STRONGLY TEMPERED HYPERSPHERICAL HAMILTONIAN SPACES

ZHENGYU MAO, CHEN WAN, AND LEI ZHANG

Abstract. In this paper, we give a complete list of strongly tempered hyperspherical Hamiltonian spaces. We show that the period integrals attached to the list contains many previously studied Rankin-Selberg integrals and period integrals, thus give a new conceptual understanding of these integrals. The list also proposes many new interesting period integrals to study.

1. Introduction

1.1. **BZSV Duality.** In [1], Ben-Zvi, Sakellaridis, and Venkatesh proposed a beautiful relative Langlands duality for hyperspherical Hamiltonian spaces (in this paper, we will call it BZSV duality). We briefly recall the datum in the duality. Throughout this paper, k is a global field, $\mathbb{A} = \mathbb{A}_k$, F is a local field, and ψ is a non-trivial additive character of $\mathbb{A}/k$ (resp. F) if we are in the global (resp. local) setting. Let G be a split connected reductive group defined over k. In Section 3 of [1], Ben-Zvi, Sakellaridis, and Venkatesh defined a special category of G-Hamiltonian spaces called the hyperspherical G-Hamiltonian spaces. Moreover, they also showed that each hyperspherical G-Hamiltonian spaces is associated to a quadruple $\Delta = (G, H, \rho_H, \iota)$ where H is a split reductive subgroup of G; ρ_H is a symplectic representation of H; and ι is a homomorphism from SL_2 into G whose image commutes with H. For the rest of this paper, we will only discuss the quadruple instead of the Hamiltonian space associated to it (we will call such quadruple BZSV quadruple in this paper).

The BZSV duality concerns a pair of dual data $(\Delta, \hat{\Delta})$ where each side contains a BZSV quadruple: $\Delta = (G, H, \rho_H, \iota)$ and $\hat{\Delta} = (\hat{G}, \hat{H}', \rho_{\hat{H}'}, \hat{\iota}')$. The map ι induces an adjoint action of $H \times \mathrm{SL}_2$ on the Lie algebra $\mathfrak{g}$ of G and we can decompose it as

$$\oplus_{k\in I} \rho_k \otimes Sym^k$$

where ρ_k is some representation of H and I is a finite subset of $\mathbb{Z}_{\geq 0}$. We let I_{odd} be the subset of I containing all the odd numbers. In order for Δ to be a BZSV quadruple, one of the (many) requirements is that the representation

$$\rho_{H,\iota} = \rho_H \oplus (\oplus_{i\in I_{odd}} \rho_i) \tag{1.1}$$

is a symplectic anomaly-free representation (see Section 5 of [1]) of H. We refer the reader to [1] for more details. Note that under BZSV duality, the group $\hat{G}$ is the Langlands dual group of G and $\hat{H}' = \hat{G}_\Delta$ can be viewed as the "dual group" of the quadruple Δ (note that the groups H and $\hat{H}'$ are not dual to each other in general, and the nilpotent orbits ι and $\hat{\iota}'$ are also not dual to each other in general). We recall the conjecture about period integrals in the BZSV duality.

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Let $\Delta = (G, H, \rho_H, \iota)$ and $\hat{\Delta} = (\hat{G}, \hat{H}', \rho_{\hat{H}'}, \hat{\iota}')$ be two quadruples that are dual to each other under the BZSV duality. We use $\rho_{H,\iota}$ and $\rho_{\hat{H}',\hat{\iota}'}$ to denote the symplectic anomaly-free representations associated to these quadruples. As we explained above, the maps ι and $\hat{\iota}'$ induce adjoint actions of $H\times \mathrm{SL}_2$ (resp. $\hat{H}'\times \mathrm{SL}_2$) on $\mathfrak{g}$ (resp. $\hat{\mathfrak{g}}$) and they can be decomposed as

$$\mathfrak{g} = \oplus_{k\in I}\rho_k \otimes Sym^k,\ \hat{\mathfrak{g}} = \oplus_{k\in \hat{I}}\hat{\rho}_k \otimes Sym^k$$

where ρ_k (resp. $\hat{\rho}_k$) are representations of H (resp. $\hat{H}'$). It is clear that the adjoint representation of H (resp. $\hat{H}'$) is a subrepresentation of ρ_0 (resp. $\hat{\rho}_0$).

For an automorphic form ϕ of $G(\mathbb{A})$ (resp. $\hat{G}(\mathbb{A})$), we can define the period integral $\mathcal{P}_{H,\iota,\rho_H}(\phi)$ (resp. $\mathcal{P}_{\hat{H}',\hat{\iota}',\rho_{\hat{H}'}}(\phi)$) of it associated to the quadruple. Let's briefly recall the definition. We have a symplectic representation $\rho_{H,\iota}: H \to \mathrm{Sp}(V)$. Let Y be a maximal isotropic subspace of V and Ω_ψ be the Weil representation of $\widetilde{\mathrm{Sp}}(V)$ on the Schwartz space $\mathcal{S}(Y(\mathbb{A}))$. The anomaly free condition on $\rho_{H,\iota}$ ensures $\widetilde{\mathrm{Sp}}(V)$ splits over $Im(\rho_{H,\iota})$ and Ω_ψ restricts to a representation of $H(\mathbb{A})$ on $\mathcal{S}(Y(\mathbb{A}))$. We define the theta series

$$\Theta_\psi^\varphi(h) = \sum_{X\in Y(k)} \Omega_\psi(h)\varphi(X),\ h\in H(\mathbb{A}), \varphi\in \mathcal{S}(Y(\mathbb{A})),$$

and we can define the period integral to be

$$\mathcal{P}_{H,\iota,\rho_H}(\phi,\varphi) = \int_{H(k)\backslash H(\mathbb{A})} \mathcal{P}_\iota(\phi)(h)\Theta_\psi^\varphi(h)dh.$$

Here $\mathcal{P}_\iota$ is the degenerate Whittaker period associated to ι (we refer the reader to Section 1.2 of [34] for its definition). To simplify the notation, we will omit the Schwartz function in the notion of the period and simply write it as $\mathcal{P}_{H,\iota,\rho_H}(\phi)$ [1]. Similarly we can also define the period integral $\mathcal{P}_{\hat{H}',\hat{\iota}',\rho_{\hat{H}'}}(\phi)$. The following conjecture is the main conjecture regarding global periods in BZSV duality.

Conjecture 1.1. *(Ben-Zvi–Sakellaridis–Venkatesh, [1])*

(1) Let π be an irreducible discrete automorphic representation of $G(\mathbb{A})$ and let $\nu: \pi \to L^2(G(k)\backslash G(\mathbb{A}))_\pi$ be an embedding. Then the period integral

$$\mathcal{P}_{H,\iota,\rho_H}(\phi),\ \phi\in Im(\nu)$$

is nonzero only if the Arthur parameter of π factors through $\hat{\iota}': \hat{H}'(\mathbb{C})\times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}(\mathbb{C})$. If this is the case, π is a lifting of a global tempered Arthur packet Π of $H'(\mathbb{A})$ (the Langlands dual group of $\hat{H}'$). Then we can choose the embedding ν so that

$$\frac{|\mathcal{P}_{H,\iota,\rho_H}(\phi)|^2}{\langle\phi,\phi\rangle} \text{``}=\text{''} \frac{L(1/2,\Pi,\rho_{\hat{H}'})\cdot \Pi_{k\in\hat{I}} L(k/2+1,\Pi,\hat{\rho}_k)}{L(1,\Pi,Ad)^2},\ \phi\in Im(\nu).$$

Here $\langle,\rangle$ is the L^2-norm, and "$=$" means the equation holds up to some Dedekind zeta functions, some global constant determined by the component group of the global L-packet associated to π, and some finite product over the ramified places (including all the archimedean places).

