defined to be

$$w_i(\omega \otimes \xi)(p_i) = \omega \otimes \xi(w_i^{-1}p_iw_i), \quad p_i \in P_i(F).$$

3.3. The orbits $P_{4,2}(F)w_iH(F)$ for $2 \leq i \leq 8$. In this subsection, we will show that the representation τ of $P_{4,2}(F)$ is not $(P_i, \delta_P^{-1/2}\delta_{P_i}^{w_i}(\omega \otimes \xi))$ -distinguished for $2 \leq i \leq 8$. Since the representation τ is $N_{4,2}(F)$ -invariant, and the character $\delta_P^{-1/2}\delta_{P_i}$ is trivial on $N_{4,2}(F) \cap P_i(F)$, it is enough to show that the character $w_i(\omega \otimes \xi)$ is nontrivial on the group $N_{4,2}(F) \cap P_i(F)$ for $2 \leq i \leq 8$.

- For $i = 2$, we have

$$N_{4,2}(F) \cap P_2(F) = \{n_2(X, Y) = \begin{pmatrix} I_2 & 0 & X \\ 0 & I_2 & Y \\ 0 & 0 & I_2 \end{pmatrix} \mid X, Y \in Mat_{2 \times 2}(F)\}.$$

The character $w_2(\omega \otimes \xi)$ on $N_{4,2}(F) \cap P_2(F)$ is defined to be

$$w_2(\omega \otimes \xi)(n_2(X, Y)) = \psi(\text{tr}(Y)),$$

which is a nontrivial character. Hence the representation τ is not $(P_2, \delta_P^{-1/2}\delta_{P_2}^{w_2}(\omega \otimes \xi))$ -distinguished.

- For $i = 3$, we have

$$N_{4,2}(F) \cap P_3(F) = \{n_3(X) = \begin{pmatrix} I_2 & 0 & X \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix} \mid X \in Mat_{2 \times 2}(F)\}.$$

The character $w_3(\omega \otimes \xi)$ on $N_{4,2}(F) \cap P_3(F)$ is defined to be

$$w_3(\omega \otimes \xi)(n_3(X)) = \psi(\text{tr}(X)),$$

which is a nontrivial character. Hence the representation τ is not $(P_3, \delta_P^{-1/2}\delta_{P_3}^{w_3}(\omega \otimes \xi))$ -distinguished.

- For $i = 4$, we have

$$N_{4,2}(F) \cap P_4(F) = \{n_4(x_{11}, x_{12}, x_{21}, x_{22}, y) = \begin{pmatrix} I_2 & 0 & X \\ 0 & I_2 & Y \\ 0 & 0 & I_2 \end{pmatrix} \mid X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, Y = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}\}.$$

The character $w_4(\omega \otimes \xi)$ on $N_{4,2}(F) \cap P_4(F)$ is defined to be

$$w_4(\omega \otimes \xi)(n_4(x_{11}, x_{12}, x_{21}, x_{22}, y)) = \psi(x_{21}),$$

which is a nontrivial character. Hence the representation τ is not $(P_4, \delta_P^{-1/2} \delta_{P_4} w_4(\omega \otimes \xi))$ -distinguished.

- For $i = 5$, we have

$$N_{4,2}(F) \cap P_5(F) = \{n_5(x_{11}, x_{12}, x_{21}, x_{22}, y) = \begin{pmatrix} I_2 & 0 & X \\ 0 & I_2 & Y \\ 0 & 0 & I_2 \end{pmatrix} \mid X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, Y = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}\}.$$

The character $w_5(\omega \otimes \xi)$ on $N_{4,2}(F) \cap P_5(F)$ is defined to be

$$w_5(\omega \otimes \xi)(n_5(x_{11}, x_{12}, x_{21}, x_{22}, y)) = \psi(\text{tr}(x_{11} + y)),$$

which is a nontrivial character. Hence the representation τ is not $(P_5, \delta_P^{-1/2} \delta_{P_5} w_5(\omega \otimes \xi))$ -distinguished.

- For $i = 6$, we have

$$N_{4,2}(F) \cap P_6(F) = \{n_6(x_1, x_2, y_1) = \begin{pmatrix} I_2 & 0 & X \\ 0 & I_2 & Y \\ 0 & 0 & I_2 \end{pmatrix} \mid X = \begin{pmatrix} 0 & x_1 \\ 0 & x_2 \end{pmatrix}, Y = \begin{pmatrix} 0 & y_1 \\ 0 & 0 \end{pmatrix}\}.$$

The character $w_6(\omega \otimes \xi)$ on $N_{4,2}(F) \cap P_6(F)$ is defined to be

$$w_6(\omega \otimes \xi)(n_6(x_1, x_2, y_1)) = \psi(y_1),$$

which is a nontrivial character. Hence the representation τ is not $(P_6, \delta_P^{-1/2} \delta_{P_6} w_6(\omega \otimes \xi))$ -distinguished.

- For $i = 7$, we have

$$N_{4,2}(F) \cap P_7(F) = \{n_7(x_1, x_2, y_1) = \begin{pmatrix} I_2 & 0 & X \\ 0 & I_2 & Y \\ 0 & 0 & I_2 \end{pmatrix} \mid X = \begin{pmatrix} 0 & x_1 \\ 0 & x_2 \end{pmatrix}, Y = \begin{pmatrix} 0 & y_1 \\ 0 & 0 \end{pmatrix}\}.$$

The character $w_7(\omega \otimes \xi)$ on $N_{4,2}(F) \cap P_7(F)$ is defined to be

$$w_7(\omega \otimes \xi)(n_7(x_1, x_2, y_1)) = \psi(x_2),$$

which is a nontrivial character. Hence the representation τ is not $(P_7, \delta_P^{-1/2} \delta_{P_7} w_7(\omega \otimes \xi))$ -distinguished.

- For $i = 8$, we have

$$N_{4,2}(F) \cap P_8(F) = \{n_8(x) = \begin{pmatrix} I_2 & 0 & X \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix} \mid X = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\}.$$

The character $w_8(\omega \otimes \xi)$ on $N_{4,2}(F) \cap P_8(F)$ is defined to be

$$w_8(\omega \otimes \xi)(n_8(x)) = \psi(x),$$

which is a nontrivial character. Hence the representation τ is not $(P_8, \delta_P^{-1/2} \delta_{P_8} w_8(\omega \otimes \xi))$ distinguished.

To summarize, we have proved that the representation τ of $P_{4,2}(F)$ is not $(P_i, \delta_P^{-1/2} \delta_{P_i}^{w_i}(\omega \otimes \xi))$ -distinguished for $2 \leq i \leq 8$.

3.4. The p-adic case. In this subsection, assume that F is p-adic. We want to show that the representation τ of $P_{4,2}(F)$ is not $(P_9, \delta_P^{-1/2} \delta_{P_9}^{w_9}(\omega \otimes \xi))$ -distinguished. If π is the parabolic induction of some essential discrete series $\tau = \tau_1 \otimes \tau_2$ of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$, let ω_{τ_i} be the central character of τ_i for $i = 1, 2$. Up to replace π by its contragredient π^\vee , we may assume that the character $\omega_{\tau_1} \omega_{\tau_2}^{-1}$ is nonnegative. Note that the multiplicities of the Ginzburg-Rallis model and the middle model will remain the same if we change the representation to its contragredient.

By the definition of w_9 , it is easy to see that

$$P_9(F) = \{p_9(a, b, c, x_{12}, x_{13}, x_{14}, x_{16}, x_{23}, x_{24}, y) = \begin{pmatrix} a & x_{12} & x_{13} & x_{14} & c & x_{16} \\ 0 & a & x_{23} & x_{24} & 0 & c \\ 0 & 0 & a & c & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & b & y \\ 0 & 0 & 0 & 0 & 0 & b \end{pmatrix} \mid \\ a, b \in F^\times, c, x_{12}, x_{13}, x_{14}, x_{16}, x_{23}, x_{24}, y \in F\}.$$

And the character $\delta_P^{-1/2} \delta_{P_9}^{w_9}(\omega \otimes \xi)$ is defined to be

$$\delta_P^{-1/2} \delta_{P_9}^{w_9}(\omega \otimes \xi)(p_9(a, b, c, x_{12}, x_{13}, x_{14}, x_{16}, x_{23}, x_{24}, y)) = \nu\left(\frac{a}{b}\right) \chi(ab) \psi\left(\frac{x_{12}}{a} + \frac{x_{23}}{a} + \frac{y}{b}\right).$$

In this case, the character is trivial on the group $N_{4,2}(F) \cap P_9(F)$, hence the argument in the previous subsection will not work. Instead, we need to study the Levi part of the model. We define a subgroup $H'_9(F)$ of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$ to be

$$H'_9(F) = \{h'_9(a, b, c, x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, y) = \begin{pmatrix} a & x_{12} & x_{13} & x_{14} \\ 0 & a & x_{23} & x_{24} \\ 0 & 0 & a & c \\ 0 & 0 & 0 & b \end{pmatrix} \times \begin{pmatrix} b & y \\ 0 & b \end{pmatrix} \mid \\ a, b \in F^\times, c, x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, y \in F\}.$$

And we define a character β' on $H'_9(F)$ to be

$$\beta'(h'_9(a, b, c, x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, y)) = \nu\left(\frac{a}{b}\right) \chi(ab) \psi\left(\frac{x_{12}}{a} + \frac{x_{23}}{a} + \frac{y}{b}\right).$$

Since the representation τ is $N_{4,2}(F)$ -invariant, it is enough to show that as a representation of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$, τ is not $(H'_9(F), \beta')$ -distinguished.

We define a subgroup $H_9(F)$ of $\mathrm{GL}_3(F) \times \mathrm{GL}_1(F) \times \mathrm{GL}_2(F)$ to be

$$H_9(F) = \{h_9(a, b, x_{12}, x_{13}, x_{23}, y) = \begin{pmatrix} a & x_{12} & x_{13} \\ 0 & a & x_{23} \\ 0 & 0 & a \end{pmatrix} \times (b) \times \begin{pmatrix} b & y \\ 0 & b \end{pmatrix} \mid \\ a, b \in F^\times, x_{12}, x_{13}, x_{23}, y \in F\}.$$

And we define a character β on $H_9(F)$ to be

$$\beta(h_9(a, b, x_{12}, x_{13}, x_{23}, y)) = \nu\left(\frac{a}{b}\right)\chi(ab)\psi\left(\frac{x_{12}}{a} + \frac{x_{23}}{a} + \frac{y}{b}\right).$$

Recall that $P_{3,1} = M_{3,1}N_{3,1}$ is the standard parabolic subgroup of GL_4 of type $(3,1)$, and $J_{N_{3,1}}(\tau_1)$ is the Jacquet module of τ_1 with respect to the unipotent subgroup $N_{3,1}$. In order to prove that τ is not $(H'_9(F), \beta')$ -distinguished, it is enough to prove the following lemma.

Lemma 3.3. *The representation $J_{N_{3,1}}(\tau_1) \otimes \tau_2$ of $\mathrm{GL}_3(F) \times \mathrm{GL}_1(F) \times \mathrm{GL}_2(F)$ is not $(H_9(F), \beta)$ -distinguished.*

Proof. If τ_1 is supercuspidal or the essential discrete series induced from some supercuspidal representation of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$, we have $J_{N_{3,1}}(\tau_1) = 0$ and the statement in the lemma is trivial.

If $\tau_1 = St_4(\eta)$ for some character η of F^\times , by the discussion in Section 2.2, we have $J_{N_{3,1}}(\tau_1) = St_3(\eta\nu) \otimes \eta\nu^{-3}$. Then the semisimple part

$$\{h_9(a, b, 0, 0, 0, 0) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \times (b) \times \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \mid a, b \in F^\times\}$$

of $H_9(F)$ acts on the representation via the central character, which is equal to $\eta^3(a)\eta(b)\omega_{\tau_2}(b)\nu^3(\frac{a}{b})$ where ω_{τ_2} is the central character of τ_2 . Then in order to prove the lemma, it is enough to show that the character $\eta^3\nu^3$ is not equal to the character $\nu\chi$. Since χ is a unitary character, we only need to show that η is a nonnegative character.

By our assumption on the central character of τ_i , we know that the character $\omega_{\tau_1}\omega_{\tau_2}^{-1} = \eta^4\omega_{\tau_2}^{-1}$ is nonnegative. On the mean time, since the central character of π is equal to χ^2 , we know that $\omega_{\tau_1}\omega_{\tau_2} = \eta^4\omega_{\tau_2} = \chi^2$ is a unitary character. This implies that η is a nonnegative character and proves the lemma when $\tau_1 = St_4(\eta)$.

Now the only remaining case is when $\tau_1 = I_{P_{2,2}}^{\mathrm{GL}_4}(\sigma_1 \otimes \sigma_2)$ for some essential discrete series $\sigma_1 \otimes \sigma_2$ of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$. If $\sigma_i = St_2(\eta_i)$ for $i = 1, 2$, then $J_{N_{3,1}}(\tau_1) = (I_{P_{2,1}}^{\mathrm{GL}_3}(St_2(\eta_1\nu^{1/2}) \otimes \eta_2\nu) \otimes \eta_2\nu^{-2}) \oplus (I_{P_{1,2}}^{\mathrm{GL}_3}(\eta_1\nu \otimes St_2(\eta_2\nu^{1/2})) \otimes \eta_1\nu^{-2})$. By the same argument as in the previous case, it is enough to show that

$$\eta_1^2\eta_2\nu^2 \neq \nu\chi, \eta_2^2\eta_1\nu^2 \neq \nu\chi.$$

Since χ is a unitary character, it is enough to show that the characters $\eta_1^2\eta_2$ and $\eta_2^2\eta_1$ are nonnegative. By our assumptions on the central characters on Section 3.1, we know that the characters $\omega_{\tau_2}\omega_{\sigma_1}^{-1} = \omega_{\tau_2}\eta_1^{-2}$ and $\omega_{\tau_2}\omega_{\sigma_2}^{-1} = \omega_{\tau_2}\eta_2^{-2}$ are not positive. On the mean time, since the central character of π is equal to χ^2 , we know that $\eta_1^2\eta_2^2\omega_{\tau_2} = \chi^2$ is a unitary character. Hence the characters $\eta_1^2\eta_2$ and $\eta_2^2\eta_1$ are nonnegative. This proves the lemma when $\sigma_i = St_2(\eta_i)$ for $i = 1, 2$.