[1]when the nilpotent orbit associated to ι is not even, the degenerate Whittaker period $\mathcal{P}_\iota$ is a Fourier-Jacobi coefficient and one also need to include an extra Schwartz function in its definition

(2) Let π be an irreducible discrete automorphic representation of $\hat{G}(\mathbb{A})$ and let $\nu : \pi \to L^2(\hat{G}(k)\backslash \hat{G}(\mathbb{A}))_\pi$ be an embedding. Then the period integral

$$\mathcal{P}_{\hat{H}',\hat{\iota}',\rho_{\hat{H}'}}(\phi),\ \phi \in Im(\nu)$$

is nonzero only if the Arthur parameter of π factors through $\iota : H(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to G(\mathbb{C})$. If this is the case, π is a lifting of a global tempered Arthur packet Π of $\hat{H}(\mathbb{A})$ (the Langlands dual of H). Then we can choose the embedding ν so that

$$\frac{|\mathcal{P}_{\hat{H}',\hat{\iota}',\rho_{\hat{H}'}}(\phi)|^2}{\langle \phi, \phi \rangle} \text{``}=\text{''} \frac{L(1/2, \Pi, \rho_H) \cdot \Pi_{k\in I} L(k/2+1, \Pi, \rho_k)}{L(1, \Pi, Ad)^2},\ \phi \in Im(\nu).$$

Remark 1.2. *The above conjecture is usually called the Ichino-Ikeda type conjecture. To state an explicit identity instead of " = ", one needs to make two adjustments on the right-hand side of the equation.*

- *In the ramified places, instead of using the local L-function, one needs to use the so-called local relative character defined by the (conjectural) Plancherel decomposition (see Section 17 of* [39] *and Section 9 of* [1]*).*
- *One also needs to add some Dedekind zeta functions on the right-hand side determined by the groups G and H (in all the known examples, those zeta functions are the L-function of the dual $M^\vee$ to the motive M associated to G, H introduced by Gross in* [22]*), as well as some global constant determined by component group of the global L-packet associated to π (see Section 14.6.4 of* [1]*) for these two quadruples.*

Remark 1.3. *In* [1]*, they also formulated many other conjectures for the duality (i.e., local/global geometric conjecture, local conjecture for Plancherel decomposition). The expectation is that those conjectures would uniquely determine the duality. In this paper we will only focus on their conjecture for period integrals. We also want to point out that given a general BZSV quadruple $\Delta = (G, H, \rho_H, \iota)$, at this moment there is no algorithm to compute the dual quadruple $\hat{\Delta}$. The only exception is for the so-called polarized case (i.e., when $\rho_H = 0$) where the algorithm is given in Section 4 of* [1] *(most quadruples considered in this paper are not polarized). As a result, given two BZSV quadruples Δ and $\hat{\Delta}$, at this moment one can only provide evidence for the duality between them by studying the various conjectures (i.e., local/global geometric conjecture, local conjecture for Plancherel decomposition, global conjecture for period integrals) in* [1].

1.2. **Strongly tempered BZSV quadruples.**

Definition 1.4. *We say the quadruple $\Delta = (G, H, \rho_H, \iota)$ is strongly tempered if $\hat{G} = \hat{H}' Z_{\hat{G}}$, i.e. the "dual group" of Δ is equal to the dual group of G up to center. We say the quadruple is reductive if ι is trivial.*

If the quadruple $\Delta = (G, H, \rho_H, \iota)$ is strongly tempered, then Conjecture 1.1(1) states that for all global tempered L-packet Π of $G(\mathbb{A})$ [2], there exists $\pi \in \Pi$ and $\nu : \pi \to L^2(G(k)\backslash G(\mathbb{A}))_\pi$ such that

$$\frac{|\mathcal{P}_{H,\iota,\rho_H}(\phi)|^2}{\langle \phi, \phi \rangle} \text{``}=\text{''} \frac{L(1/2, \Pi, \rho_{\hat{H}'})}{L(1, \Pi, Ad)},\ \phi \in Im(\nu). \tag{1.2}$$

[2] when $\hat{G} \neq \hat{H}'$, we need to make some assumptions on the central character of Π so that its Langlands parameter factors through $\hat{H}'$

In other words, it means that the norm square of the period integral $\mathcal{P}_{H,\iota,\rho_H}(\phi)$ is essentially equal to the central value of an automorphic L-function on every tempered global L-packet.

The most well-known example of strongly tempered quadruple is the Gross-Prasad model $(G,H,\rho_H,\iota)=(\mathrm{SO}_{2n+1}\times\mathrm{SO}_{2n},\mathrm{SO}_{2n},0,1)$. In this case the dual quadruple is given by

$$(\hat{G},\hat{G},\hat{\rho},1)=(\mathrm{Sp}_{2n}\times\mathrm{SO}_{2n},\mathrm{Sp}_{2n}\times\mathrm{SO}_{2n},std_{\mathrm{Sp}_{2n}}\otimes std_{\mathrm{SO}_{2n}},1).$$

In this case, Conjecture 1.1(1) is just the Ichino-Ikeda conjecture in [25] and Conjecture 1.1(2) is just the Rallis inner product formula for the theta correspondence between Sp_{2n} and SO_{2n}.

Remark 1.5. *Conjecturally the quadruple is strongly tempered if and only if the integral*

$$\int_{H(F)}\mathcal{P}_\iota(\phi)(h)\varphi(h)dh \tag{1.3}$$

is absolutely convergent for all tempered matrix coefficient ϕ of $G(F)$. Here $F=k_v$ is a local field for some $v\in|k|$, $\mathcal{P}_\iota$ is the local analogue of the global degenerate Whittaker period, and $\varphi(h)$ is a matrix coefficient of the local Weil representation of $H(F)$ associated to the symplectic representation ρ_H (although the unipotent integral $\mathcal{P}_\iota$ is not necessarily convergent and it needs to be regularized, see examples in [2, 31, 42, 43, 44]*). The local relative character in Remark 1.2 is given by the integral* (1.3) *where ϕ is the matrix coefficient of π_v; and π_v is the local component of π at v which is a tempered representation of $G(F)$.*

In [34], we proposed a relative trace formula comparison that relates the periods $\mathcal{P}_{H,\iota,\rho_H}(\phi)$ associated to any BZSV quadruple (G,H,ρ_H,ι) to the periods $\mathcal{P}_{H_0,\iota_0,\rho_{H_0}}(\phi_0)$ associated to a strongly tempered BZSV quadruple $(G_0,H_0,\rho_{H_0},\iota_0)$. Thus it is natural to consider Conjecture 1.1 first for the strongly tempered BZSV quadruples. In this paper we provide and study a complete list of strongly tempered BZSV quadruples (and hence a complete list of strongly tempered hyperspherical Hamiltonian spaces).

By duality, in order to classify the strongly tempered quadruple Δ, it is enough to classify its dual quadruple

$$\hat{\Delta}=(\hat{G},\hat{H}',\hat{\rho},1).$$

Since $\hat{H}'Z_{\hat{G}}=\hat{G}$, it is enough to classify all the BZSV quadruples of the form

$$(\hat{G},\hat{G},\hat{\rho},1).$$

By [1], a quadruple $\hat{\Delta}=(\hat{G},\hat{G},\hat{\rho},1)$ is a BZSV quadruple if it satisfies the following three conditions.

(1) The symplectic representation $\hat{\rho}$ is anomaly-free (see [1, Section 5]).
(2) The symplectic representation $\hat{\rho}$ is multiplicity free.
(3) The generic stabilizer of the representation $\hat{\rho}$ of $\hat{G}$ is connected.

The set of multiplicity-free symplectic representations were classified by Knop [28] and Losev [33] independently. In this paper we will use the list in [28]. By [28, Theorem 2.3], the classification is reduced to that of symplectic representations that are *saturated* and multiplicity free, which are listed in Table 1, 2, 11, 12, 22, S of [28]. In this paper we write down the strongly tempered quadruples that are (up to isogeny) the duals of $(\hat{G},\hat{G},\hat{\rho},1)$ when $\hat{\rho}$ is the symplectic representations listed in Knop's tables. In order to find the dual quadruple, we will provide a systematic way to write down H and ι (see Property 2.9). On the other hand the choice of ρ_H has been done in an ad hoc way at this moment.

Remark 1.6. *Condition (3) above is related to the Type N spherical root. Whenever this condition fails, we should expect some covering group to appear in the dual quadruple* $\Delta = (G, H, \rho_H, \iota)$*. This is not covered in BZSV's framework at this moment. Nonetheless, for some of the cases in* [28] *that do not satisfy (3), we are still able to write down a candidate for the dual of the quadruple* $\hat{\Delta}$ *from some existing automorphic integrals in previous literatures.* [3]*.*

1.3. **Statement of main results.** We first consider representations not in Table S of [28] (because Table S of [28] is an infinite table), i.e. consider all quadruples $\hat{\Delta} = (\hat{G}, \hat{G}, \hat{\rho}, 1)$ satisfy the following two conditions:

(1) The symplectic representation $\hat{\rho}$ is anomaly-free.
(2) The symplectic representation $\hat{\rho}$ appears in Table 1, 2, 11, 12, 22 of [28].

For each of them, we will write down a quadruple $\Delta = (G, H, \rho_H, \iota)$ and claim it is dual to $\hat{\Delta}$ up to isogeny, or more precisely it is dual to $(\hat{G}, \widehat{G/Z_\Delta}, \hat{\rho}, 1)$ where $Z_\Delta = Z_G \cap ker(\rho_H)$ and Z_G is the center of G. To support the claim we provide evidence through the three main theorems below. Our results are summarized in the 6 tables at the end of this paper (Table 21, 22, 23, 24, 25 and 26, the first two tables are for reductive cases while the last four tables are for non-reductive cases).

Theorem 1.7. *For all the reductive cases (Table 21 and 22) except the quadruple* $(\mathrm{GL}_6 \times \mathrm{GL}_2, \mathrm{GL}_2 \times S(\mathrm{GL}_4 \times \mathrm{GL}_2), \wedge^2 \otimes std_{\mathrm{GL}_2})$*, and for all quadruples in Table 23 and 24, the local relative character of the period integral* $\mathcal{P}_{H,\rho_H,\iota}$ *is equal to the* $L-$*value in Conjecture 1.1(1) at unramified places, namely equals* $\frac{L(1/2,\Pi,\hat{\rho})}{L(1,\Pi,Ad)}$ *for the unramified representation* Π.

Recall that the local relative character at unramified places is defined in (1.3) with ϕ and φ being unramified matrix coefficients normalized to be 1 at identity, and with suitably chosen Haar measures. It is easy to check for all cases in Table 21 – 26, the integral (1.3) is absolutely convergent.

Remark 1.8. *For the quadruple* $(\mathrm{GL}_6 \times \mathrm{GL}_2, \mathrm{GL}_2 \times S(\mathrm{GL}_4 \times \mathrm{GL}_2), \wedge^2 \otimes std_{\mathrm{GL}_2})$ *and for all quadruples in Table 25 and 26, as far as we know, their local relative characters have not been computed at unramified places. Although we believe they can be computed by the same method as in* [25] *and* [44]*.*

Theorem 1.9. *For the quadruples in Table 21, 23 and 25, Conjecture 1.1(2) holds, if we assume (when applicable) the global period integral conjectures in* [12, 13, 25] *for Gan-Gross-Prasad models.*

Remark 1.10. *In most cases for Theorem 1.9 and some cases for Theorem 1.7 we utilize the theta correspondence. We summarize the results needed for theta correspondence in Section 2.2.*

Remark 1.11. *In* [13]*, the authors only formulated a global conjecture regarding the non-vanishing of the period integrals for non-tempered Arthur L-packets (Conjecture 9.11 of* [13]*). An Ichino-Ikeda type conjecture for the period is not available in* [13] *because of the difficulty in the definition of local relative character in the non-tempered case (see the last paragraph of Section 9 of* [13]*). Thus strictly speaking, for some cases in Theorem 1.9 we can only*

[3]In this paper, we will not check the connectedness condition for representations in [28], we will leave it as an exercise for the reader.

claim the nonvanishing part of Conjecture 1.1(2). However the identity in Conjecture 1.1(2) disregards the local factors at bad places, thus to prove it we only need an Ichino-Ikeda type conjecture without specifying the local factors at bad places. The formulation of such a conjecture is well known and we assume this version of the conjecture in Theorem 1.9.

Beside the above two theorems, we provide one further evidence for the duality for all the non-reductive quadruples. To state the evidence, we need to say a little more about the Hamiltonian space associated to the quadruple. Let $\Delta = (G, H, \rho_H, \iota)$ be a BZSV quadruple. Let M be the centralizer of $\{\iota(diag(t, t^{-1}))|\ t \in \mathrm{GL}_1\}$ in G. It is easy to see that M is a Levi of G and $H \subset M$. We define

$$\Delta_{red} = (M, H, 1, \rho_{H,\iota})$$

where the representation $\rho_{H,\iota}$ has been defined in (1.1). The Hamiltonian G-space associated to Δ is defined by certain induction of the Hamiltonian M-space associated to Δ_{red} (see Section 3 of [1] for details). In Section 4.2.2 of [1], they proposal a conjecture about the relation between the dual quadruples of Δ and Δ_{red}. We will recall this conjecture in Conjecture 2.8. Now we are ready to state the third evidence.

Theorem 1.12. *For any quadruple $\Delta = (G, H, \rho_H, \iota)$ in Table 23, 24, 25 and 26, the corresponding quadruple $\Delta_{red} = (M, H, 1, \rho_{H,\iota})$ is a quadruple in Table 21 and 22. Moreover, the duality for the quadruples Δ and Δ_{red} [4] is compatible with Conjecture 2.8.*

Remark 1.13. *Most of the quadruples in Table 21 and 22 come from Tables 1, 11, 2, 12, 22 of* [28]*. There are some exceptions; the quadruples given in* (5.5)*,* (6.3)*,* (6.4)*,* (7.7) *and* (7.8) *are strongly tempered and dual to $\hat{\rho}$ from Table S in* [28].

Remark 1.14. *For quadruples in Table 23, 24 and 25, Theorem 1.7 and 1.9 already provide strong evidence for the duality of (G, H, ρ_H, ι). Combining with Theorem 1.12, we get strong evidence of Conjecture 2.8 for quadruples in these three tables.*

Lastly we consider Table S of [28]. The representations coming out of this table are glued together from various representations of this table that already appeared in Table 1, 2, 11, 12, 22 of [28]. Since the length can be arbitrary (i.e. we can glue any number of certain representations together), so this table produces infinitely many representations. In Section 9, for all the representations $\hat{\rho}$ coming from Table S that are anomaly-free and with connected generic stabilizer, we will describe a way to glue the dual quadruples which gives the dual of the quadruple $(\hat{G}, \hat{G}, \hat{\rho}, 1)$.

More precisely, given representations $(\hat{G}_i, \hat{\rho}_i)$ in Table S of [28], and let $(\hat{G}, \hat{\rho})$ be the gluing of those representations. Assume that $\hat{\rho}$ is anomaly-free and its generic stabilizer is connected. We will describe the dual quadruple Δ of $\hat{\Delta} = (\hat{G}, \hat{G}, \hat{\rho}, 1)$ in terms of the dual quadruple Δ_i of $(\hat{G}_i, \hat{G}_i, \hat{\rho}_i, 1)$. Roughly speaking, Δ is glued from Δ_i where the gluing process will be described in Section 9. To justify our construction, we will prove the following theorem.

Theorem 1.15. *With the notation above, Conjecture 1.1 for $(\Delta, \hat{\Delta})$ follows from Conjecture 1.1 for $(\Delta_i, \hat{\Delta}_i)$.*

[4] here by saying the duality for Δ (resp. Δ_{red}) we mean the duality between Δ (resp. Δ_{red}) and the quadruple $(\hat{G}, \hat{G}, 1, \hat{\rho})$ (resp. $(\hat{M}, \hat{M}, 1, \hat{\rho})$) where $\hat{\rho}$ is the corresponding symplectic representation in Table 21-26

In this paper, we provide the evidence of duality mainly through the period integral aspect, i.e., Conjecture 1.1. As we mentioned in Remark 1.3, there are other ways to justify the duality, for example from the geometric conjectures (e.g. [9, 11, 3, 4, 41, 10]) and local Plancherel conjectures (e.g. [9], [11]). We will not consider those conjectures in this paper. We just want to remark that Theorem 1.7 provides numerical evidence for the local Plancherel conjecture in Proposition 9.2.1 of [1], but we will not digress in these directions here.