If $\sigma_1 = St_2(\eta_1)$ and σ_2 is supercuspidal, then $J_{N_{3,1}}(\tau_1) = I_{P_{1,2}}^{\mathrm{GL}_3}(\eta_1\nu \otimes \sigma_2\nu^{1/2}) \otimes \eta_1\nu^{-2}$. Let ω_{σ_2} be the central character of σ_2 . By the same

argument as in the previous case, it is enough to show that the character $\eta_1\omega_{\sigma_2}$ is nonnegative. But this just follows from the facts that the character $\omega_{\tau_2}\omega_{\sigma_2}^{-1}$ is not positive and the character $\eta_1^2\omega_{\sigma_2}\omega_{\tau_2} = \chi^2$ is unitary. This proves the lemma when $\sigma_1 = St_2(\eta_1)$ and σ_2 is supercuspidal.

If $\sigma_2 = St_2(\eta_2)$ and σ_1 is supercuspidal, the argument is similar to the previous case and we will skip it here. Finally, if σ_1 and σ_2 are supercuspidal, then $J_{N_{3,1}}(\tau_1) = 0$ and the statement in the lemma is trivial. This finishes the proof of the lemma, and hence the proof of Proposition 3.2 when F is p-adic. \square

3.5. The archimedean case. In this subsection, we assume that $F = \mathbb{R}$. We want to show that the representation τ of $P_{4,2}(F)$ is not $(P_9, \delta_P^{-1/2}\delta_{P_9}^{w_9}(\omega \otimes \xi))$ -distinguished. By the same argument as in the previous subsection, it is enough to prove Lemma 3.3 for the case when $F = \mathbb{R}$. In other words, we want to prove the following lemma.

Lemma 3.4. *The representation $J_{N_{3,1}}(\tau_1) \otimes \tau_2$ of $GL_3(F) \times GL_1(F) \times GL_2(F)$ is not $(H_9(F), \beta)$ -distinguished.*

Proof. Recall that $\tau_1 = I_{P_{2,2}}^{GL_4}(\sigma_1 \otimes \sigma_2)$ and $\tau_2 = \sigma_3$ where $\sigma_i = St(\eta_i, \alpha_i, n_i)$ are some essential discrete series of $GL_2(F)$ for $i = 1, 2, 3$. By the discussion in Section 2.2, we have $J_{N_{3,1}}(\tau_1) = (I_{P_{2,1}}^{GL_3}(St(\eta_1\nu^{1/2}, \alpha_1, n_1) \otimes \eta_2\alpha_2\nu^{\frac{n_2+1}{2}}) \otimes \eta_2\nu^{-\frac{n_2+3}{2}}) \oplus (I_{P_{1,2}}^{GL_3}(\eta_1\alpha_1\nu^{\frac{n_1+1}{2}} \otimes St(\eta_2\nu^{1/2}, \alpha_2, n_2)) \otimes \eta_1\nu^{-\frac{n_1+3}{2}})$. Then the semisimple part

$$\{h_9(a, b, 0, 0, 0, 0) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \times (b) \times \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \mid a, b \in F^\times\}$$

of $H_9(F)$ acts on the representation via the central character, which is either

$$\eta_1^2(a)\alpha_1(a)\eta_2(a)\alpha_2(a)\eta_2(b)\eta_3^2(b)\alpha_3(b)\nu^{\frac{n_2+3}{2}}\left(\frac{a}{b}\right),$$

or

$$\eta_2^2(a)\alpha_2(a)\eta_1(a)\alpha_1(a)\eta_1(b)\eta_3^2(b)\alpha_3(b)\nu^{\frac{n_1+3}{2}}\left(\frac{a}{b}\right).$$

Then in order to prove the lemma, it is enough to show that

$$\eta_2\eta_3^2\alpha_3\nu^{-\frac{n_2+3}{2}} \neq \nu^{-1}\chi, \quad \eta_1\eta_3^2\alpha_3\nu^{-\frac{n_1+3}{2}} \neq \nu^{-1}\chi.$$

But this just follows from our assumptions on the central characters (i.e. (3.1)) in Section 3.1. This finishes the proof of the lemma, and hence the proof of Proposition 3.2 when $F = \mathbb{R}$. \square

3.6. The quaternion case. In this subsection, F is a p-adic field or \mathbb{R} . Let π_D be an irreducible admissible representation of $G_D(F) = GL_3(D)$ with central character χ^2 . Let $P_{2,1,D}(F) = M_{2,1,D}(F)N_{2,1,D}(F)$ be the standard parabolic subgroup of $G_D(F)$ of type $(2, 1)$. Assume that $\pi_D =$

$I_{P_{2,1,D}}^{G_D}(\tau_D)$ where $\tau_D = \tau_{1,D} \otimes \tau_{2,D}$ is an irreducible admissible representation of $\mathrm{GL}_2(D) \times \mathrm{GL}_1(D)$. Let $m(\pi_D)$ (resp. $m(\tau_D)$) be the multiplicity of the Ginzburg-Rallis model (resp. middle model). By a similar (but much easier) argument as in the previous subsections, we can prove the following proposition which is an analogue of Proposition 3.2 for the quaternion case. We will skip the proof here.

Proposition 3.5. *With the notations above, we have*

$$m(\pi_D) \neq 0 \Rightarrow m(\tau_D) \neq 0.$$

4. THE RELATIONS BETWEEN THE MIDDLE MODEL AND THE TRILINEAR GL_2 MODEL

In this section, we study the relations between the middle model and the trilinear GL_2 model. The goal is to show that if some induced representation is distinguished by the middle model, then the representation of the Levi subgroup is distinguished by the trilinear GL_2 model. In Section 4.1, we recall the definition of the trilinear GL_2 model and state the problem. In Section 4.2, we study the double cosets $P_{2,2}(F) \times \mathrm{GL}_2(F) \backslash G_1(F) / H_1(F)$. Then we consider the p-adic case in Section 4.3, and the archimedean case in Section 4.4. Finally in Section 4.5, we will consider the quaternion case.

4.1. The problem. We first recall the definition of the trilinear GL_2 model. Let $G_0(F) = \mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$, and let $H_0(F)$ be the image of the diagonal embedding of $\mathrm{GL}_2(F)$ into $G_0(F)$. As in Section 1, the character χ induces a character ω on $H_0(F)$. For an irreducible generic representation σ of $G_0(F)$ whose central character equals χ^2 on $Z_{H_0}(F)$, we define the multiplicity of the trilinear GL_2 model to be $m(\sigma) := \dim(\mathrm{Hom}_{H_0(F)}(\sigma, \omega))$. Similarly, we can also define the quaternion version $(G_{0,D}(F), H_{0,D}(F))$ with $G_{0,D}(F) = \mathrm{GL}_1(D) \times \mathrm{GL}_1(D) \times \mathrm{GL}_1(D)$. The trilinear GL_2 model has been studied by Prasad in [P90].

Now let $\tau = \tau_1 \otimes \tau_2$ be an irreducible generic representation of $G_1(F) = \mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$. Assume that τ_2 is an essential discrete series of $\mathrm{GL}_2(F)$, and $\tau_1 = I_{P_{2,2}}^{\mathrm{GL}_4}(\sigma_1 \otimes \sigma_2)$ is the parabolic induction of some essential discrete series $\sigma_1 \otimes \sigma_2$ of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$. Let ω_i be the central character of σ_i for $i = 1, 2$. Up to change the order of σ_1 and σ_2 , we may assume that $\omega_1 \omega_2^{-1}$ is a nonnegative character.

Let $\sigma = \sigma_1 \otimes \sigma_2 \otimes \tau_2$ which is an essential discrete series of $G_0(F)$. Let $m(\tau)$ (resp. $m(\sigma)$) be the multiplicity of the middle model (resp. trilinear GL_2 model). The goal of this section is to prove the following proposition.

Proposition 4.1. *With the notations above, we have*

$$m(\tau) \neq 0 \Rightarrow m(\sigma) \neq 0.$$

4.2. The double cosets $P_{2,2}(F) \times \mathrm{GL}_2(F) \backslash G_1(F) / H_1(F)$. By a similar (but much easier) argument as in Section 3.2, we can write down the double cosets for $P_{2,2}(F) \times \mathrm{GL}_2(F) \backslash G_1(F) / H_1(F)$. It contains three elements $\{(P_{2,2}(F) \times \mathrm{GL}_2(F))w_i H_1(F), i = 1, 2, 3\}$ where

$$w_1 = I_4 \times I_2, w_2 = w_{(1324)} \times I_2, w_3 = w_{(3412)} \times I_2.$$

The open orbit is $(P_{2,2}(F) \times \mathrm{GL}_2(F))w_3 H_1(F)$. It is easy to see that the model associated to the open orbit is the trilinear GL_2 model. Hence in order to prove Proposition 4.1, it is enough to show that the orbits $(P_{2,2}(F) \times \mathrm{GL}_2(F))w_i H_1(F)$ are not distinguished for $i = 1, 2$.

For the orbit $(P_{2,2}(F) \times \mathrm{GL}_2(F))w_1 H_1(F)$, we have $w_1^{-1} H_1(F) w_1 \cap (P_{2,2}(F) \times \mathrm{GL}_2(F)) = H_1(F)$. Also it is easy to see that $\delta_{P_{2,2} \times \mathrm{GL}_2}|_{H_1} = \delta_{H_1} = 1$. Hence it is enough to show that the representation $\sigma_1 \otimes \sigma_2 \otimes \tau_2$ of $P_{2,2}(F) \times \mathrm{GL}_2(F)$ is not $(H_1, \omega \otimes \xi_1)$ -distinguished. But this follows from the fact that the representation is $N_{2,2}(F)$ -invariant and the character $\omega \otimes \xi_1$ is nontrivial on $N_{2,2}(F)$.

Hence it remains to prove that the orbit $(P_{2,2}(F) \times \mathrm{GL}_2(F))w_2 H_1(F)$ is not distinguished. In this case, we have

$$\begin{aligned} & w_2^{-1} H_1(F) w_2 \cap (P_{2,2}(F) \times \mathrm{GL}_2(F)) \\ = & \{h_2(a, b, c, x, y, z) := \begin{pmatrix} a & x & c & z \\ 0 & a & 0 & c \\ 0 & 0 & b & y \\ 0 & 0 & 0 & b \end{pmatrix} \times \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a, b \in F^\times, c, x, y, z \in F\}. \end{aligned}$$

We will use $H_2(F)$ to denote this group. It is easy to check that the modular character $\delta_{H_2}(\delta_P^{-1/2})|_{H_2}$ is trivial. Hence the character on $H_2(F)$ is $w_2(\omega \otimes \xi_1)$ defined by

$$w_2(\omega \otimes \xi_1)(h_2(a, b, c, x, y, z)) = \omega \otimes \xi(w_2 h_2(a, b, c, x, y, z) w_2^{-1}) = \chi(ab) \psi\left(\frac{x}{a} + \frac{y}{b}\right).$$

We want to show that the representation $\sigma_1 \otimes \sigma_2 \otimes \tau_2$ of $P_{2,2}(F) \times \mathrm{GL}_2(F)$ is not $(H_2, w_2(\omega \otimes \xi))$ -distinguished. We will prove this in Section 4.3 for the p-adic case; and in Section 4.4 for the archimedean case.

4.3. The p-adic case. In this subsection, assume that F is p-adic. We want to show that the representation $\sigma_1 \otimes \sigma_2 \otimes \tau_2$ of $P_{2,2}(F) \times \mathrm{GL}_2(F)$ is not $(H_2, w_2(\omega \otimes \xi))$ -distinguished. Let

$$H'_2(F) = \{h'_2(a, b, c, x, y) = \begin{pmatrix} a & x \\ 0 & a \end{pmatrix} \times \begin{pmatrix} b & y \\ 0 & b \end{pmatrix} \times \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a, b \in F^\times, c, x, y \in F\}$$

be a subgroup of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$, and define a character β'_2 on $H'_2(F)$ to be

$$\beta'_2(h'_2(a, b, c, x, y)) = \chi(ab) \psi\left(\frac{x}{a} + \frac{y}{b}\right).$$

Since the representation $\sigma_1 \otimes \sigma_2 \otimes \tau_2$ is $N_{2,2}(F)$ -invariant, it is enough to show that the representation $\sigma_1 \otimes \sigma_2 \otimes \tau_2$ of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$ is

not (H'_2, β'_2) -distinguished. Define a subgroup $H''_2(F)$ of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_1(F) \times \mathrm{GL}_1(F)$ to be

$$H''_2(F) = \{h''_2(a, b, x, y) = \begin{pmatrix} a & x \\ 0 & a \end{pmatrix} \times \begin{pmatrix} b & y \\ 0 & b \end{pmatrix} \times (a) \times (b) \mid a, b \in F^\times, x, y \in F\}.$$

And define a character β''_2 on $H''_2(F)$ to be

$$\beta''_2(h''_2(a, b, x, y)) = \chi(ab)\psi\left(\frac{x}{a} + \frac{y}{b}\right).$$

Then in order to show the representation $\sigma_1 \otimes \sigma_2 \otimes \tau_2$ of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$ is not (H'_2, β'_2) -distinguished, it is enough to prove the following lemma.

Lemma 4.2. *The representation $\sigma_1 \otimes \sigma_2 \otimes J_{N_2}(\tau_2)$ of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_1(F) \times \mathrm{GL}_1(F)$ is not (H''_2, β''_2) -distinguished.*

Proof. If τ_2 is supercuspidal, then $J_{N_2}(\tau_2) = 0$ and the lemma is trivial. If $\tau_2 = St(\eta)$ for some character η of F^\times , the Jacquet module $J_{N_2}(\tau_2)$ is equal to $\eta\nu \otimes \eta\nu^{-1}$. Then the semisimple part

$$\{h''_2(a, b, 0, 0) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \times \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \times (a) \times (b) \mid a, b \in F^\times, x, y \in F\}.$$

of $H''_2(F)$ acts on the representation $\sigma_1 \otimes \sigma_2 \otimes J_{N_2}(\tau_2)$ via the central character, which is equal to $\omega_1(a)\omega_2(b)\eta(a)\eta(b)\nu\left(\frac{a}{b}\right)$ where ω_i is the central character of σ_i for $i = 1, 2$.

On the mean time, we have $\beta''_2(h''_2(a, b, 0, 0)) = \chi(ab)$. As a result, in order to prove the lemma, it is enough to show that the character $\omega_1\eta\nu$ is not equal to the character χ . Since χ is a unitary character, it is enough to show that $\omega_1\eta$ is a nonnegative character.