1.4. **Rankin-Selberg integrals and special values of period integrals.** To end this introduction, we would like to point out that the list of strongly tempered quadruples we found in this paper recovers many existing integrals such as the Rankin-Selberg integrals in [5], [6], [7], [8], [14], [15], [16], [17], [26], [27], [35], [36] and the period integrals in [12], [21], [44]. It also produces many new interesting period integrals for studying.

A simple example that leads to a Rankin-Selberg integral is the quadruple (4.1):

$$(\mathrm{GL}_n\times\mathrm{GL}_n, \mathrm{GL}_n, T(std_{\mathrm{GL}_n}), 1)$$

which is dual to

$$(\mathrm{GL}_n\times\mathrm{GL}_n, \mathrm{GL}_n\times\mathrm{GL}_n, T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n}), 1).$$

The attached period integral is

$$\int_{\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A})}\phi_1(g)\phi_2(g)\Theta^{\Phi}(g)\ dg$$

where $\phi_1\in\pi_1,\phi_2\in\pi_2$ are cusp forms in irreducible unitary cuspidal automorphic representations π_1 and π_2 on GL_n and $\Theta^{\Phi}(g)$ is a theta series on GL_n explicitly given by

$$\Theta^{\Phi}(g)=|\det g|^{-\frac{1}{2}}\sum_{\xi\in k^n}\Phi(\xi g).$$

Let $\xi_0=(0,0,\dots,0,1)$, then we can identify $\Phi(g)$ with the sum of $|\det g|^{-\frac{1}{2}}\Phi(0)$ and a mirabolic Eisenstein series

$$E^{\Phi}(g)=|\det g|^{-\frac{1}{2}}\sum_{\gamma\in P_0(k)\backslash\mathrm{GL}_n(k)}\Phi(\gamma g)$$

where P_0 is the mirabolic subgroup that fixes ξ_0. This period integral is just the specialization of the well-known Rankin-Selberg integral for tensor product $L-$function [26] evaluated at a specified value.

The theory of Rankin-Selberg integrals is a very successful theory, producing many integral representations to study L-functions. A noted drawback of this theory is that the integrals are mostly developed in an ad hoc way. The list provided in this paper can actually fit many of the Rankin-Selberg integrals into the framework of BZSV duality. To be precise, those Rankin-Selberg integrals (evaluated at certain value) are simply the period integrals attached to some strongly tempered BZSV quadruples whose dual is closely related to the L-functions associated to the Rankin-Selberg integrals. The following is a list of such Rankin-Selberg integrals.

- Integrals for exterior square $L-$functions by Bump-Friedberg [5].
- Integrals for Spin $L-$function by Bump-Ginzburg [6], [7] and [16].
- Integrals for standard $L-$functions of exceptional groups E_6 by Ginzburg [14].
- Multivariable Rankin-Selberg integrals by Ginzburg-Hundley [17] and Pollack-Shah [36].

- Rankin-Selberg convolution by Jacquet-Piatetski-Shapiro-Shalika [26].
- Integrals for exterior square $L-$functions by Jacquet-Shalika [27].

The above list exhausts all currently known Rankin-Selberg integrals utilizing the mirobolic Eisenstein series. There are also examples above that use the Eisenstein series of other types (e.g., the ones in [17] and [36]).

Our list provides more candidates for Rankin-Selberg integrals. For example, Model 12 of Table 26 suggests considering the following Rankin-Selberg integral of $G = \mathrm{GSO}_8$, which should produce the standard L-function and the Half-Spin L-function. Let π be a generic cuspidal automorphic representation of $\mathrm{GSO}_8(\mathbb{A})$, $\phi \in \pi$ and $P = MN$ be a maximal parabolic subgroup GSO_8 with its Levi subgroup $M = \mathrm{GL}_2 \times \mathrm{GSO}_4$. Let $H = S(\mathrm{GL}_2 \times \mathrm{GSO}_4)$ be a subgroup of M and let $E(h, s_1, s_2)$ be an automorphic function on H induced from the trivial function on GL_2 and the Borel Eisenstein series of GSO_4 (s_1, s_2 are the parameter of the Eisenstein series). It is easy to see that one can take a Fourier-Jacobi coefficient of ϕ along the unipotent subgroup N that produces an automorphic function on H. We will denote it by $\mathcal{P}_N(\phi)$. Then, the integral associated to Model 12 of Table 26 is just

$$\int_{H(k)\backslash H(\mathbb{A})/Z_G(\mathbb{A})} \mathcal{P}_N(\phi)(h) E(h, s_1, s_2) dh.$$

In the spirit of Conjecture 1.1, we expect this to be the integral representation of the L-function $L(s_1, \pi, \rho_1) L(s_2, \pi, \rho_2)$ where ρ_1 (resp. ρ_2) is the standard representation (resp. Half-Spin representation) of $\mathrm{Spin}_8(\mathbb{C})$.

Meanwhile the majority of the quadruples in our list have period integrals that cannot be considered as specializations of Rankin-Selberg integrals. In some cases, the identities between the periods and the $L-$values in Conjecture 1.1 are consequences of Gan-Gross-Prasad conjectures [12, 13, 25]) and the Conjectures in [44]. There is also one case where the integral is predicted by the work of Ginzburg-Jiang-Rallis [21] on the central value of symmetric cube $L-$functions. Of more interest are the many cases where the conjectured identity in Conjecture 1.1 is new and unrelated to the conjectures mentioned above. For example each of the quadruple in tables 25 and 26 gives such a new conjecture.

We now list one example from Table 22 that not only provides a new Ichino-Ikeda type conjecture for a strongly tempered quadruple but also can be used to explain the Rankin-Selberg in [17]. The example is Model 3 of Table 22. The quadruple is reductive and is given by

$$\Delta = (G, H, \rho_H) = (\mathrm{GSp}_4 \times \mathrm{GSpin}_8 \times \mathrm{GL}_2, S(\mathrm{GSpin}_8 \times G(\mathrm{Sp}_4 \times \mathrm{SL}_2)), std_{\mathrm{Sp}_4} \otimes std_{\mathrm{Spin}_8} \oplus \mathrm{HSpin}_8 \otimes std_{\mathrm{SL}_2}).$$

Let π be a cuspidal generic automorphic representation of $G(\mathbb{A})$, $\phi \in \pi$ and Θ_{ρ_H} be the theta series associated to the symplectic representation ρ_H. Then the period integral is given by

$$\mathcal{P}_\Delta(\phi) = \int_{H(k)\backslash H(\mathbb{A})/Z_\Delta(\mathbb{A})} \phi(h) \Theta_{\rho_H}(h) dh.$$

In the spirit of Conjecture 1.1, we expect the square of this period integral to be equal to

$$\frac{L(1/2, \Pi, \hat{\rho})}{L(1, \Pi, Ad)}$$

where $\hat{\rho}$ is the representation $std_{\mathrm{Sp}_4} \otimes std_{\mathrm{Spin}_8} \oplus \mathrm{HSpin}_8 \otimes std_{\mathrm{SL}_2}$ of $\widehat{G/Z_\Delta}(\mathbb{C})$. This is a new period integral that has not been considered before. If we replace the cusp form on GSp_4

and GL_2 by Borel Eisenstein series, then the period integral $\mathcal{P}_\Delta$ becomes the Rankin-Selberg integral in [17].

Remark 1.16. *In this paper we will also encounter some representations $\hat{\rho}$ whose generic stabilizer is not connected. While these representations do not fit in the current framework of BZSV quadruple, one can still consider the associated period integrals and they are related to the previously studied integrals on covering groups, in* [8, 15, 21, 23, 35, 40].

1.5. **Organization of the paper.** In Section 2, we will explain our strategy for writing down the dual quadruple. In Sections 3-7, we will consider Tables 1, 2, 11, 12, and 22 of [28]. In Section 8 we summarize our findings in six tables. In Section 9 we will discuss Table S of [28].

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2. Our strategy

2.1. **Notation and convention.** In this paper, for a group G of Type A_n (resp. B_n, C_n, D_n, G_2, E_6, E_7), we use std_G to denote the n-dimensional (resp. $2n+1$-dimensional, $2n$-dimensional, $2n$-dimensional, 7-dimensional, 27-dimensional, 56-dimensional) standard representation of G. We use Spin_{2n} (resp. Spin_{2n+1}) to denote the Spin representation of the reductive group of Type D_n (resp. B_n) and we use HSpin_{2n} to denote the Half-Spin representation of reductive group with Type D_n. We use Sym^n (resp. $\wedge^n$) to denote the n-th symmetric power (resp. exterior power) of a reductive group of Type A. We use $\wedge_0^3$ to denote the third fundamental representation of a reductive group of Type C_3. Lastly, for a representation ρ of G, we use $\rho^\vee$ to denote the dual representation and $T(\rho)$ to denote $\rho \oplus \rho^\vee$.

In this paper, we always use l to denote the similitude character of a similitude group. If we have two similitude group GH_1 and GH_2, we let

$$G(H_1 \times H_2) = \{(h_1, h_2) \in GH_1 \times GH_2 |\ l(h_1) = l(h_2)\},$$

$$S(GH_1 \times GH_2) = \{(h_1, h_2) \in GH_1 \times GH_2 |\ l(h_1)l(h_2) = 1\}.$$

Similarly we can also define $G(H_1 \times \cdots \times H_n)$ and $S(GH_1 \times \cdots \times GH_n)$. For example,

$$S(\mathrm{GL}_2^3) = S(\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2) = \{(h_1, h_2, h_3) \in \mathrm{GL}_2^3 |\ \det(h_1 h_2 h_3) = 1\}.$$

All the nilpotent orbits considered in this paper are principal in a Levi subgroup (this is also the case in [1]). As a result, we will use the Levi subgroup or just the root type of the Levi subgroup to denote the nilpotent orbit (the zero nilpotent orbit is denoted by 1). For a split reductive group G, we will use T_G to denote a maximal split torus of G (a minimal Levi subgroup).

For a BZSV quadruple $\hat{\Delta} = (\hat{G}, \hat{G}, \hat{\rho}, 1)$, there are many other quadruples that is essentially equal to $\hat{\Delta}$ up to some central isogeny. To be specific, one can take any group $\hat{H}$ of the same root Type as $\hat{G}$ such that the representation $\hat{\rho}$ can also be defined on $\hat{H}$. Then one can choose any group $\hat{G}'$ containing $\hat{H}$ such that $\hat{G}' = \hat{H}Z_{\hat{G}'}$. The quadruple $(\hat{G}', \hat{H}, \hat{\rho}, 1)$ is essentially equal to $\hat{\Delta}$ up to some central isogeny. For example, both $(\mathrm{PGL}_2^3, \mathrm{PGL}_2, 0, 1)$ and $(\mathrm{GL}_2^3, \mathrm{GL}_2, 0, 1)$ can be viewed as trilinear GL_2-model. The dual quadruple of them are $(\mathrm{SL}_2^3, \mathrm{SL}_2^3, \hat{\rho}, 1)$ and $(\mathrm{GL}_2^3, S(\mathrm{GL}_2^3), \hat{\rho}, 1)$ where $\hat{\rho}$ is the tensor product of SL_2^3 and $S(\mathrm{GL}_2^3)$ respectively, and they are equal to each other up to some central isogeny. While there are various choices of dual quadruples pairs $(\Delta, \hat{\Delta})$ associated to $\hat{\rho}$ due to the isogeny issue, in this paper, for each representation $\hat{\rho}$ in [28], we will only write down one quadruple $\Delta = (G, H, \rho_H, \iota)$ whose dual quadruple $\hat{\Delta}$ is $(\hat{G}, \widehat{G/Z_\Delta}, \hat{\rho}, 1)$ where $Z_\Delta = Z_G \cap ker(\rho_H)$.

Remark 2.1. *In our proof of Theorem 1.7, we frequently quote the unramified computation in* [25] *and* [44]*. The settings in* [25] *and* [44] *may actually differ from ours through finite isogeny or central isogeny. It is clear that the computation can be adapted and the results there still apply. For example, in* [25]*, they computed the local relative character for the Gross-Prasad model* $(\mathrm{SO}_{n+1} \times \mathrm{SO}_n, \mathrm{SO}_n)$ *at unramified places. Their results can be also applied to models like* $(\mathrm{GL}_4 \times \mathrm{GSp}_4, \mathrm{GSp}_4)$ *(which is essentially the Gross-Prasad model* $(\mathrm{SO}_6 \times \mathrm{SO}_5, \mathrm{SO}_5)$ *up to some central isogeny).*

2.2. **Theta correspondence for classical groups.** In this paper we will frequently use theta correspondence for classical groups. We will briefly review it in this subsection. We start with the theta correspondence for the general linear group. Let $n \geq m \geq 1$ and $G = H_1 \times H_2 = \mathrm{GL}_n \times \mathrm{GL}_m$. We use V to denote the underlying vector space of the representation $\rho = std_{\mathrm{GL}_n} \otimes std_{\mathrm{GL}_m}$ of G. For $\varphi \in \mathcal{S}(V(\mathbb{A}))$, we define the theta function

$$\Theta_\psi^\varphi(g) = \sum_{X \in V(k)} \rho(g)\varphi(X),\ g \in G(\mathbb{A})$$

which is an automorphic function on $G(\mathbb{A}) = H_1(\mathbb{A}) \times H_2(\mathbb{A})$. Let π be a cuspidal automorphic representation of $H_2(\mathbb{A})$. For $\phi \in L^2(H_2(k)\backslash H_2(\mathbb{A}))_\pi$, the integral

$$\int_{H_2(k)\backslash H_2(\mathbb{A})} \Theta_\psi^\varphi(h_1, h_2)\phi(h_2)dh_2$$

gives an automorphic function on $H_1(\mathbb{A})$ which will be denoted by $\Theta(\phi)$.

Theorem 2.2. *(*[32]*) We have*

$$\{\Theta(\phi)|\ \phi \in L^2(H_2(k)\backslash H_2(\mathbb{A}))_\pi\} = \{E(\phi', 1)|\ \phi' \in L^2(H_2(k)\backslash H_2(\mathbb{A}))_\pi\}$$

where $E(\phi', 1)$ *is the Eisenstein series on* $H_1(\mathbb{A}) = \mathrm{GL}_n(\mathbb{A})$ *induced from* ϕ' *and the identity function on* $\mathrm{GL}_{n-m}(\mathbb{A})$*. Moreover, for* $\phi_1, \phi_2 \in L^2(H_2(k)\backslash H_2(\mathbb{A}))_\pi$*, we have the Rallis inner product formula*

$$\int_{H_2(k)\backslash H_2(\mathbb{A})/Z_{H_2}(\mathbb{A})} \int_{H_1(k)\backslash H_1(\mathbb{A})} \int_{H_1(k)\backslash H_1(\mathbb{A})} \Theta_\psi^\varphi(h_1, h_2)\Theta_\psi^\varphi(h_1', h_2)E(\phi_1, 1)(h_1)E(\phi_2, 1)(h_1')dh_1dh_1'dh_2$$

$$\text{``}=\text{''}\mathrm{Res}_{s=\frac{n-m}{2}} L(s + \frac{1}{2}, \pi) \cdot \int_{H_2(k)\backslash H_2(\mathbb{A})/Z_{H_2}(\mathbb{A})} \phi_1(h_2)\phi_2(h_2)dh_2.$$

Remark 2.3. *When* $m = 1$*, the above theorem implies that if we integrate the theta series on* GL_n *associated to the symplectic representation* $T(std_n)$ *over the center of* GL_n *we will get the mirabolic Eisenstein series of* GL_n*. We will frequently use this fact in later discussions.*

For the unramified computation, we also need the local theta correspondence for unramified representation. Let F be a p-adic local field that is a local place of k. We use $\phi_\rho(h_1,h_2)$ to denote the local spherical matrix coefficient of the Weil representation with $\phi_\rho(1,1)=1$. Let π be a tempered unramified representation of $H_2(F)$, ϕ_π (resp. $\phi_{\pi,1}$) be the unramified matrix coefficient of π (resp. $Ind_{\mathrm{GL}_m\times\mathrm{GL}_{n-m}}^{\mathrm{GL}_n}(\pi\otimes 1)$) with $\phi_\pi(1)=\phi_{\pi,1}(1)=1$.