Since the product of the central characters of σ_1, σ_2 and τ_2 is equal to χ^2 , we have $\omega_1\omega_2\eta^2 = \chi^2$ which is a unitary character. By our assumption in Section 4.1, the character $\omega_1\eta(\omega_2\eta)^{-1} = \omega_1\omega_2^{-1}$ is a nonnegative character. Therefore $\omega_1\eta$ is a nonnegative character. This finishes the proof of the lemma and hence the proof of Proposition 4.1 when F is p-adic. \square

4.4. The archimedean case. In this subsection, assume that $F = \mathbb{R}$. We want to show that the representation $\sigma_1 \otimes \sigma_2 \otimes \tau_2$ of $P_{2,2}(F) \times \mathrm{GL}_2(F)$ is not $(H_2, w_2(\omega \otimes \xi))$ -distinguished. By the same argument as in the previous subsection, we only need to prove Lemma 4.2 when $F = \mathbb{R}$.

Lemma 4.3. *The representation $\sigma_1 \otimes \sigma_2 \otimes J_{N_2}(\tau_2)$ of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_1(F) \times \mathrm{GL}_1(F)$ is not (H''_2, β''_2) -distinguished.*

Proof. Since $F = \mathbb{R}$, the representation τ_2 is of the form $\tau_2 = St(\eta, \alpha, n)$ for some character η of F^\times . Then the Jacquet module $J_{N_2}(\tau_2)$ is equal to $\eta\alpha\nu^{\frac{n+1}{2}} \otimes \eta\nu^{-\frac{n+1}{2}}$.

By the same argument as in the proof of Lemma 4.2, we only need to show that the character $\omega_1\eta\alpha\nu^{\frac{n+1}{2}}$ is not equal to the character χ . Since χ

and α are unitary characters, it is enough to show that $\omega_1\eta$ is a nonnegative character, which follows from the same argument as in the p-adic case. This finishes the proof of the lemma and hence the proof of Proposition 4.1 when $F = \mathbb{R}$. \square

4.5. The quaternion case. In this subsection, F is a p-adic field or \mathbb{R} . Let $\tau_D = \tau_{1,D} \otimes \tau_{2,D}$ be an irreducible admissible representation of $G_{1,D}(F) = \mathrm{GL}_2(D) \times \mathrm{GL}_1(D)$ whose central character is equal to χ^2 on $Z_{H_{1,D}}(F)$. Assume that $\tau_{1,D} = I_{P_{1,1,D}}^{\mathrm{GL}_2(D)}(\sigma_{1,D} \otimes \sigma_{2,D})$ where $P_{1,1,D} = M_{1,1,D}N_{1,1,D}$ is the upper triangular minimal parabolic subgroup of $\mathrm{GL}_2(D)$, and $\sigma_{1,D} \otimes \sigma_{2,D}$ is an irreducible admissible representation of $\mathrm{GL}_1(D) \otimes \mathrm{GL}_1(D)$. Then $\sigma_D = \sigma_{1,D} \otimes \sigma_{2,D} \otimes \tau_{2,D}$ is a representation of $\mathrm{GL}_1(D) \times \mathrm{GL}_1(D) \times \mathrm{GL}_1(D)$. Let $m(\tau_D)$ (resp. $m(\sigma_D)$) be the multiplicity of the middle model (resp. trilinear GL_2 model). By a similar (but much easier) argument as in the previous subsections, we can prove the following proposition which is an analogue of Proposition 4.1 for the quaternion case. We will skip the proof here.

Proposition 4.4. *With the notations above, we have*

$$m(\tau_D) \neq 0 \Rightarrow m(\sigma_D) \neq 0.$$

5. THE PROOF OF THEOREM 1.2

In this section, we are going to prove Theorem 1.2. Let π be a generic representation of $G(F) = \mathrm{GL}_6(F)$ with central character χ^2 . Assume that π_D exists. Our goal is to show that

$$(5.1) \quad m(\pi) + m(\pi_D) = 1.$$

By Remark 1.3, we know that π is either an essentially discrete series, or the parabolic induction of some essential discrete series of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$ or $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$. If π is an essential discrete series, up to twist π by some characters, we may assume that π is a discrete series. Then (5.1) has already been proved in the previous papers [Wan15] and [Wan16a].

If π is the parabolic induction of an essential discrete series $\tau = \tau_1 \otimes \tau_2$ of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$, then F must be a p-adic field since $\mathrm{GL}_4(\mathbb{R})$ does not have essential discrete series. Let $\tau_i = \sigma_i \nu^{s_i}$ ($i = 1, 2$) with σ_i being a discrete series and $s_i \in \mathbb{R}$. If $s_1 = s_2$, then π is an essential tempered representation. Up to twist π by some character, we may assume that π is tempered and (5.1) has already been proved in the previous papers [Wan15] and [Wan16a]. From now on, we only consider the case when $s_1 \neq s_2$. By the theory of the local Jacquet-Langlands correspondence, we know that π_D is the parabolic induction of the essential discrete series $\tau_D = \tau_{1,D} \otimes \tau_{2,D}$ of $\mathrm{GL}_2(D) \times \mathrm{GL}_1(D)$ where $\tau_{1,D}$ (resp. $\tau_{2,D}$) is the local Jacquet-Langlands correspondence of τ_1 (resp. τ_2) to $\mathrm{GL}_2(D)$ (resp. $\mathrm{GL}_1(D)$). We let $m(\tau)$ and $m(\tau_D)$ be the multiplicities of the middle model.

By Proposition 3.2 and Proposition 3.5, we know that

$$(5.2) \quad m(\pi) \neq 0 \Rightarrow m(\tau) \neq 0, \quad m(\pi_D) \neq 0 \Rightarrow m(\tau_D) \neq 0.$$

On the other hand, by the same argument as in Section 5.1 and 6.2 of the previous paper [Wan16b], together with the assumption that $s_1 \neq s_2$, we have

$$(5.3) \quad m(\tau) \neq 0 \Rightarrow m(\pi) \neq 0, \quad m(\tau_D) \neq 0 \Rightarrow m(\pi_D) \neq 0.$$

Note that although we only considered the archimedean case in [Wan16b], the argument in Section 5.1 and 6.2 of the loc. cit. also works for the p-adic case. Combining (5.2) and (5.3), we have

$$m(\pi) \neq 0 \Leftrightarrow m(\tau) \neq 0, \quad m(\pi_D) \neq 0 \Leftrightarrow m(\tau_D) \neq 0.$$

Since $m(\pi), m(\pi_D), m(\tau), m(\tau_D) \leq 1$, the relations above imply that

$$m(\pi) = m(\tau), \quad m(\pi_D) = m(\tau_D).$$

Hence in order to prove (5.1), it is enough to show that

$$(5.4) \quad m(\tau) + m(\tau_D) = 1.$$

But since τ is an essential discrete series and the multiplicity of the middle model is invariant under the unramified twist, up to twist τ by some characters, we may assume that τ is a discrete series. Then (5.4) has already been proved in the previous papers [Wan15] and [Wan16b]. This proves (5.1).

If π is the parabolic induction of an essential discrete series $\tau = \tau_1 \otimes \tau_2 \otimes \tau_3$ of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$, let $\tau_i = \sigma_i \nu^{s_i}$ ($i = 1, 2, 3$) with σ_i being a discrete series and $s_i \in \mathbb{R}$. If $s_1 = s_2 = s_3$, then π is an essential tempered representation. Up to twist π by some character, we may assume that π is tempered and (5.1) has already been proved in the previous papers [Wan15] and [Wan16a]. If $s_1 = s_2 \neq s_3$, let τ_4 be the parabolic induction of τ_1 and τ_2 . Then $\tau' := \tau_4 \otimes \tau_3$ is an essential tempered representation of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$ and π is the parabolic induction of τ' . Let τ'_D be the local Jacquet-Langlands correspondence of τ' to $\mathrm{GL}_2(D) \times \mathrm{GL}_1(D)$. Also let $m(\tau')$ and $m(\tau'_D)$ be the multiplicities of the middle model. By a similar argument as in the previous case, we can show that

$$m(\pi) = m(\tau'), \quad m(\pi_D) = m(\tau'_D).$$

Hence in order to prove (5.1), it is enough to show that

$$(5.5) \quad m(\tau') + m(\tau'_D) = 1.$$

But since τ' is an essential tempered representations, up to twist τ by some characters, we may assume that τ' is a tempered representation. Then (5.5) has already been proved in the previous papers [Wan15] and [Wan16b]. This proves (5.1). Similarly we can also prove (5.1) for the cases when $s_1 = s_3 \neq s_2$ or $s_2 = s_3 \neq s_1$.

Hence it remains to consider the case when $s_i \neq s_j$ for $i \neq j$. By the theory of the local Jacquet-Langlands correspondence, we know that π_D is the parabolic induction of the essential discrete series $\tau_D = \tau_{1,D} \otimes \tau_{2,D} \otimes \tau_{3,D}$

of $\mathrm{GL}_1(D) \times \mathrm{GL}_1(D) \times \mathrm{GL}_1(D)$ where $\tau_{i,D}$ are the local Jacquet-Langlands correspondence of τ_i to $\mathrm{GL}_1(D)$ for $i = 1, 2, 3$. We let $m(\tau)$ and $m(\tau_D)$ be the multiplicities of the trilinear GL_2 model. Combining the results in Proposition 3.2, Proposition 3.5, Proposition 4.1 and Proposition 4.4, we have

$$(5.6) \quad m(\pi) \neq 0 \Rightarrow m(\tau) \neq 0, \quad m(\pi_D) \neq 0 \Rightarrow m(\tau_D) \neq 0.$$

On the mean time, by the same argument as in the previous paper [Wan16b], together with the assumption that $s_i \neq s_j$ for $i \neq j$, we have

$$(5.7) \quad m(\tau) \neq 0 \Rightarrow m(\pi) \neq 0, \quad m(\tau_D) \neq 0 \Rightarrow m(\pi_D) \neq 0.$$

Combining (5.6) and (5.7), together with the fact that $m(\pi), m(\pi_D), m(\tau), m(\tau_D) \leq 1$, we have

$$(5.8) \quad m(\pi) = m(\tau), \quad m(\pi_D) = m(\tau_D).$$

By the works of Prasad in [P90] for the trilinear GL_2 model, we have

$$(5.9) \quad m(\tau) + m(\tau_D) = 1.$$

Then (5.1) follows from (5.8) and (5.9). This finishes the proof of Theorem 1.2.

6. THE REDUCED MODELS

For the rest part of this paper, assume that F is p-adic. We are going to prove Theorem 1.6. In this section, we will consider several reduced models for the Ginzburg-Rallis model, and the goal is to show that the multiplicities of those reduced models are nonzero (we refer the readers to Appendix B of the previous paper [Wan15] for the definition of reduced models). The main tools of our proof are the Rankin-Selberg integrals of $\mathrm{GL}_3 \times \mathrm{GL}_3$ and $\mathrm{GL}_3 \times \mathrm{GL}_2$ in [JPSS], and the Rankin-Selberg integrals of the exterior square L-function of GL_4 and GL_5 in [K11]. In the next section, by applying the open orbit method, together with the results in this section, we are going to prove Theorem 1.6.

6.1. The reduced model for $\mathrm{GL}_5(F)$. Let σ be an irreducible generic representation of $\mathrm{GL}_5(F)$, and let $H_5(F) = H_{5,0}(F) \times U_5(F)$ be a subgroup of $\mathrm{GL}_5(F)$ with

$$H_{5,0}(F) = \{h_5(a, c) = \mathrm{diag}\left(\begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix}, 1\right) \mid a \in F^\times, c \in F\},$$

$$U_5(F) = \{u_5(X, y_1, y_2, z_1, z_2) = \begin{pmatrix} I_2 & X & Y \\ 0 & I_2 & Z \\ 0 & 0 & 1 \end{pmatrix} \mid X \in \mathrm{Mat}_{2 \times 2}(F), Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \\ Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, y_1, y_2, z_1, z_2 \in F\}.$$

We define a character $\omega_{5,s}$ (resp. ξ_5) of $H_{5,0}(F)$ (resp. $U_5(F)$) to be $\omega_{5,s}(h_5(a,c)) = \nu^s \alpha(a)$ (resp. $\xi_5(u_5(X, y_1, y_2, z_1, z_2)) = \psi(\text{tr}(X) + z_2)$) where α is some unitary character of F^\times and $s \in \mathbb{C}$. Then $\omega_{5,s} \otimes \xi_5$ is a character of $H_5(F)$. We want to study the multiplicity

$$m(\sigma) := \dim(\text{Hom}_{H_5(F)}(\sigma, \omega_{5,s} \otimes \xi_5)).$$

Proposition 6.1. *Assume that σ is a discrete series and $\text{Re}(s) < 2$. Then $m(\sigma) \neq 0$.*

Proof. Let $W(\sigma, \psi)$ be the Whittaker model of σ and let $w_5 = \text{diag}(1, w_{(21)}, I_2) = w_{(13245)} \in \text{GL}_5(F)$. For $a \in F^\times$, let $t_5(a) = \text{diag}(a, 1, a, 1, 1)$. For $W \in W(\sigma, \psi)$ and $t \in \mathbb{C}$, define

$$J_5(W, t) = \int_{F^\times} \int_{V_2(F) \backslash \text{Mat}_{2 \times 2}(F)} W(w_5 \begin{pmatrix} I_2 & X & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} t_5(a)) \psi(-\text{tr}(X)) \alpha^{-1}(a) |a|^t dX da$$

where $V_2(F) = \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \text{Mat}_{2 \times 2}(F) \mid x_{21} = 0 \right\}$. By the proof of Theorem 6.1 of [K11], the integral above is absolutely convergent when $\text{Re}(t) > -2$. Also it is easy to see from the definition that

$$(6.1) \quad J_5(\sigma(h_5(a,c)u_5(X, y_1, y_2, z_1, z_2))W, t) = \psi(\text{tr}(X) + z_2) \alpha(a) |a|^{-t} \cdot J_5(W, t).$$

Define a linear map $l : W(\sigma, \psi) \rightarrow \mathbb{C}$ to be

$$l(W) := J_5(-s, W), \quad W \in W(\sigma, \psi).$$

Then we know that the map l is well defined when $\text{Re}(s) < 2$. Moreover, by (6.1), we know that $l \in \text{Hom}_{H_5(F)}(\sigma, \omega_{5,s} \otimes \xi_5)$. Hence it remains to prove that $l \neq 0$.