Theorem 2.4. *([32]) With the notation above, we have*

$$\int_{H_2(F)}\phi_\rho(h_1,h_2)\phi_\pi(h_2)dh_2=L(\frac{n-m+1}{2},\pi)\cdot\phi_{\pi,1}(h_1).$$

Next we study the theta correspondence between SO_{2n} and Sp_{2m} with $n\geq m\geq 1$. Let $G=H_1\times H_2=\mathrm{SO}_{2n}\times\mathrm{Sp}_{2m}$ and we use V to denote the underlying vector space of the representation $\rho=std_{\mathrm{SO}_{2n}}\otimes std_{\mathrm{Sp}_{2m}}$ of G. Let Y be a maximal isotropic subspace of V, we can define $\Theta_\psi^\varphi(g)$ an automorphic function on $G(\mathbb{A})$ as in the introduction, for any Schwartz function φ on Y.

Let Π be a cuspidal tempered global Arthur packet of $H_2(\mathbb{A})=\mathrm{Sp}_{2m}(\mathbb{A})$ and let Π' be its lifting to $H_1(\mathbb{A})=\mathrm{SO}_{2n}(\mathbb{A})$ under the map $\mathrm{SO}_{2m+1}(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})\to\mathrm{SO}_{2n}(\mathbb{A})$ whose restrict to SL_2 is the principal embedding from SL_2 to $\mathrm{SO}_{2n-2m-1}$ (if $n>m$ then Π' is a non-tempered Arthur L-packet) [5]. For $\phi\in L^2(H_2(k)\backslash H_2(\mathbb{A}))_\pi$, the integral

$$\int_{H_2(k)\backslash H_2(\mathbb{A})}\Theta_\psi^\varphi(h_1,h_2)\phi(h_2)dh_2$$

gives an automorphic function on $H_1(\mathbb{A})=\mathrm{SO}_{2n}(\mathbb{A})$ which will be denoted by $\Theta(\phi)$. Then the following theorem holds.

Theorem 2.5. *([30, 45, 19]) With the notation above, the representation*

$$\{\Theta(\phi)|\ \phi\in L^2(\mathrm{Sp}_{2m}(k)\backslash\mathrm{Sp}_{2m}(\mathbb{A}))_\Pi\}$$

of $\mathrm{SO}_{2n}(\mathbb{A})$ *is a direct sum of some distinct irreducible representations belonging to the Arthur L-packet* Π' *of* $H_1(\mathbb{A})=\mathrm{SO}_{2n}(\mathbb{A})$. *Moreover, for* $\phi_1,\phi_2\in\Pi'$, *we have the Rallis inner product formula*

$$\int_{H_2(k)\backslash H_2(\mathbb{A})}\int_{H_1(k)\backslash H_1(\mathbb{A})}\int_{H_1(k)\backslash H_1(\mathbb{A})}\Theta_\psi^\varphi(h_1,h_2)\Theta_\psi^\varphi(h_1',h_2)\phi_1(h_1)\phi_2(h_1')dh_1dh_1'dh_2$$

$$\text{“}=\text{”}\mathrm{Res}_{s=\frac{2n-2m-1}{2}}L(s+\frac{1}{2},\Pi')\cdot\int_{H_1(k)\backslash H_1(\mathbb{A})}\phi_1(h_1)\phi_2(h_1)dh_1.$$

For the unramified computation, we also need the local theta correspondence for unramified representation. Let F be a p-adic local field that is a local place of k. We use $\phi_\rho(h_1,h_2)$ to denote the local spherical matrix coefficient of the Weil representation with $\phi_\rho(1,1)=1$. Let π be a tempered unramified representation of $H_2(F)$ and π' be its lifting to $H_1(F)$ (which is also unramified). Let ϕ_π (resp. $\phi_{\pi'}$) be the unramified matrix coefficient of π (resp. π') with $\phi_\pi(1)=\phi_{\pi'}(1)=1$.

Theorem 2.6. *([32]) With the notation above, we have*

$$\int_{H_2(F)}\phi_\rho(h_1,h_2)\phi_\pi(h_2)dh_2=L(n-m,\pi')\cdot\phi_{\pi'}(h_1).$$

[5]in fact here Π' should be an Arthur packet of $O_{2n}(\mathbb{A})$ which is the union of two Arthur packets of $\mathrm{SO}_{2n}(\mathbb{A})$ differed by the outer automorphism

The theta correspondence between SO_{2m} and Sp_{2n} (resp. GSO_{2n} and GSp_{2m}, GSO_{2m} and GSp_{2n}) is similar and we will skip it here.

2.3. **A conjecture of the duality under certain induction.** We recall the notion from the introduction. Let $\Delta = (G, H, \rho_H, \iota)$ be a BZSV quadruple. Let M be the centralizer of $\{\iota(diag(t, t^{-1}))|\ t \in \mathrm{GL}_1\}$ in G. It is easy to see that M is a Levi of G and $H \subset M$. We define

$$\Delta_{red} = (M, H, 1, \rho_{H,\iota})$$

where the representation $\rho_{H,\iota}$ has been defined in 1.1. It is clear that Δ is reductive if and only if $\Delta = \Delta_{red}$.

In Section 4.2.2 of [1], Ben-Zvi–Sakellaridis–Venkatesh made a conjecture about the relation between the dual of Δ and Δ_{red}. To state their conjecture, we first need a definition.

Definition 2.7. *Let M be a Levi subgroup of G and ρ be an irreducible representation of M with the highest weight ϖ_M. There exists a Weyl element w of G such that $w\varpi_M$ is a dominant weight of G* [6]*. We define $(\rho)_M^G$ to be the irreducible representation of G whose highest weight is $w\varpi_M$. In general, if $\rho = \oplus_i \rho_i$ is a finite-dimensional representation of M with ρ_i irreducible, we define*

$$(\rho)_M^G = \oplus_i (\rho_i)_M^G.$$

Now we are ready to state the conjecture.

Conjecture 2.8. *With the notation above. If the dual of Δ_{red} is given by $\hat{\Delta}_{red} = (\hat{M}, \hat{H}'_M, \rho', \hat{\iota}')$, then the dual of Δ is given by*

$$(\hat{G}, \hat{H}', (\rho')_{\hat{H}'_M}^{\hat{H}'}, \hat{\iota}')$$

where $\hat{H}'$ is generated by $\hat{H}'_M$ and $\{Im(\iota_\alpha)|\ \alpha \in \Delta_{\hat{G}} - \Delta_{\hat{M}}\}$. Here $\Delta_{\hat{G}}$ (resp. $\Delta_{\hat{M}}$) is the set of simple roots of $\hat{G}$ (resp. $\hat{M}$) and $\iota_\alpha : \mathrm{SL}_2 \to \hat{G}$ is the embedding associated to α.

2.4. **General strategy.** Let $\hat{\Delta} = (\hat{G}, \hat{G}, \hat{\rho}, 1)$ be a quadruple such that $\hat{\rho}$ is an anomaly-free symplectic representation of $\hat{G}$, and it appears in Table 1, 2, 11, 12, 22 of [28]. Our goal is to write down a dual quadruple (up to isogeny) $\Delta = (G, H, \rho_H, \iota)$.