For $W \in W(\sigma, \psi)$, let $W'(g) = W\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right)$ with $g \in \text{GL}_4(F)$. Then W' is a left (N_4, ψ) -invariant function on $\text{GL}_4(F)$. We let $W_4(\sigma, \psi)$ be the space of all such W' . We also let $W_{4,c}(\psi)$ be the space of all the left (N_4, ψ) -invariant functions on $\text{GL}_4(F)$ that are compactly supported mod $N_4(F)$. Then according to [GK], we know that

$$(6.2) \quad W_{4,c}(\psi) \subset W_4(\sigma, \psi).$$

For $W \in W(\sigma, \psi)$, $a \in F^\times$ and $x \in F$, let $f_W(a, x) = W(w_5 \begin{pmatrix} I_2 & X & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} t_5(a))$

with $X = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$. This defines a function on $F^\times \times F$, and we have

$$J_5(W, t) = \int_{F^\times} \int_F f_W(a, x) \alpha^{-1}(a) |a|^t dx da.$$

By (6.2), we know that for all $f \in C_c^\infty(F^\times \times F)$, there exists $W \in W(\sigma, \psi)$ such that $f_W = f$. Hence we know that for all $t \in \mathbb{C}$ with $\text{Re}(t) > -2$, there

exists $W \in W(\sigma, \psi)$ such that $J_5(W, t) \neq 0$. This implies that $l \neq 0$ and finishes the proof of the proposition. \square

6.2. The reduced model for $\mathrm{GL}_4(F)$. Let σ be an irreducible generic representation of $\mathrm{GL}_4(F)$, and let $H_4(F) = H_{4,0}(F) \times U_4(F)$ be a subgroup of $\mathrm{GL}_4(F)$ with

$$H_{4,0}(F) = \{h_4(a, c) = \mathrm{diag}\left(\begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix}\right) \mid a \in F^\times, c \in F\},$$

$$U_4(F) = \{u_4(X) = \begin{pmatrix} I_2 & X \\ 0 & I_2 \end{pmatrix} \mid X \in \mathrm{Mat}_{2 \times 2}(F)\}.$$

We define a character $\omega_{4,s}$ (resp. ξ_4) on $H_{4,0}(F)$ (resp. $U_4(F)$) to be $\omega_{4,s}(h_4(a, c)) = \nu^s \alpha(a)$ (resp. $\xi_4(u_4(X)) = \psi(\mathrm{tr}(X))$) where α is some unitary character of F^\times and $s \in \mathbb{C}$. Then $\omega_{4,s} \otimes \xi_4$ is a character of $H_4(F)$. We want to study the multiplicity

$$m(\sigma) := \dim(\mathrm{Hom}_{H_4(F)}(\sigma, \omega_{4,s} \otimes \xi_4)).$$

Proposition 6.2. *Assume that σ is a discrete series and $\mathrm{Re}(s) < 1$. Then $m(\sigma) \neq 0$.*

Proof. Let $W(\sigma, \psi)$ be the Whittaker model of σ and let $w_4 = \mathrm{diag}(1, w_{(21)}, 1) = w_{(1324)} \in \mathrm{GL}_4(F)$. For $a \in F^\times$, let $t_4(a) = \mathrm{diag}(a, 1, a, 1)$. For $W \in W(\sigma, \psi)$ and $t \in \mathbb{C}$, define

$$J_4(W, t) = \int_{F^\times} \int_{V_2(F) \backslash \mathrm{Mat}_{2 \times 2}(F)} W(w_4 \begin{pmatrix} I_2 & X \\ 0 & I_2 \end{pmatrix} t_4(a)) \psi(-\mathrm{tr}(X)) \alpha^{-1}(a) |a|^t dX da$$

where $V_2(F) = \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathrm{Mat}_{2 \times 2}(F) \mid x_{21} = 0 \right\}$. By Lemma 4.1 of [K11], we know that the integral above is absolutely convergent for $\mathrm{Re}(t) > -1$. Also it is easy to see from the definition that

$$(6.3) \quad J_4(\sigma(h_4(a, c)u_4(X))W, t) = \psi(\mathrm{tr}(X)) \alpha(a) |a|^{-t} \cdot J_4(W, t).$$

Define a linear map $l : W(\sigma, \psi) \rightarrow \mathbb{C}$ to be

$$l(W) := J_4(-s, W), \quad W \in W(\sigma, \psi).$$

Then we know that the map l is well defined when $\mathrm{Re}(s) < 1$. Moreover, by (6.3), we know that $l \in \mathrm{Hom}_{H_4(F)}(\sigma, \omega_{4,s} \otimes \xi_4)$. Hence it remains to prove that $l \neq 0$. The argument is similar to the GL_5 -case and we will skip it here. This finishes the proof of the proposition. \square

6.3. The reduced model for $\mathrm{GL}_3(F) \times \mathrm{GL}_3(F)$. Given an irreducible generic representation $\sigma = \sigma_1 \otimes \sigma_2$ of $\mathrm{GL}_3(F) \times \mathrm{GL}_3(F)$. Define a subgroup $H_{(3,3),0}(F) = H_{(3,3),0}(F) \times U_{(3,3)}(F)$ of $\mathrm{GL}_3(F) \times \mathrm{GL}_3(F)$ to be

$$H_{(3,3),0}(F) = \{h_{3,3}(a, c) = \begin{pmatrix} a & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} a & 0 & 0 \\ 0 & a & c \\ 0 & 0 & 1 \end{pmatrix} \mid a \in F^\times, c \in F\},$$

$$U_{(3,3)}(F) = \{u_{3,3}(x_1, x_2, y_1, y_2) = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & y_1 & y_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid x_1, x_2, y_1, y_2 \in F\}.$$

We then define a character $\omega_{(3,3),s}$ (resp. $\xi_{(3,3)}$) on $H_{(3,3),0}(F)$ (resp. $U_{(3,3)}(F)$) to be $\omega_{(3,3),s}(h_{3,3}(a, c)) = \nu^s(a)$ (resp. $\xi_{(3,3)}(u_{3,3}(x_1, x_2, y_1, y_2)) = \psi(x_2 - y_1)$) where $s \in \mathbb{C}$. Then $\omega_{(3,3),s} \otimes \xi_{(3,3)}$ is a character of $H_{(3,3)}(F)$. We want to study the multiplicity

$$m(\sigma) := \dim(\text{Hom}_{H_{(3,3)}(F)}(\sigma, \omega_{(3,3),s} \otimes \xi_{(3,3)})).$$

Proposition 6.3. *Assume that σ is a discrete series and $\text{Re}(s) < 2$. Then $m(\sigma) \neq 0$.*

Proof. Let $W(\sigma_1, \psi)$ (resp. $W(\sigma_2, \bar{\psi})$) be the Whittaker model of σ_1 (resp. σ_2). For $W_1 \in W(\sigma_1, \psi)$, $W_2 \in W(\sigma_2, \bar{\psi})$, and $t \in \mathbb{C}$, define

$$(6.4) \quad J_{3,3}(W_1, W_2, t) := \int_{F^\times} W_1\left(\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) W_2\left(\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) |a|^t da.$$

We first prove that the integral above is absolutely convergent when $\text{Re}(t) > -2$. In fact, since the Whittaker model for the contragredient of σ_2 is $\{W_2'(g) = W_2(w_{(321)}(g^t)^{-1}) \mid W_2 \in W(\sigma_2, \bar{\psi})\}$, by changing σ_2 to its contragredient, it is enough to show that the integral

$$\int_{F^\times} W_1\left(\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) W_2\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & a^{-1} \end{pmatrix}\right) |a|^t da$$

is absolutely convergent when $\text{Re}(t) > -2$. Since the central character of σ_2 is a unitary character, it is enough to show that the integral

$$(6.5) \quad \int_{F^\times} W_1\left(\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) W_2\left(\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) |a|^t da$$

is absolutely convergent when $\text{Re}(t) > -2$. By the proof of Proposition 8.3 of [JPSS], the integral

$$\int_{N_2(F) \backslash \text{GL}_2(F)} W_1\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right) W_2\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right) |\det(g)|^t dg$$

is absolutely convergent when $\text{Re}(t) > -1$. By the Iwasawa decomposition, we can rewrite the integral above as

$$\int_{K_2} \int_{F^\times} \int_{F^\times} W_1\left(\begin{pmatrix} ab & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} W_2\left(\begin{pmatrix} ab & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} |a|^{t-1} |b|^{2t} da db dk$$

where $K_2 = \text{GL}_2(\mathcal{O}_F)$. This implies that the integral (6.5) (and hence the integral (6.4)) is absolutely convergent when $\text{Re}(t) > -2$. Meanwhile, it is

easy to see from the definition that

(6.6)

$$J_{3,3}(\sigma_1\left(\begin{pmatrix} a & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)W_1, \sigma_2\left(\begin{pmatrix} a & 0 & 0 \\ 0 & a & c \\ 0 & 0 & 1 \end{pmatrix}\right)W_2, t) = |a|^{-t} J_{3,3}(W_1, W_2, t),$$

$$J_{3,3}(\sigma_1\left(\begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}\right)W_1, \sigma_2\left(\begin{pmatrix} 1 & y_1 & y_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)W_2, t) = \psi(x_2 - y_1) J_{3,3}(W_1, W_2, t).$$

For $\operatorname{Re}(s) < 2$, define a linear map $l : W(\sigma_1, \psi) \times W(\sigma_2, \bar{\psi}) \rightarrow \mathbb{C}$ to be $l(W_1, W_2) := J_{3,3}(-s, W_1, W_2)$ with $W_1 \in W(\sigma_1, \psi)$ and $W_2 \in W(\sigma_2, \bar{\psi})$. By the discussion above, we know that l is well defined and belongs to the Hom space $\operatorname{Hom}_{H_{(3,3)}(F)}(\sigma, \omega_{(3,3),s} \otimes \xi_{(3,3)})$. So it remains to show that $l \neq 0$. The argument is similar to the GL_5 -case and we will skip it here. This finishes the proof of the proposition. \square

6.4. The reduced model for $\operatorname{GL}_3(F) \times \operatorname{GL}_2(F)$. Given an irreducible generic representation $\sigma = \sigma_1 \otimes \sigma_2$ of $\operatorname{GL}_3(F) \times \operatorname{GL}_2(F)$. Define a subgroup $H_{(3,2)}(F) = H_{(3,2),0}(F) \times U_{(3,2)}(F)$ of $\operatorname{GL}_3(F) \times \operatorname{GL}_2(F)$ to be

$$H_{(3,2),0}(F) = \{h_{3,2}(a, c) = \begin{pmatrix} a & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix} \mid a \in F^\times, c \in F\},$$

$$U_{(3,2)}(F) = \{u_{3,2}(x, y) = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \times I_2 \mid x, y \in F\}.$$

We then define a character $\omega_{(3,2),s}$ (resp. $\xi_{(3,2)}$) on $H_{(3,2),0}(F)$ (resp. $U_{(3,2)}(F)$) to be $\omega_{(3,2),s}(h_{3,2}(a, c)) = \nu^s(a)$ (resp. $\xi_{(3,2)}(u_{3,2}(x, y)) = \psi(y)$) where $s \in \mathbb{C}$. Then $\omega_{(3,2),s} \otimes \xi_{(3,2)}$ is a character of $H_{(3,2)}(F)$. We want to study the multiplicity

$$m(\sigma) := \dim(\operatorname{Hom}_{H_{(3,2)}(F)}(\sigma, \omega_{(3,2),s} \otimes \xi_{(3,2)}).$$

The goal is to prove the following proposition.

Proposition 6.4. (1) *If σ_1 is supercuspidal, then $m(\sigma) \neq 0$.*

(2) *If σ_1 and σ_2 are discrete series, assume that $\operatorname{Re}(s) < \frac{3}{2}$. Then $m(\sigma) \neq 0$.*

Before we prove the proposition, we first recall some results from [JPSS] for the Rankin-Selberg integral of $\operatorname{GL}_3(F) \times \operatorname{GL}_2(F)$. Let $W(\sigma_1, \psi)$ (resp. $W(\sigma_2, \bar{\psi})$) be the Whittaker model of σ_1 (resp. σ_2). For $W_1 \in W(\sigma_1, \psi)$, $W_2 \in W(\sigma_2, \bar{\psi})$, and $t \in \mathbb{C}$, define

$$I_{3,2}(t, W_1, W_2) := \int_{N_2(F) \backslash \operatorname{GL}_2(F)} W_1\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right) W_2(g) |\det(g)|^{t-1/2} dg.$$

In general, the integral above is only absolutely convergent when $\operatorname{Re}(t) \gg 0$, and it has a meromorphic continuation to the whole complex plane. The following proposition has been proved in Section 8 of [JPSS].

Proposition 6.5. (1) *If σ_1 is supercuspidal, then the integral $I_{3,2}(t, W_1, W_2)$ is absolutely convergent for all $t \in \mathbb{C}$. Moreover, for all $t \in \mathbb{C}$, there exists $W_1 \in W(\sigma_1, \psi)$ and $W_2 \in W(\sigma_2, \bar{\psi})$ such that $I_{3,2}(t, W_1, W_2) \neq 0$.*

(2) *If σ_1 and σ_2 are discrete series, then the integral $I_{3,2}(t, W_1, W_2)$ is absolutely convergent when $\operatorname{Re}(t) > 0$. Moreover, for all $t \in \mathbb{C}$ with $\operatorname{Re}(t) > 0$, there exists $W_1 \in W(\sigma_1, \psi)$ and $W_2 \in W(\sigma_2, \bar{\psi})$ such that $I_{3,2}(t, W_1, W_2) \neq 0$.*

We can decompose the integral $I_{3,2}(t, W_1, W_2)$ as

$$I_{3,2}(t, W_1, W_2) = \int_K \int_{F^\times} \int_{F^\times} W_1 \left(\begin{pmatrix} ab & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \right) W_2 \left(\begin{pmatrix} ab & 0 \\ 0 & b \end{pmatrix} k \right) |a|^{t-3/2} |b|^{2t-1} da db dk.$$

For $W_1 \in W(\sigma_1, \psi)$ and $W_2 \in W(\sigma_2, \bar{\psi})$, define

$$J_{3,2}(t, W_1, W_2) := \int_{F^\times} W_1 \left(\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) W_2 \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) |a|^t da.$$

Then it is easy to see from the definition that

$$J_{3,2}(t, \sigma_1 \left(\begin{pmatrix} a & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) W_1, \sigma_2 \left(\begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix} \right) W_2) = |a|^{-t} J_{3,2}(t, W_1, W_2),$$

$$(6.7) \quad J_{3,2}(t, \sigma_1 \left(\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right) W_1, W_2) = \psi(y) J_{3,2}(t, W_1, W_2).$$

Meanwhile, the above proposition implies the following corollary.

Corollary 6.6. (1) *If σ_1 is supercuspidal, then the integral $J_{3,2}(t, W_1, W_2)$ is absolutely convergent for all $t \in \mathbb{C}$. Moreover, for all $t \in \mathbb{C}$, there exists $W_1 \in W(\sigma_1, \psi)$ and $W_2 \in W(\sigma_2, \bar{\psi})$ such that $J_{3,2}(t, W_1, W_2) \neq 0$.*

(2) *If σ_1 and σ_2 are discrete series, then the integral $J_{3,2}(t, W_1, W_2)$ is absolutely convergent when $\operatorname{Re}(t) > -\frac{3}{2}$. Moreover, for all $t \in \mathbb{C}$ with $\operatorname{Re}(t) > -\frac{3}{2}$, there exists $W_1 \in W(\sigma_1, \psi)$ and $W_2 \in W(\sigma_2, \bar{\psi})$ such that $J_{3,2}(t, W_1, W_2) \neq 0$.*

Now we are ready to prove Proposition 6.4. We first consider the case when σ_1 is supercuspidal, we want to show that $m(\sigma) \neq 0$. Define a linear map $l : W(\sigma_1, \psi) \times W(\sigma_2, \bar{\psi}) \rightarrow \mathbb{C}$ to be

$$l(W_1, W_2) := J(-s, W_1, W_2), \quad W_1 \in W(\sigma_1, \psi), W_2 \in W(\sigma_2, \bar{\psi}).$$

By Corollary 6.6(1), we know that l is well defined and nonzero for all $s \in \mathbb{C}$. Hence it is enough to show that l belongs to the Hom space $\text{Hom}_{H_{(3,2)}(F)}(\sigma, \omega_{(3,2),s} \otimes \xi_{(3,2)})$. But this just follows from (6.7).

We then consider the case when σ_1 and σ_2 are discrete series. Assume that $\text{Re}(s) < \frac{3}{2}$. We still define the linear map $l : W(\sigma_1, \psi) \times W(\sigma_2, \bar{\psi}) \rightarrow \mathbb{C}$ as above. By Corollary 6.6(2), we know that l is well defined and nonzero. Meanwhile, (6.7) shows that l belongs to the Hom space $\text{Hom}_{H_{(3,2)}(F)}(\sigma, \omega_{(3,2),s} \otimes \xi_{(3,2)})$. This implies that $m(\sigma) \neq 0$ and finishes the proof of Proposition 6.4.

7. THE PROOF OF THEOREM 1.6

7.1. The problem. In this section, we are going to prove Theorem 1.6. Let π be an irreducible generic representation of $\text{GL}_6(F)$ with central character χ^2 . Assume that $\pi_D = 0$. Our goal is to prove $m(\pi) = 1$. Since $m(\pi) \leq 1$, it is enough to show that

$$(7.1) \quad m(\pi) \neq 0.$$

Since π is generic, π is the parabolic induction of an essential discrete series $\otimes_{i=1}^k \sigma_i \nu^{s_i}$ of $\times_{i=1}^k \text{GL}_{n_i}(F)$ where σ_i is a discrete series of $\text{GL}_{n_i}(F)$, $s_i \in \mathbb{R}$ and $\sum_{i=1}^k n_i = 6$. Since $\pi_D = 0$, there exists $1 \leq i \leq k$ such that n_i is an odd number. Without loss of generality, we assume that $n_1 \geq n_2 \geq \cdots \geq n_k$.

We first deal with an easy case. If $n_1 \leq 2$, then we can find a generic representation $\tau = \tau_1 \otimes \tau_2 \otimes \tau_3$ of $\text{GL}_2(F) \times \text{GL}_2(F) \times \text{GL}_2(F)$ such that π is the parabolic induction of τ . Let τ_D be the local Jacquet-Langlands correspondence of τ to $\text{GL}_1(D) \otimes \text{GL}_1(D) \otimes \text{GL}_1(D)$. Since $\pi_D = 0$, we have $\tau_D = 0$. By the result of Prasad for the trilinear GL_2 model, we know that the multiplicity $m(\tau)$ for the trilinear GL_2 model is equal to 1. Then by the same argument as in Section 5.1 of the previous paper [Wan16b], together with our assumption on the generalized Jacquet integrals, we have $m(\tau) \neq 0 \Rightarrow m(\pi) \neq 0$. This implies that $m(\pi) \neq 0$ and proves Theorem 1.6 in this case.

From now on, we assume that $n_1 > 2$. Then there are only five possibilities for the set $\{n_1, n_2, \dots, n_k\}$:

$$\{5, 1\}, \{4, 1, 1\}, \{3, 3\}, \{3, 2\}, \{3, 1, 1, 1\}.$$

We will study this five cases for the rest of this section. The idea is to apply the open orbit method together with the results in Section 6 for the reduced models to prove (7.1).

7.2. The $\{5, 1\}$ case. In this subsection, we study the case when $\{n_1, n_2, \dots, n_k\} = \{5, 1\}$. Then we know that $\pi = I_{\bar{P}_{5,1}}^G(\sigma \nu^{s_1} \otimes \eta \nu^{s_2})$ where σ is a discrete series of $\text{GL}_5(F)$ and η is a unitary character of F^\times . Since the central character of π is unitary, we have $s_2 = -5s_1$. If we set $s = -s_1$, we then have $\pi = I_{\bar{P}_{5,1}}^G(\sigma \nu^{-s} \otimes \eta \nu^{5s})$. If $s = 0$, π is a tempered representation and (7.1) has already been proved in the previous paper [Wan16a]. So from now on,

we assume that $s \neq 0$. Without loss of generality, we may assume that $s > 0$ (otherwise we just need to change π to its contragredient). We first study the reduced model. Define the following subgroups of $\mathrm{GL}_5(F)$:

$$H'_{5,0}(F) = \{h'_5(a, b, c) = \mathrm{diag}\left(\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}, \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}, a\right) \mid a, b \in F^\times, c \in F\}, \quad H'_5 = H'_{5,0} \rtimes U_5$$

where the subgroup U_5 has been defined in Section 6.1. We define a character $\omega'_5 \otimes \xi'_5$ on $H'_5(F)$ to be

$$\omega'_5 \otimes \xi'_5(h'_5(a, b, c)u_5(X, y_1, y_2, z_1, z_2)) := \chi\nu^{-3/2+3s}(a)\chi\eta^{-1}\nu^{3/2-3s}(b)\psi(\mathrm{tr}(X)+z_1).$$

Lemma 7.1. *The Hom space $\mathrm{Hom}_{H'_5(F)}(\sigma, \omega'_5 \otimes \xi'_5)$ is nonzero.*

Proof. By conjugating $H'_5(F)$ by the Weyl element $\mathrm{diag}(w_{(21)}, w_{(21)}, 1)$, we can change the group $H'_5(F)$ to $H''_5(F) = H''_{5,0}(F) \rtimes U_5(F)$ with

$$H''_{5,0}(F) = \{h''_5(a, b, c) = \mathrm{diag}\left(\begin{pmatrix} b & c \\ 0 & a \end{pmatrix}, \begin{pmatrix} b & c \\ 0 & a \end{pmatrix}, a\right) \mid a, b \in F^\times, c \in F\}.$$

And the character $\omega''_5 \otimes \xi''_5$ on $H''_5(F)$ is defined to be

$$\omega''_5 \otimes \xi''_5(h''_5(a, b, c)u_5(X, y_1, y_2, z_1, z_2)) := \chi\nu^{-3/2+3s}(a)\chi\eta^{-1}\nu^{3/2-3s}(b)\psi(\mathrm{tr}(X)+z_2).$$

Note that the group $H''_5(F)$ is the product of the group $H_5(F)$ in Section 6.1 and the center of $\mathrm{GL}_5(F)$. Since the central character of π is χ^2 , the central character of σ is $\chi^2\eta^{-1}$. Hence by eliminate the center of $\mathrm{GL}_5(F)$ in $H''_5(F)$, we only need to show that the Hom space $\mathrm{Hom}_{H_5(F)}(\sigma, \omega_{5,0} \otimes \xi_{5,0})$ is nonzero where the character $\omega_{5,0} \otimes \xi_{5,0}$ is defined to be

$$\omega_{5,0} \otimes \xi_{5,0}(h_5(a, c)u_5(X, y_1, y_2, z_1, z_2)) = \nu^{3/2-3s}\chi\eta^{-1}(a)\psi(\mathrm{tr}(X) + z_2).$$

Since $s > 0$ and $\chi\eta^{-1}$ is a unitary character, we know the Hom space is nonzero by Proposition 6.1. This finishes the proof of the lemma. \square

Fix a nonzero element $l_0 \in \mathrm{Hom}_{H'_5(F)}(\sigma, \omega'_5 \otimes \xi'_5)$. We realize the representation π on the function space

$$\begin{aligned} \pi = \{ & f : \mathrm{GL}_6(F) \rightarrow \sigma \mid f \text{ locally constant, } f\left(\begin{pmatrix} h & 0 \\ X & a \end{pmatrix}g\right) = \nu^{-1/2-s}(\det(h))\nu^{5/2+5s}\eta(a) \\ & \cdot \sigma(h)f(g), \forall g \in \mathrm{GL}_6(F), h \in \mathrm{GL}_5(F), a \in F^\times, X \in \mathrm{Mat}_{1 \times 5}(F)\}, \end{aligned}$$

while the $\mathrm{GL}_6(F)$ -action is just the right translation. Define

$$U'(F) = \{u'(x_1, x_2, x_3, x_4) = \begin{pmatrix} I_4 & 0 & X_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid X_0 = (x_1, x_2, x_3, x_4)^t, x_i \in F\}.$$

And we define a character ξ' on $U'(F)$ to be $\xi'(u'(x_1, x_2, x_3, x_4)) = \psi(-x_4)$. Then we define a map $l_1 : \pi \rightarrow \sigma$ to be

$$l_1(f) := \int_{U'(F)} f(u)\xi'(u)du, \quad f \in \pi.$$

Since $s > 0$, the integral above is absolutely convergent by Proposition 2.1 of [Wan16b]. We then define the map $l_2 : \pi \rightarrow \mathbb{C}$ to be $l_2(f) := l_0(l_1(f))$.

Lemma 7.2. (1) $l_2(\pi(\text{diag}(\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}, \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}, \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}))f) = \chi(ab)\nu(\frac{b}{a})l_2(f)$

for all $f \in \pi$, $a, b \in F^\times$ and $c \in F$.

(2) $l_2(\pi(u)f) = \xi(u)l_2(f)$ for all $f \in \pi$ and $u \in U(F)$.

Proof. (1) By the definition of l_1 and an easy change of variable, we know that

$$l_1(\pi(\text{diag}(a, b, a, b, a, b))f) = \nu^{3s-1/2}(\frac{b}{a})\eta(b) \cdot \sigma(\text{diag}(a, b, a, b, a))l_1(f)$$

for all $f \in \pi$ and $a, b \in F$. Then since $l_2(f) = l_0(l_1(f))$ and $l_0 \in \text{Hom}_{H'_5(F)}(\sigma, \omega'_5 \otimes \xi'_5)$, we have

$$l_2(\pi(\text{diag}(a, b, a, b, a, b))f) = \chi(ab)\nu(\frac{b}{a})l_2(f).$$

Hence it remains to prove that for all $f \in \pi$ and $c \in F$, we have

$$(7.2) \quad l_2(\pi(n(c))f) = l_2(f)$$

where $n(c) = \text{diag}(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix})$. Fix $f \in \pi$ and $c \in F$.

Choose $k > 0$ large enough such that $c \in \varpi_F^{-k}\mathcal{O}_F$. For $n > 0$, let $U'_n(F) = \{u'(x_1, x_2, x_3, x_4) \mid x_1, x_3 \in \varpi_F^{-n}\mathcal{O}_F, x_2, x_4 \in \varpi_F^{-2n}\mathcal{O}_F\}$. Then $U'_n(F)$ is an open compact subgroup of $U'(F)$ with $U'(F) = \cup_{n>0}U'_n(F)$. By the definition of l_2 , we have

$$l_2(f) = \lim_{n \rightarrow \infty} l_0\left(\int_{U'_n(F)} f(u)\xi'(u)du\right),$$

$$l_2(\pi(n(c))f) = \lim_{n \rightarrow \infty} l_0\left(\int_{U'_n(F)} f(un(c))\xi'(u)du\right).$$

Hence in order to prove (7.2), it is enough to show that for all $n > k$, we have

$$l_0\left(\int_{U'_n(F)} f(u)\xi'(u)du\right) = l_0\left(\int_{U'_n(F)} f(un(c))\xi'(u)du\right).$$

Since $U'_n(F)$ is compact and f is locally constant, we only need to show that for all $n > k$, we have

$$(7.3) \quad \int_{U'_n(F)} l_0(f(u))\xi'(u)du = \int_{U'_n(F)} l_0(f(un(c)))\xi'(u)du.$$

For $u = u'(x_1, x_2, x_3, x_4)$, we have

$$un(c) = n(c)\text{diag}(u'_5(0, cx_1, cx_2 - c^2x_1, cx_3, cx_4 - c^2c_3), 1)u'(x_1, x_2 - cx_1, x_3, x_4 - cx_3).$$

This implies that $f(un(c)) = \sigma(h'_5(1, 1, c)u'_5(0, cx_1, cx_2 - c^2x_1, cx_3, cx_4 - c^2c_3))f(u'(x_1, x_2 - cx_1, x_3, x_4 - cx_3))$. Hence we have

$$l_0(f(un(c))) = \psi(cx_3)l_0(f(u'(x_1, x_2 - cx_1, x_3, x_4 - cx_3))).$$

As a result, the right hand side of (7.3) is equal to

$$\int_{\varpi_F^{-n}\mathcal{O}_F} \int_{\varpi_F^{-n}\mathcal{O}_F} \int_{\varpi_F^{-2n}\mathcal{O}_F} \int_{\varpi_F^{-2n}\mathcal{O}_F} l_0(f(u'(x_1, x_2 - cx_1, x_3, x_4 - cx_3)))\psi(cx_3 - x_4)dx_2dx_4dx_1dx_3.$$

If we change variables $y_2 = x_2 - cx_1$ and $y_4 = x_4 - cx_3$, we will still have $y_2, y_4 \in \varpi_F^{-2n}\mathcal{O}_F$ since $c \in \varpi_F^{-n}\mathcal{O}_F$. Hence the right hand side of (7.3) is equal to

$$\begin{aligned} & \int_{\varpi_F^{-n}\mathcal{O}_F} \int_{\varpi_F^{-n}\mathcal{O}_F} \int_{\varpi_F^{-2n}\mathcal{O}_F} \int_{\varpi_F^{-2n}\mathcal{O}_F} l_0(f(u'(x_1, y_2, x_3, y_4)))\psi(-y_4)dy_2dy_4dx_1dx_3 \\ &= \int_{U'_n(F)} l_0(f(u))\xi'(u)du. \end{aligned}$$

This proves (7.3) and finishes the proof of (1).