The data in Knop's tables of [28], besides $(\hat{G}, \hat{\rho})$, also contains the following two items: a Levi subgroup $\hat{L}$ of $\hat{G}$ and a Weyl group $\hat{W}_V$ written in the form of $W_{\hat{H}}$ where $\hat{H}$ is the root type (e.g. A_n, B_n, C_n, etc). (In [28] the notations are L, G, W_V in place of $\hat{L}, \hat{G}, \hat{W}_V$ respectively.) Our key observation is that two data (H, ι) of the dual quadruple $\Delta = (G, H, \rho_H, \iota)$ are given by the following properties.

Property 2.9. *(1) The root type of H is dual to the root type of $\hat{W}_V$ in the tables of* [28].
(2) The nilpotent orbit $\mathcal{O}_\iota$ associated to ι is the principal nilpotent orbit of L where L is the dual Levi of $\hat{L}$.

Remark 2.10. *Basically, the Weyl group $\hat{W}_V$ can be viewed as the "little Weyl group" of the quadruple $\hat{\Delta} = (\hat{G}, \hat{G}, \hat{\rho}, 1)$, and $\hat{\mathfrak{l}}$ in tables of* [28] *is an analogue of $\hat{\mathfrak{l}}_X$ in Table 3 of* [29].

[6] the choice of w is not unique but $w\varpi_M$ is uniquely determined by ϖ_M

As a result, it remains to find out what is ρ_H. We do not have a systematic way to write down ρ_H. Instead we propose a ρ_H in an ad hoc way and then provide evidence for the duality between $\Delta = (G, H, \rho_H, \iota)$ and $(\hat{G}, \widehat{G/Z_\Delta}, \hat{\rho}, 1)$.

We provide two strong evidences for the duality. The first one is evidence for Conjecture 1.1, i.e., Theorem 1.7 and 1.9. The second evidence is for non-reductive models. For those models, we will show that the duality is compatible with Conjecture 2.8.

In the sections that follow, we will go through Knop's list of representations $\hat{\rho}$. For each ρ we write down a quadruple (G, H, ρ_H, ι). When the quadruple is not reductive, we will also write down Δ_{red} which is dual to another representation $(\hat{M}, \hat{\rho}_M)$ in Knop's list and verify that Theorem 1.12 holds. For cases in Table 21, 22, 23 and 24, we give references where the local relative character is calculated in the unramified places, thus verifying Theorem 1.7. We also verify Theorem 1.9 for the global periods associated to the dual side $\hat{\Delta}$ for cases in Table 21, 23 and 25.

3. Models in Table 1 of [28]

In this section we will consider Table 1 of [28], this is for the case when $\hat{\rho}$ is an irreducible representation of $\hat{G}$. It is easy to check that the representations in (1.2), (1.8), (1.9) and (1.10) of [28] are not anomaly free and the representation in (1.1) of [28] is only anomaly free when $p = 2n$ is even. Hence it remains to consider the following cases. Note that we only write the root type of $\hat{\mathfrak{l}}$ and we write 0 if it is abelian. Also we separate the cases when $\hat{\mathfrak{l}}$ is abelian and when $\hat{\mathfrak{l}}$ is not abelian. These are precisely the cases where the dual quadruple is reductive/non-reductive (see Property 2.9).

Number in [28]	$(\hat{G}, \hat{\rho})$	$\hat{W}_V$	$\hat{\mathfrak{l}}$
(1.1), p=2m	$(\mathrm{Sp}_{2m} \times \mathrm{SO}_{2m}, std_{\mathrm{Sp}_{2m}} \otimes std_{\mathrm{SO}_{2m}})$	D_m	0
(1.1), p=2m+2	$(\mathrm{Sp}_{2m} \times \mathrm{SO}_{2m+2}, std_{\mathrm{Sp}_{2m}} \otimes std_{\mathrm{SO}_{2m+2}})$	C_m	0
(1.3), m=2	$(\mathrm{Spin}_5 \otimes \mathrm{Spin}_7, \mathrm{Spin}_5 \otimes \mathrm{Spin}_7)$	$C_2 \times A_1$	0
(1.3), m=3	$(\mathrm{Sp}_6 \otimes \mathrm{Spin}_7, std_{\mathrm{Sp}_6} \otimes \mathrm{Spin}_7)$	$C_3 \times B_3$	0
(1.3), m=4	$(\mathrm{Sp}_8 \otimes \mathrm{Spin}_7, std_{\mathrm{Sp}_8} \otimes \mathrm{Spin}_7)$	$D_4 \times B_3$	0
(1.6)	(SL_2, Sym^3)	A_1	0

Table 1. Reductive models in Table 1 of [28]

3.1. **The reductive case.** In this subsection we consider the reductive cases, i.e., the ones in Table 1. The nilpotent orbit ι is trivial for all these cases so we will ignore it.

For (1.1) with $p = 2m$ (resp. $p = 2m+2$), the associated quadruple Δ is

$$(G, H, \rho_H) = (\mathrm{SO}_{2m+1} \times \mathrm{SO}_{2m}, \mathrm{SO}_{2m}, 0) \tag{3.1}$$

$$(\text{resp.} (G, H, \rho_H) = (\mathrm{SO}_{2m+1} \times \mathrm{SO}_{2m+2}, \mathrm{SO}_{2m+1}, 0)) \tag{3.2}$$

which is just the reductive Gross-Prasad model. The unramified computations in [25] prove Theorem 1.7 in these two cases. For the dual side, Theorem 2.5 applied to the theta correspondence between $\mathrm{SO}_{2m} \times \mathrm{Sp}_{2m}$ (resp. $\mathrm{SO}_{2m+2} \times \mathrm{Sp}_{2m}$) implies Conjecture 1.1(2) and this proves Theorem 1.9.

Number in [28]	$(\hat{G},\hat{\rho})$	$\hat{W}_V$	$\hat{\mathfrak{l}}$
(1.1), $p=2n<2m$	$(\mathrm{Sp}_{2m}\times \mathrm{SO}_{2n}, std_{\mathrm{Sp}_{2m}}\otimes std_{\mathrm{SO}_{2n}})$	D_n	C_{m-n}
(1.1), $p=2n>2m+2$	$(\mathrm{Sp}_{2m}\times \mathrm{SO}_{2n}, std_{\mathrm{Sp}_{2m}}\otimes std_{\mathrm{SO}_{2n}})$	C_m	D_{n-m}
(1.3), m=1	$(\mathrm{SL}_2\times \mathrm{Spin}_7, std_{\mathrm{SL}_2}\otimes \mathrm{Spin}_7)$	A_1	A_2
(1.3), $m>4$	$(\mathrm{Sp}_{2m}\otimes \mathrm{Spin}_7, std_{\mathrm{Sp}_{2m}}\otimes \mathrm{Spin}_7)$	$D_4\times B_3$	C_{m-4}
(1.4)	$(\mathrm{SL}_2\times \mathrm{Spin}_9, std_{\mathrm{SL}_2}\otimes \mathrm{Spin}_9)$	$A_1\times A_1$	A_2
(1.5), n=11	$(\mathrm{Spin}_{11}, \mathrm{Spin}_{11})$	A_1	A_4
(1.5), n=12	$(\mathrm{Spin}_{12}, \mathrm{HSpin}_{12})$	A_1	A_5
(1.5), n=13	$(\mathrm{Spin}_{13}, \mathrm{Spin}_{13})$	B_2	$A_2\times A_2$
(1.7)	$(SL_6,\wedge^3)$	A_1	$A_2\times A_2$
(1.11)	(E_7, std_{E_7})	A_1	E_6

Table 2. Non-reductive models in Table 1 of [28]

For (1.3) with $m=2$, the associated quadruple Δ is

$$(G,H,\rho_H)=(\mathrm{GSp}_6\times \mathrm{GSp}_4, G(\mathrm{Sp}_4\times \mathrm{Sp}_2),0)$$

which is the model $(\mathrm{GSp}_6\times\mathrm{GSp}_4, G(\mathrm{Sp}_4\times\mathrm{Sp}_2))$ studied in [44]. The unramified computations in [44] prove Theorem 1.7 in this case.