(2) For $u \in U(F)$, we can write $u = u'u''$ with $u' \in U'(F)$ and $u'' \in U''(F) := \{\text{diag}(u_5, 1) \mid u_5 \in U_5(F)\}$. Hence we only need to verify (2) for u' and u'' . For $u' \in U'(F) \subset U(F)$, by the definition of l_1 , we have

$$l_1(\pi(u')f) = \xi'(u')^{-1}l_1(f) = \xi(u')l_1(f), \quad \forall f \in \pi.$$

This implies that $l_2(\pi(u')f) = l_0(l_1(\pi(u')f)) = \xi(u')l_2(f)$.

On the other hand, for $u'' = \text{diag}(u_5, 1) \in U''(F)$, since $U''(F)$ normalizes the group $U'(F)$ and preserves the character ξ' of $U'(F)$, by the definition of l_1 and an easy change of variable, we know that

$$l_1(\pi(u'')f) = \sigma(u_5)l_1(f).$$

Since $l_0 \in \text{Hom}_{H'_5(F)}(\sigma, \omega'_5 \otimes \xi'_5)$, we have

$$l_2(\pi(u'')f) = l_0(\sigma(u_5)l_1(f)) = \xi'_5(u_5)l_0(l_1(f)) = \xi(u'')l_2(f).$$

This proves (2) and finishes the proof of the lemma. \square

Now we are ready to prove (7.1). Define the map $l : \pi \rightarrow \mathbb{C}$ to be

$$l(f) = \int_{K_2} l_2(\pi(\text{diag}(k, k, k))f)\chi^{-1}(\det(k))dk, \quad f \in \pi.$$

Then by Lemma 7.2 and Proposition 2.4, we know that

$$l(\pi(h)f) = \omega \otimes \xi(h)l(f), \quad \forall h \in H(F), f \in \pi.$$

As a result, in order to show that $m(\pi) \neq 0$, it is enough to show that $l \neq 0$. Choose $v_0 \in \sigma$ such that $l_0(v_0) = 1$. Fix $n > 0$ such that the character ψ is trivial on $\varpi_F^n\mathcal{O}_F$ and

$$\sigma(k)v_0 = v_0, \quad \forall k \in I_5 + \varpi_F^n \text{Mat}_{5 \times 5}(F).$$

We define an element $f_0 \in \pi$ to be

$$f_0(g) = \nu^{-1/2-s}(\det(h))\nu^{5/2+5s}\eta(a) \cdot \sigma(h)v_0$$

if $g = \begin{pmatrix} h & 0 \\ X & a \end{pmatrix} u$ with $h \in \mathrm{GL}_5(F)$, $a \in F^\times$, $X \in \mathrm{Mat}_{5 \times 1}(F)$, $u \in N_{5,1}(\varpi_F^n \mathcal{O}_F)$, and $f_0(g) = 0$ otherwise. Here $N_{5,1}(\varpi_F^n \mathcal{O}_F) = \left\{ \begin{pmatrix} I_5 & X \\ 0 & 1 \end{pmatrix} \mid X \in \mathrm{Mat}_{5 \times 1}(\varpi_F^n \mathcal{O}_F) \right\}$. Then the function f_0 is right $N_{5,1}(\varpi_F^n \mathcal{O}_F)$ -invariant. We want to show that $l(f_0) \neq 0$. We first prove a lemma.

Lemma 7.3. *For all $h \in \mathrm{GL}_2(F)$, we have*

$$(7.4) \quad l_2(\pi(\mathrm{diag}(h, h, h))f_0)\chi^{-1}(\det(h)) \geq 0.$$

Proof. By Lemma 7.2, it is enough to consider the case when $h = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. In this case, the left hand side of (7.4) is equal to $l_0(l_1(\pi(\mathrm{diag}(h, h, h))f_0))$. Hence it is enough to show that there exists $c \geq 0$ such that

$$(7.5) \quad l_1(\pi(\mathrm{diag}(h, h, h))f_0) = cv_0.$$

By the definition of l_1 , we have

$$(7.6) \quad l_1(\pi(\mathrm{diag}(h, h, h))f_0) = \int_{U'(F)} f_0(u \cdot \mathrm{diag}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)) \xi'(u) du.$$

For $u = u'(x_1, x_2, x_3, x_4)$, we have $u \cdot \mathrm{diag}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \mathrm{diag}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, I_2\right) u'(x_1, x_2 - xx_1, x_3, x_4 - xx_3) \mathrm{diag}(I_2, I_2, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})$.

By the definition of f_0 , we know that $f_0(u \cdot \mathrm{diag}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right))$ is nonzero only if

$$x, x_1, x_2 - xx_1, x_3, x_4 - xx_3 \in \varpi_F^n \mathcal{O}_F \iff x, x_1, x_2, x_3, x_4 \in \varpi_F^n \mathcal{O}_F.$$

Moreover, if this holds, we have

$$\begin{aligned} f_0(u \cdot \mathrm{diag}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)) &= f_0(\mathrm{diag}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, I_2\right)) \\ &= \sigma(\mathrm{diag}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1\right)) v_0 = v_0, \end{aligned}$$

and $\xi'(u) = \psi(-x_4) = 1$. As a result, when $x \in \varpi_F^n \mathcal{O}_F$, we can rewrite (7.6) as

$$(7.7) \quad l_1(\pi(\mathrm{diag}(h, h, h))f_0) = \int_{U'(\varpi_F^n \mathcal{O}_F)} v_0 du = \mathrm{vol}(\varpi_F^n \mathcal{O}_F)^4 v_0$$

where $U'(\varpi_F^n \mathcal{O}_F) = \{u'(x_1, x_2, x_3, x_4) \mid x_1, x_2, x_3, x_4 \in \varpi_F^n \mathcal{O}_F\}$. And when $x \notin \varpi_F^n \mathcal{O}_F$, we have $l_1(\pi(\mathrm{diag}(h, h, h))f_0) = 0$. This proves (7.5) and finishes the proof of the lemma. \square

Now we are ready to show $l(f_0) \neq 0$. By the definition of l , we have

$$l(f_0) = \int_{K_2} l_2(\pi(\text{diag}(k, k, k))f_0)\chi^{-1}(\det(k))dk.$$

By the lemma above, we have

$$l_2(\pi(\text{diag}(k, k, k))f_0)\chi^{-1}(\det(k)) \geq 0$$

for all $k \in K_2$. Moreover, when $k = I_2$, we have

$$l_2(\pi(\text{diag}(k, k, k))f_0)\chi^{-1}(\det(k)) = l_2(f_0) = \text{vol}(\varpi_F^n \mathcal{O}_F)^4 l_0(v_0) = \text{vol}(\varpi_F^n \mathcal{O}_F)^4 \neq 0.$$

This proves that $l(f_0) > 0$ and finishes the proof of (7.1) for the $\{5, 1\}$ case.

7.3. The $\{4, 1, 1\}$ case. In this subsection, we study the case when $\{n_1, n_2, \dots, n_k\} = \{4, 1, 1\}$. Then $\pi = I_{\bar{P}_{4,2}}^G(\tau_1 \nu^{s_1} \otimes \tau_2)$ where τ_1 is a discrete series of $\text{GL}_4(F)$ and $\tau_2 = I_{B_2}^{\text{GL}_2}(\eta_1 \nu^{s_2} \otimes \eta_2 \nu^{s_3})$ is an irreducible generic representation of $\text{GL}_2(F)$ with η_1, η_2 being unitary characters of F^\times and $s_i \in \mathbb{R}$. Let $\tau = \tau_1 \nu^{s_1} \otimes \tau_2$ and let $m(\tau)$ be the multiplicity of the middle model. Then by the same argument as in Section 5.1 of [Wan16b], together with our assumption on the generalized Jacquet integrals, we have $m(\tau) \neq 0 \Rightarrow m(\pi) \neq 0$. Hence it is enough to show that

$$(7.8) \quad m(\tau) \neq 0.$$

Since the multiplicity of the middle model is invariant under the unramified twist, we may assume that $s_1 = 0$ and $s_2 + s_3 = 0$. Without loss of generality, we assume that $s = s_2 \geq 0$, then $\tau = \tau_1 \otimes \tau_2$ with $\tau_2 = I_{B_2}^{\text{GL}_2}(\eta_1 \nu^s \otimes \eta_2 \nu^{-s})$. Then the central character ω_{τ_1} of τ_1 is $\chi^2(\eta_1 \eta_2)^{-1}$. We realize the representation τ_2 on the function space

$$\begin{aligned} \tau_2 &= \{f : \text{GL}_2(F) \rightarrow \mathbb{C} \mid f \text{ locally constant, } f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} g\right) \\ &= \eta_1(a)\eta_2(b)\nu\left(\frac{a}{b}\right)^{s+1/2} f(g), \forall g \in \text{GL}_2(F), a, b \in F^\times, x \in F\}, \end{aligned}$$

while the $\text{GL}_2(F)$ -action is just the right translation. We define the following subgroups of $\text{GL}_4(F)$:

$$H'_{4,0}(F) = \{h'_4(a, b, c) = \text{diag}\left(\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}, \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}\right) \mid a, b \in F^\times, c \in F\}, \quad H'_4 = H'_{4,0} \rtimes U_4$$

where the subgroup U_4 has been defined in Section 6.2. We define a character $\omega'_4 \otimes \xi_4$ on $H'_4(F)$ to be

$$\omega'_4 \otimes \xi_4(h'_4(a, b, c)u_4(X)) := \chi\eta_1^{-1}\nu^{1/2-s}(a)\chi\eta_2^{-1}\nu^{s-1/2}(b)\psi(\text{tr}(X)).$$

The following lemma is an easy consequence of Proposition 6.2.

Lemma 7.4. *The Hom space $\text{Hom}_{H'_4(F)}(\tau_1, \omega'_4 \otimes \xi_4)$ is nonzero.*

By the lemma above, we can choose a nonzero element $l_0 \in \text{Hom}_{H'_4(F)}(\tau_1, \omega'_4 \otimes \xi_4)$, and we define a linear map $l_1 : \tau_1 \otimes \tau_2 \rightarrow \mathbb{C}$ to be

$$l_1(v \otimes f) := l_0(v_1)f(1), \quad v \in \tau_1, f \in \tau_2.$$

Then l satisfies the following two equations

$$(7.9) \quad l_1(\tau_1(h'_4(a, b, c))v \otimes \tau_2\left(\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}\right)f) = \nu\left(\frac{a}{b}\right)\chi(ab)l_1(v \otimes f),$$

$$(7.10) \quad l_1(u_4(X)v \otimes f) = \psi(\text{tr}(X))l_1(v \otimes f).$$

Finally, we define the linear map $l : \tau_1 \otimes \tau_2 \rightarrow \mathbb{C}$ to be

$$l(v \otimes f) := \int_{K_2} l_1(\tau_1\left(\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\right)v \otimes \tau_2(k)f)\chi^{-1}(\det(k))dk, \quad v \in \tau_1, f \in \tau_2.$$

Combining (7.9), (7.10) and Proposition 2.2, we have

$$(7.11) \quad l(\tau_1\left(\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}\right)v \otimes \tau_2(g)f) = \chi(\det(g))l(v \otimes f),$$

$$(7.12) \quad l(u_4(X)v \otimes f) = \psi(\text{tr}(X))l(v \otimes f).$$

The equations (7.11) and (7.12) imply that l is an element in the Hom space for the middle model. Hence in order to prove (7.8), we only need to show that $l \neq 0$. The argument is similar to the $\{5, 1\}$ case and we will skip it here. This proves (7.8) and finishes the proof of (7.1) for the $\{4, 1, 1\}$ case.

7.4. The $\{3, 3\}$ case. In this subsection, we study the case when $\{n_1, n_2, \dots, n_k\} = \{3, 3\}$. Then $\pi = I_{P_{3,3}}^G(\sigma_1\nu^{s_1} \otimes \sigma_2\nu^{s_2})$ where σ_i is a discrete series of $\text{GL}_3(F)$ and $s_i \in \mathbb{R}$. Since the central character of π is unitary, we have $s_2 = -s_1$. Set $s = -s_1$. We have $\pi = I_{P_{3,3}}^G(\sigma_1\nu^{-s} \otimes \sigma_2\nu^s)$. If $s = 0$, π is a tempered representation and (7.1) has already been proved in the previous paper [Wan16a]. From now on, we assume that $s \neq 0$. Without loss of generality, we assume that $s > 0$. By a similar argument as in the $\{5, 1\}$ case, in order to prove (7.1), we only need to show that the reduced model associated to $\text{GL}_3(F) \times \text{GL}_3(F)$ has nonzero multiplicity. In this case, the reduced model is defined as follows. Let $H'_{(3,3)}(F) = H'_{(3,3),0}(F) \times U_{(3,3)}(F)$ be a subgroup of $\text{GL}_3(F) \times \text{GL}_3(F)$ where

$$H'_{(3,3),0}(F) = \{h'_{3,3}(a, b, c) = \begin{pmatrix} a & 0 & 0 \\ c & b & 0 \\ 0 & 0 & a \end{pmatrix} \times \begin{pmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & c & b \end{pmatrix} \mid a, b \in F^\times, c \in F\},$$

and the subgroup $U_{(3,3)}$ has been defined in Section 6.3. The character $\omega'_{(3,3)} \otimes \xi'_{(3,3)}$ on $H_{3,3}(F) := H_{3,3,0}(F) \times U_{3,3}(F)$ is given by

$$\omega'_{(3,3)} \otimes \xi'_{(3,3)} : h(a, b, c)u(x_1, x_2, y_1, y_2) \mapsto \nu^{3/2}\left(\frac{b}{a}\right)\chi(ab)\psi(x_1 + y_2).$$

The reduced model is just $(H'_{3,3}(F), \omega'_{(3,3)} \otimes \xi'_{(3,3)})$. Then as in $\{5, 1\}$ case,

after we conjugate the reduced model by the Weyl element $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times$

$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and then get rid of the center, the reduced model is equivalent

to the model $(H_{(3,3)}(F), \omega''_{(3,3)} \otimes \xi_{(3,3)})$ where the group $H_{(3,3)}$ is defined in Section 6.3, and the character $\omega''_{(3,3)} \otimes \xi_{(3,3)}$ is given by

$$\omega''_{(3,3)} \otimes \xi_{(3,3)} : h_{3,3}(a, c)u_{3,3}(x_1, x_2, y_1, y_2) \mapsto \nu^{3/2}(a)\chi(a)\psi(x_2 - y_1).$$

Up to twist σ_1 by the unitary character χ^{-1} , we can eliminate the character χ in the expression of $\omega''_{(3,3)} \otimes \xi_{(3,3)}$. We need to show that the representation $\sigma_1\nu^{-s} \otimes \sigma_2\nu^s$ is $(H_{(3,3)}(F), \omega''_{(3,3)} \otimes \xi_{(3,3)})$ -distinguished. This is equivalent to show that the representation $\sigma_1 \otimes \sigma_2$ is $(H_{(3,3)}(F), \omega_{(3,3), 3/2-s} \otimes \xi_{(3,3)})$ -distinguished where the character $\omega_{(3,3), 3/2-s} \otimes \xi_{(3,3)}$ is defined in Section 6.3. Since $s < 0$, this will be a consequence of Proposition 6.3. Hence we have proved (7.1) for the $\{3, 3\}$ case.

7.5. The $\{3, 2, 1\}$ case. In this subsection, we study the case when $\{n_1, n_2, \dots, n_k\} = \{3, 2, 1\}$. Then π is the parabolic induction of some essential discrete series $\sigma_1\nu^{s_1} \otimes \sigma_2\nu^{s_2} \otimes \eta\nu^{s_3}$ of $\mathrm{GL}_3(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_1(F)$ where σ_1 (resp. σ_2) is a discrete series of $\mathrm{GL}_3(F)$ (resp. $\mathrm{GL}_2(F)$), η is a unitary character of F^\times and $s_i \in \mathbb{R}$. Up to change π to its contragredient, we may assume that $s_1 \leq s_3$. Since the central character of π is unitary, we have $3s_1 + 2s_2 + s_3 = 0$, together with the assumption that $s_1 \leq s_3$, we have

$$(7.13) \quad s_1 + s_2 + s_3 \geq 0.$$

Let $\tau_1 = I_{\bar{P}_{3,1}}^{\mathrm{GL}_4}(\sigma_1\nu^{s_1} \otimes \eta\nu^{s_3})$, $\tau_2 = \sigma_2\nu^{s_2}$ and $\tau = \tau_1 \otimes \tau_2$. Then τ is an irreducible generic representation of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$ and π is the parabolic induction of τ . Let $m(\tau)$ be the multiplicity for the middle model. As in the $\{4, 1, 1\}$ case, it is enough to show that

$$(7.14) \quad m(\tau) \neq 0.$$

Then by a similar argument as in the $\{5, 1\}$ case, in order to prove (7.14) we only need to show that the reduced model has nonzero multiplicity.

Remark 7.5. *In this case, it is not clear to us how to directly relate the multiplicity of the Ginzburg-Rallis model with the reduced model associated to $\mathrm{GL}_3(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_1(F)$. This is why we reduce to the middle model case first.*

The reduced model is defined as follows. Define the following subgroups of $\mathrm{GL}_3(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_1(F)$:

$$H_{3,2,1,0}(F) = \{h_{3,2,1}(a, b, c) = \begin{pmatrix} a & 0 & 0 \\ c & b & 0 \\ 0 & 0 & a \end{pmatrix} \times \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \times (b) \mid a, b \in F^\times, c \in F\},$$

$$U_{3,2,1,0}(F) = \{u_{3,2,1}(x, y) = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \times I_2 \times (1) \mid x, y \in F\}.$$

The character $\omega_{3,2,1} \otimes \xi_{3,2,1}$ on $H_{3,2,1}(F) := H_{3,2,1,0}(F) \rtimes U_{3,2,1}(F)$ is given by

$$\omega_{3,2,1} \otimes \xi_{3,2,1} : h_{3,2,1}(a, b, c)u_{3,2,1}(x, y) \mapsto \nu\left(\frac{b}{a}\right)\chi(ab)\psi(x).$$

The reduced model is just $(H_{3,2,1}(F), \omega_{3,2,1} \otimes \xi_{3,2,1})$. After we conjugate the reduced model by the Weyl element $w_{(213)} \times w_{(21)} \times (1)$ and get rid of the center part and the GL_1 -part, the reduced model is equivalent to the model $(H_{(3,2)}(F), \omega'_{(3,2)} \otimes \xi_{(3,2)})$ of $\mathrm{GL}_3(F) \times \mathrm{GL}_2(F)$ where the subgroup $H_{(3,2)}(F)$ is defined in Section 6.4, and the character $\omega'_{(3,2)} \otimes \xi_{(3,2)}$ is given by

$$\omega'_{(3,2)} \otimes \xi_{(3,2)} : h_{3,2}(a, c)u_{3,2}(x, y) \mapsto \nu^{1-s_3}\chi\eta^{-1}(a)\psi(y).$$

We need to show that the representation $\sigma_1\nu^{s_1} \otimes \sigma_2\nu^{s_2}$ is $(H_{(3,2)}(F), \omega'_{(3,2)} \otimes \xi_{(3,2)})$ -distinguished. This is equivalent to show that the representation $\sigma_1 \otimes \sigma_2$ is $(H_{(3,2)}(F), \omega''_{(3,2)} \otimes \xi_{(3,2)})$ -distinguished with

$$\omega''_{(3,2)} \otimes \xi_{(3,2)} : h_{3,2}(a, c)u_{3,2}(x, y) \mapsto \nu^{1-s_1-s_2-s_3}\chi\eta^{-1}(a)\psi(y).$$

Up to twist σ_2 by the unitary character $\eta\chi^{-1}$, we can eliminate the character $\chi\eta^{-1}$ in the expression of $\omega''_{(3,2)} \otimes \xi_{(3,2)}$. Hence we need to show that the representation $\sigma_1 \otimes \sigma_2$ is $(H_{(3,2)}(F), \omega_{(3,2),1-s_1-s_2-s_3} \otimes \xi_{(3,2)})$ -distinguished where the character $\omega_{(3,2),1-s_1-s_2-s_3} \otimes \xi_{(3,2)}$ is defined in Section 6.4. This just follows from Proposition 6.4(2) and (7.13). Hence we have proved (7.1) for the $\{3, 2, 1\}$ case.

7.6. The $\{3, 1, 1, 1\}$ case. In this subsection, we study the case when $\{n_1, n_2, \dots, n_k\} = \{3, 1, 1, 1\}$. Then π is the parabolic induction of an essential discrete series $\sigma\nu^{s_1} \otimes \eta_1\nu^{s_2} \otimes \eta_2\nu^{s_3} \otimes \eta_3\nu^{s_4}$ of $\mathrm{GL}_3(F) \times \mathrm{GL}_1(F) \times \mathrm{GL}_1(F) \times \mathrm{GL}_1(F)$ where σ is a discrete series of $\mathrm{GL}_3(F)$, η_i are unitary characters of $\mathrm{GL}_1(F)$ and $s_i \in \mathbb{R}$. If $s_1 = s_2 = s_3 = s_4 = 0$, then π is a tempered representation and (7.1) has already been proved in the previous paper [Wan16a]. So from now on, we assume that at least one s_i is nonzero. Since the central character of π is equal to χ^2 which is a unitary character, we have

$$(7.15) \quad 3s_1 + s_2 + s_3 + s_4 = 0.$$

We first need a lemma.

Lemma 7.6. *Up to change (s_1, s_2, s_3, s_4) to $(-s_1, -s_2, -s_3, -s_4)$ (which is equivalent to change π to its contragredient), we can find $2 \leq i \leq 4$ such that $s_i > s_1$ and $s_j + 2s_1 \neq \frac{1}{2}$ for all $j \neq 1, i$.*

Proof. Up to change (s_1, s_2, s_3, s_4) to $(-s_1, -s_2, -s_3, -s_4)$, we may assume that $s_1 \leq 0$. Since at least one s_i is nonzero, together with (7.15), we know that at least one s_i is positive. We may assume that $s_2 > 0$. If $s_3 + 2s_1 \neq \frac{1}{2}$ and $s_4 + 2s_1 \neq \frac{1}{2}$, we just need to let $i = 2$ and this proves the lemma. Otherwise, assume that $s_3 + 2s_1 = \frac{1}{2}$. Then

$$(7.16) \quad s_3 = \frac{1}{2} - 2s_1 > 0 \geq s_1.$$

If $s_2 + 2s_1 \neq \frac{1}{2}$ and $s_4 + 2s_1 \neq \frac{1}{2}$, we just need to let $i = 3$ and this proves the lemma. Otherwise, at least one of $s_2 + 2s_1$ and $s_4 + 2s_1$ must be equal to $\frac{1}{2}$. If $s_4 + 2s_1 = \frac{1}{2}$, combining with (7.15) and (7.16), we know that $s_2 = s_1 - 1 < s_1 \leq 0$ which is a contradiction. So the only possibility is $s_2 + 2s_1 = \frac{1}{2}$. In this case, we have $s_4 = s_1 - 1$. Now if we let $t_i = -s_i$ for $1 \leq i \leq 4$, we have

$$t_4 = t_1 + 1 > t_1, \quad t_2 + 2t_1 = t_3 + 2t_1 = -\frac{1}{2} \neq \frac{1}{2}.$$

Hence we just need to change (s_1, s_2, s_3, s_4) to $(-s_1, -s_2, -s_3, -s_4)$ and let $i = 4$. This finishes the proof of the lemma. \square

By the lemma above, up to change π to its contragredient and switch the order of η_i , we may assume that

$$(7.17) \quad s_1 < s_2, \quad s_3 + 2s_1 \neq \frac{1}{2}, \quad s_4 + 2s_1 \neq \frac{1}{2}.$$

Let $\tau_1 = I_{\bar{P}_{3,1}}^{\mathrm{GL}_4}(\sigma_1 \nu^{s_1} \otimes \eta_1 \nu^{s_2})$, $\tau_2 = I_{B_2}^{\mathrm{GL}_2}(\eta_2 \nu^{s_3} \otimes \eta_3 \nu^{s_4})$, and $\tau = \tau_1 \otimes \tau_2$. Then τ is an irreducible generic representation of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$ and π is the parabolic induction of τ . Let $m(\tau)$ be the multiplicity for the middle model. As in the previous cases, it is enough to show that $m(\tau) \neq 0$. Then by the same argument as in the $\{3, 2, 1\}$ case, we only need to show that the representation $\sigma_1 \otimes \tau_2$ of $\mathrm{GL}_3(F) \times \mathrm{GL}_2(F)$ is $(H_{(3,2)}(F), \omega_{(3,2),0} \otimes \xi_{(3,2)})$ -distinguished where the group $H_{(3,2)}(F)$ is defined in Section 6.4 and the character $\omega_{(3,2),0} \otimes \xi_{(3,2)}$ is given by

$$\omega_{(3,2),0} \otimes \xi_{(3,2)} : h_{3,2}(a, c) u_{3,2}(x, y) \mapsto \nu^{1-s_1-s_2} \chi \eta_1^{-1}(a) \psi(y).$$

Remark 7.7. *In this case, it is not clear to us how to relate the multiplicity of the Ginzburg-Rallis model with the reduced model associated to $\mathrm{GL}_3(F) \times \mathrm{GL}_1(F) \times \mathrm{GL}_1(F) \times \mathrm{GL}_1(F)$. The reason is that when we apply the open orbit method, unlike the previous cases, we need to integrate over $T_2(F) \backslash \mathrm{GL}_2(F)$ (instead of $B_2(F) \backslash \mathrm{GL}_2(F)$) which is not compact. Instead, we will only relate it to the reduced model associated to $\mathrm{GL}_3(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_1(F)$ as in the $\{3, 2, 1\}$ case. But then we would need to prove a similar statement*

as in Proposition 6.4(2) when σ_2 is a principal series (see Proposition 7.8 below).

When σ_1 is a supercuspidal representation of $\mathrm{GL}_3(F)$. Up to twist σ_1 by the character $\eta_1\chi^{-1}$, we can eliminate the character $\chi\eta_1^{-1}$ in the expression of $\omega_{(3,2),0} \otimes \xi_{(3,2)}$. Then by Proposition 6.4(1), we know that the representation $\sigma_1 \otimes \tau_2$ is $(H_{(3,2)}(F), \omega_{(3,2),0} \otimes \xi_{(3,2)})$ -distinguished. This proves (7.1).

Now the only case left is when $\sigma_1 = St_3(\eta)$ is a Steinberg representation where η is a unitary character of F^\times . Since the central character of π is χ^2 , we have

$$(7.18) \quad \eta^3 \eta_1 \eta_2 \eta_3 = \chi^2.$$

Also since the parabolic induction of $\sigma\nu^{s_1} \otimes \eta_1\nu^{s_2} \otimes \eta_2\nu^{s_3} \otimes \eta_3\nu^{s_4}$ is irreducible, by the result of [Z80], we have

$$(7.19) \quad \frac{\eta_1\nu^{s_2}}{\eta\nu^{s_1}} \neq \nu^2.$$

By [Z80], we know that σ_1 is the unique subrepresentation of induced representation $I_{\bar{P}_{2,1}}^{\mathrm{GL}_3}(St_2(\nu^{-1/2}\eta) \otimes \eta\nu)$, and the quotient $I_{\bar{P}_{2,1}}^{\mathrm{GL}_3}(St_2(\nu^{-1/2}\eta) \otimes \eta\nu)/\sigma_1$ is an irreducible representation of $\mathrm{GL}_3(F)$ which is the unique quotient of the induced representation $I_{\bar{P}_{2,1}}^{\mathrm{GL}_3}(((\eta\nu^{1/2}) \circ \det) \otimes \eta\nu^{-1})$. As a result, in order to show that the representation $\sigma_1 \otimes \tau_2$ is $(H_{(3,2)}(F), \omega_{(3,2),0} \otimes \xi_{(3,2)})$ -distinguished, it is enough to prove the following proposition.

Proposition 7.8. (1) *The representation $I_{\bar{P}_{2,1}}^{\mathrm{GL}_3}(St_2(\nu^{-1/2}\eta) \otimes \eta\nu) \otimes \tau_2$ is $(H_{(3,2)}(F), \omega_{(3,2),0} \otimes \xi_{(3,2)})$ -distinguished.*
(2) *The representation $I_{\bar{P}_{2,1}}^{\mathrm{GL}_3}(((\eta\nu^{1/2}) \circ \det) \otimes \eta\nu^{-1}) \otimes \tau_2$ is not $(H_{(3,2)}(F), \omega_{(3,2),0} \otimes \xi_{(3,2)})$ -distinguished.*

Proof. (1) We will use open orbit method to prove the first part. Define the subgroup

$$H_{2,2}(F) = \{h_{2,2}(a, c) = \begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix} \mid a \in F^\times, c \in F\}$$

of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$. And define the character $\omega_{2,2}$ on $H_{2,2}(F)$ to be

$$\omega_{2,2}(h_{2,2}(a, c)) = \nu^{1/2-s_1-s_2} \chi\eta_1^{-1}(a).$$

We first prove a claim.

Claim 7.9. *The representation $St_2(\nu^{-1/2}\eta) \otimes \tau_2$ is $(H_{2,2}(F), \omega_{2,2})$ -distinguished.*

Proof. Since the central character of $St_2(\nu^{-1/2}\eta)$ (resp. τ_2) is $\nu^{-1}\eta^2$ (resp. $\eta_2\eta_3\nu^{s_3+s_4}$), it is enough to show that the representation $St_2(\nu^{-1/2}\eta) \otimes \tau_2$ is $(H'_{2,2}(F), \omega'_{2,2})$ -distinguished where

$$H'_{2,2}(F) = \{h'_{2,2}(a, b, c) = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \times \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a, b \in F^\times, c \in F\},$$

and the character $\omega'_{2,2}$ is defined to be

$$\omega'_{2,2}(h'_{2,2}(a, b, c)) = \nu^{1/2-s_1-s_2} \chi \eta_1^{-1}(a) \nu^{s_1+s_2+s_3+s_4-3/2} \eta^2 \eta_1 \eta_2 \eta_3 \chi^{-1}(b).$$

By (7.15) and (7.18), we can rewrite the character $\omega'_{2,2}$ as

$$\omega'_{2,2}(h'_{2,2}(a, b, c)) = \nu^{1/2-s_1-s_2} \eta_1^{-1} \chi(a) \nu^{-2s_1-3/2} \eta^{-1} \chi(b).$$

By the Frobenius reciprocity, it is enough to show that the Hom space

$$\text{Hom}_{\text{GL}_2(F)}((St_2(\nu^{-1/2}\eta) \otimes \tau_2)|_{\text{GL}_2(F)^{\text{diag}}}, \tau_3)$$

is nonzero where $\tau_3 = \text{Ind}_{B_2}^{\text{GL}_2}(\nu^{1/2-s_1-s_2} \eta_1^{-1} \chi \otimes \nu^{-2s_1-3/2} \eta^{-1} \chi)$ is a representation of $\text{GL}_2(F)$. It is easy to see that the Hom space above is isomorphic to the Hom space for the trilinear GL_2 model of the representation $St_2(\nu^{-1/2}\eta) \otimes \tau_2 \otimes \tilde{\tau}_3$ of $\text{GL}_2(F) \times \text{GL}_2(F) \times \text{GL}_2(F)$ where $\tilde{\tau}_3$ is the contragredient of τ_3 . Since $s_1 < s_2$, we have

$$\frac{\nu^{1/2-s_1-s_2} \eta_1^{-1} \chi}{\nu^{-2s_1-3/2} \eta^{-1} \chi} = \nu^{2+s_1-s_2} \eta \eta_1^{-1} \neq \nu^2.$$

Moreover, by (7.19), we also have

$$\frac{\nu^{1/2-s_1-s_2} \eta_1^{-1} \chi}{\nu^{-2s_1-3/2} \eta^{-1} \chi} = \nu^{2+s_1-s_2} \eta \eta_1^{-1} \neq 1.$$

This implies that τ_3 is an irreducible generic principal series of $\text{GL}_2(F)$. By [P90], we know that the multiplicity of $St_2(\nu^{-1/2}\eta) \otimes \tau_2 \otimes \tilde{\tau}_3$ for the trilinear GL_2 model is nonzero. This proves the claim. \square

Now back to the proof of (1). Choose a nonzero element $l_0 \in \text{Hom}_{H_{2,2}(F)}(St_2(\nu^{-1/2}\eta) \otimes \tau_2, \omega_{2,2})$. We realize the representation $I_{\tilde{P}_{2,1}}^{\text{GL}_3}(St_2(\nu^{-1/2}\eta) \otimes \eta\nu)$ on the function space

$$\{f : \text{GL}_3(F) \rightarrow St_2(\nu^{-1/2}\eta) \mid f \text{ locally constant, } f\left(\begin{pmatrix} h & 0 \\ X & a \end{pmatrix} g\right) = \eta\nu^2(a) \nu^{-1/2}(\det(h))$$

$$\cdot St_2(\nu^{-1/2}\eta) f(g), \forall g \in \text{GL}_3(F), h \in \text{GL}_2(F), a \in F^\times, X \in \text{Mat}_{1 \times 2}(F)\},$$

while the $\text{GL}_3(F)$ -action is just the right translation. We define a map $J : I_{\tilde{P}_{2,1}}^{\text{GL}_3}(St_2(\nu^{-1/2}\eta) \otimes \eta\nu) \rightarrow St_2(\nu^{-1/2}\eta)$ to be

$$J(f) := \int_{N_{2,1}(F)} f(u) \xi_{2,1}(u)^{-1} du, \quad f \in I_{\tilde{P}_{2,1}}^{\text{GL}_3}(St_2(\nu^{-1/2}\eta) \otimes \eta\nu)$$

where the character $\xi_{2,1}$ is defined to be $\xi_{2,1}(u) = \xi_{(3,2)}(u \otimes I_2)$. By Proposition 2.1 of [Wan16b], the integral above is absolutely convergent. Also by a similar argument as in the proof of Proposition 5.1 of [Wan16b], we know that the map J is surjective. We then define a map $l : I_{\tilde{P}_{2,1}}^{\text{GL}_3}(St_2(\nu^{-1/2}\eta) \otimes \eta\nu) \otimes \tau_2 \rightarrow \mathbb{C}$ to be

$$l(f \otimes v) := l_0(J(f) \otimes v), \quad f \in I_{\tilde{P}_{2,1}}^{\text{GL}_3}(St_2(\nu^{-1/2}\eta) \otimes \eta\nu), v \in \tau_2.$$

Since J is surjective and $l_0 \neq 0$, we have $l \neq 0$. Moreover, it is easy to see from the definition that l belongs to the Hom space

$$\mathrm{Hom}_{H_{(3,2)}(F)}(I_{\bar{P}_{2,1}}^{\mathrm{GL}_3}(St_2(\nu^{-1/2}\eta) \otimes \eta\nu) \otimes \tau_2, \omega_{(3,2),0} \otimes \xi_{(3,2)}).$$

This proves (1).

(2) We will apply the orbit method to prove the second part. We first study the double cosets

$$(\bar{P}_{2,1}(F) \times \mathrm{GL}_2(F)) \backslash \mathrm{GL}_3(F) \times \mathrm{GL}_2(F) / H_{3,2}(F).$$

By the Bruhat decomposition, this double cosets contain three elements

$$\begin{aligned} & (\bar{P}_{2,1}(F) \times \mathrm{GL}_2(F))H_{(3,2)}(F), \quad (\bar{P}_{2,1}(F) \times \mathrm{GL}_2(F))(w_{(321)} \times I_2)H_{(3,2)}(F), \\ & (\bar{P}_{2,1}(F) \times \mathrm{GL}_2(F))(w_{(132)} \times I_2)H_{(3,2)}(F). \end{aligned}$$

We need to show that all three orbits are not distinguished. For the nonopen orbits $(\bar{P}_{2,1}(F) \times \mathrm{GL}_2(F))(w_{(321)} \times I_2)H_{(3,2)}(F)$ and $(\bar{P}_{2,1}(F) \times \mathrm{GL}_2(F))(w_{(321)} \times I_2)H_{(3,2)}(F)$, this is trivial since the character $\xi_{(3,2)}$ of $U_{3,2}(F)$ is nontrivial on $(w_{(321)}\bar{U}_{2,1}(F)w_{(321)} \times (I_2)) \cap U_{3,2}(F)$ and $(w_{(132)}\bar{U}_{2,1}(F)w_{(132)} \times (I_2)) \cap U_{3,2}(F)$.

Hence it remains to show the open orbit $(\bar{P}_{2,1}(F) \times \mathrm{GL}_2(F))H_{(3,2)}(F)$ is not distinguished. After eliminate the $\bar{U}_{2,1}(F)$ -part, we only need to show that the representation $((\eta\nu^{1/2}) \circ \det) \otimes \tau_2$ of $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$ is not $(H_{2,2}(F), \omega_{2,2})$ -distinguished. Here $(H_{2,2}(F), \omega_{2,2})$ is defined in the proof of (1). Since the representation $(\eta\nu^{1/2}) \circ \det$ of the first $\mathrm{GL}_2(F)$ is a character, it is enough to show that the representation τ_2 of $\mathrm{GL}_2(F)$ is not $(H_2(F), \omega_2)$ -distinguished where

$$H_2(F) = \{h_2(a, c) := \begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix} \mid a \in F^\times, c \in F\}, \quad \omega_2(h_2(a, c)) = \nu^{-s_1-s_2}\eta_1^{-1}\eta^{-1}\chi(a).$$

Since $\tau_2 = I_{B_2}^{\mathrm{GL}_2}(\eta_2\nu^{s_3} \otimes \eta_3\nu^{s_4})$, the Jacquet module $J_{N_2}(\tau_2)$ is

$$(\eta_2\nu^{s_3+1/2} \otimes \eta_3\nu^{s_4-1/2}) \oplus (\eta_3\nu^{s_4+1/2} \otimes \eta_2\nu^{s_3-1/2}).$$

Hence in order to show the representation τ_2 of $\mathrm{GL}_2(F)$ is not $(H_2(F), \omega_2)$ -distinguished, it is enough to show that the character $\nu^{-s_1-s_2}\eta_1^{-1}\eta^{-1}\chi$ is not equal to $\eta_2\nu^{s_3+1/2}$ or $\eta_3\nu^{s_4+1/2}$. Since χ, η, η_i are unitary characters, it is enough to show that

$$\nu^{-s_1-s_2} \neq \nu^{s_3+1/2}, \quad \nu^{-s_1-s_2} \neq \nu^{s_4+1/2}.$$

But this just follows from (7.15) and (7.17). This finishes the proof of the proposition and hence the proof of (7.1) for the $\{3, 1, 1, 1\}$ case. \square

Now the proof of Theorem 1.6 is finally complete.

APPENDIX A. THE MIDDLE MODEL CASE

As we mentioned in Remark 1.9, our method in this paper can also be applied to the middle model case, and it will give us the analogy results of Theorem 1.2, 1.6 and Theorem 1.7 for the middle model. To be specific, let $\tau = \tau_1 \otimes \tau_2$ be an irreducible generic representation of $G_1(F) = \mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$ whose central character equals χ^2 on $Z_{H_0}(F)$. Let $\tau_D = \tau_{1,D} \otimes \tau_{2,D}$ be the local Jacquet-Langlands correspondence of τ to $\mathrm{GL}_2(D) \times \mathrm{GL}_1(D)$ if it exists; otherwise let $\tau_D = 0$. Let $m(\tau)$ and $m(\tau_D)$ be the multiplicities for the middle model. The following theorem is an analogue of Theorem 1.2, 1.6 and Theorem 1.7 for the middle model, its proof is similar to the Ginzbrug-Rallis model case and we will skip it here.

Theorem A.1. (1) *If $\tau_{1,D}$ exists, we have*

$$m(\tau) + m(\tau_D) = 1.$$

(2) *Assume that the generalized Jacquet integrals in Section 7 of [Wan16b] have holomorphic continuation. Then*

$$m(\tau) + m(\tau_D) = 1$$

for all irreducible generic representations τ of $G_1(F)$. In particular, $m(\tau) = 1$ when $\tau_D = 0$.

Remark A.2. *As in the Ginzburg-Rallis model case, we can also prove the epsilon dichotomy conjecture. To be specific, when $F = \mathbb{R}$, for all irreducible generic representations $\tau = \tau_1 \otimes \tau_2$ of $G_1(F)$ whose central character equals χ^2 on $Z_{H_0}(F)$, we have*

$$m(\tau) = 1 \iff \epsilon\left(\frac{1}{2}, \wedge^2(\phi_{\tau_1}) \otimes \phi_{\tau_2} \otimes \chi^{-1}\right) = 1, \quad m(\tau) = 0 \iff \epsilon\left(\frac{1}{2}, \wedge^2(\phi_{\tau_1}) \otimes \phi_{\tau_2} \otimes \chi^{-1}\right) = -1.$$

Here ϕ_{τ_i} are the Langlands parameters of τ_i . When F is p -adic, then the above relations hold when τ is not an essential discrete series.

APPENDIX B. HOLOMORPHIC CONTINUATION OF THE GENERALIZED JACQUET INTEGRALS

In this appendix, we will brief discuss our assumption on the generalized Jacquet integral. Let F be a p -adic field or \mathbb{R} as usual, G be a reductive group defined over F and $Q = LN$ be a proper parabolic subgroup of G . Let $\xi : N(F) \rightarrow \mathbb{C}^\times$ be a generic character on $N(F)$ and let $\bar{Q} = L\bar{N}$ be the opposite parabolic subgroup of Q . Given an irreducible admissible representation σ of $L(F)$ and $s \in \mathbb{C}$, for $f_s \in I_Q^G(\delta_Q^s \sigma)$, we define the generalized Jacquet integral to be

$$J_N(f_s) := \int_{N(F)} f_s(n) \xi(n) dn.$$

In general, the integral above will only convergent when $Re(s)$ large. The hypothesis we need in this paper is the following.

Hypothesis: Assume that N is abelian, then the generalized Jacquet integral $J_N(f_s)$ has holomorphic continuation.

Remark B.1. *In general, we also expect this hypothesis holds when N is not abelian. However, as for this paper, we only need the hypothesis holds for two cases. One is when $G = \mathrm{GL}_6$ and $Q = P_{4,2}$ is the standard parabolic subgroup of type $(4, 2)$. The other case is when $G = \mathrm{GL}_4$ and $Q = P_{2,2}$ is the standard parabolic subgroup of type $(2, 2)$.*

Remark B.2. *When Q is the Borel subgroup of G , the integral $J_N(f)$ is the original Jacquet integral studied by Jacquet in [Ja67]. In the loc. cit., Jacquet proved that the Jacquet integral has holomorphic continuation.*

Remark B.3. *When $F = \mathbb{R}$, this hypothesis has been proved by Gomez and Wallach in [GW12] for the case when the stabilizer of the unipotent character is compact, and proved by Gomez in [G] for the general case. The second paper is still in preparation. In the p -adic, as far as we know, this hypothesis is still open, but is expected.*

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