For (1.3) with $m=3$, the associated quadruple Δ is

$$(G,H,\rho_H)=(\mathrm{GSp}_6\times \mathrm{GSpin}_7, S(\mathrm{GSp}_6\times \mathrm{GSpin}_7), std_{\mathrm{Sp}_6}\otimes \mathrm{Spin}_7).$$

For (1.3) with $m=4$, the associated quadruple Δ is

$$(G,H,\rho_H)=(\mathrm{GSp}_6\times \mathrm{GSpin}_9, S(\mathrm{GSp}_6\times \mathrm{GSpin}_8), std_{\mathrm{Sp}_6}\otimes \mathrm{HSpin}_8). \tag{3.3}$$

Theorem 1.7 and 1.9 for two cases can be established by the same argument as Model (11.11) of [28] (see (5.4) and (5.3) of Section 5.1) together with the triality of D_4.

For (1.6), it is clear that the generic stabilizer of $\hat{\rho}$ in $\hat{G}$ is not connected, hence it does not belong to the current framework of BZSV duality. However, for this specific case, by the work of [21], we expect there is an associated quadruple of the form $(\mathrm{GL}_2,\mathrm{GL}_2,\rho_H,1)$ where ρ_H is no longer an anomaly free symplectic representation, but rather we understand that ρ_H corresponds to the theta series on $H=\mathrm{GL}_2$ defined via the cubic covering of GL_2 as in [21]. There is a covering group involved in the integral since the generic stabilizer is not connected. In [21] it is established that the nonvanishing of $\mathcal{P}_{H,\iota,\rho_H}(\phi)$ is equivalent to the nonvanishing of $L(1/2,\Pi,\hat{\rho})$. We expect further that Conjecture 1.1(1) holds in this case.

By the discussion above, the strongly tempered quadruple associated to Table 1 (without the row corresponding to (1.6)) is given as follows. Note that ι is trivial for all these cases.

(G, H, ρ_H)	$\hat{\rho}$
$(\mathrm{SO}_{2m+1}\times\mathrm{SO}_{2m},\ \mathrm{SO}_{2m},\ 0)$	$std_{\mathrm{Sp}_{2m}}\otimes std_{\mathrm{SO}_{2m}}$
$(\mathrm{SO}_{2m+2}\times\mathrm{SO}_{2m+1},\ \mathrm{SO}_{2m+1},\ 0)$	$std_{\mathrm{Sp}_{2m}}\otimes std_{\mathrm{SO}_{2m+2}}$
$(\mathrm{GSp}_6\times\mathrm{GSp}_4,\ G(\mathrm{Sp}_4\times\mathrm{Sp}_2),0)$	$\mathrm{Spin}_5\otimes\mathrm{Spin}_7$
$(\mathrm{GSp}_6\times\mathrm{GSpin}_7,\ S(\mathrm{GSp}_6\times\mathrm{GSpin}_7),\ std_{\mathrm{Sp}_6}\otimes\mathrm{Spin}_7)$	$std_{\mathrm{Sp}_6}\otimes\mathrm{Spin}_7$
$(\mathrm{GSp}_6\times\mathrm{GSpin}_9, S(\mathrm{GSp}_6\times\mathrm{GSpin}_8), std_{\mathrm{Sp}_6}\otimes\mathrm{HSpin}_8)$	$std_{\mathrm{Sp}_8}\otimes\mathrm{Spin}_7$

Table 3. Dual quadruples of Table 1

3.2. **The non-reductive case.** In this subsection we consider the non-reductive cases, i.e., the ones in Table 2.

For (1.1) with $p=2n<2m$, the associated quadruple Δ is

$$(\mathrm{SO}_{2m+1}\times \mathrm{SO}_{2n}, \mathrm{SO}_{2n}, 0, (\mathrm{GL}_1)^n\times \mathrm{SO}_{2m-2n+1}\times T_{\mathrm{SO}_{2n}})$$

and it is the Gross-Prasad period for $\mathrm{SO}_{2m+1}\times \mathrm{SO}_{2n}$. For (1.1) with $p=2n>2m+2$, the associated quadruple Δ is

$$(\mathrm{SO}_{2m+1}\times \mathrm{SO}_{2n}, \mathrm{SO}_{2m+1}, 0, T_{\mathrm{SO}_{2m+1}}\times (\mathrm{GL}_1)^m\times \mathrm{SO}_{2n-2m})$$

and it is still the Gross-Prasad period for $\mathrm{SO}_{2m+1}\times \mathrm{SO}_{2n}$. In these two cases Δ_{red} are given by (3.1), (3.2). It is clear that Theorem 1.12 holds in these two cases. The unramified computation in [25] proves Theorem 1.7 for these two cases. Theorem 2.5 applied to the theta correspondence between $\mathrm{SO}_{2n}\times \mathrm{Sp}_{2m}$ implies Conjecture 1.1(2) and proves Theorem 1.9 for these two cases.

For (1.3) when $m=1$, the associated quadruple Δ is

$$(\mathrm{GSp}_6\times \mathrm{GL}_2, \mathrm{GL}_2, 0, (\mathrm{GL}_3\times \mathrm{GL}_1)\times T_{\mathrm{GL}_2}) \tag{3.4}$$

and it is the model $(\mathrm{GSp}_6\times \mathrm{GL}_2, \mathrm{GL}_2\ltimes U)$ studied in [44]. In this case $\Delta_{red}=((\mathrm{GL}_2)^3, \mathrm{GL}_2, 0, 1)$ (which a special case of (3.2) with $m=1$). It is clear that Theorem 1.12 holds in this case and the unramified computation in [44] proves Theorem 1.7 in this case.

For (1.3) when $m>4$, the associated quadruple Δ is

$$(\mathrm{GSpin}_{2m+1}\times \mathrm{GSp}_6, S(\mathrm{GSpin}_8\times \mathrm{GSp}_6), std_{\mathrm{Sp}_6}\otimes \mathrm{HSpin}_8, L)$$

where L is the Levi subgroup whose projection to GSpin_{2m+1} (resp. GSp_6) is of the form $(\mathrm{GL}_1)^4\times \mathrm{GSpin}_{2m-7}$ (resp. the maximal torus). The nilpotent orbit induces a Bessel period for the unipotent radical of the parabolic subgroup $P=MU$ with $M=(\mathrm{GL}_1)^{m-4}\times \mathrm{GSpin}_9\times \mathrm{GSp}_6$ whose stabilizer is $\mathrm{GSpin}_8\times \mathrm{GSp}_6$ and we can naturally embed H into the stabilizer. In this case Δ_{red} is given by (3.3) and it is clear that Theorem 1.12 holds. Theorem 1.7 and 1.9 for this model can be established by the same argument as (5.8) in Section 5.2 together with the triality of D_4.

For (1.4), the associated quadruple Δ is

$$(\mathrm{GSp}_8\times \mathrm{GL}_2, G(\mathrm{SL}_2\times \mathrm{SL}_2), 0, \mathrm{GL}_3\times \mathrm{GL}_1\times \mathrm{GL}_1\times T_{\mathrm{GL}_2}). \tag{3.5}$$

The nilpotent orbit induces a Bessel period for the unipotent radical of the parabolic subgroup $P=MU$ with $M=\mathrm{GL}_2\times \mathrm{GSp}_4\times \mathrm{GL}_2$ whose stabilizer is $G(\mathrm{SL}_2\times \mathrm{SL}_2)\times \mathrm{GL}_2$. We embeds H into the stabilizer so that the induced embedding from H into M is given by the natural embeddings of H into GSp_4 and into $\mathrm{GL}_2\times \mathrm{GL}_2$. In this case $\Delta_{red}=(\mathrm{GSp}_4\times \mathrm{GL}_2\times \mathrm{GL}_2, G(\mathrm{SL}_2\times \mathrm{SL}_2), 0, 1)$ which is essentially the Gross-Prasad model for $\mathrm{SO}_5\times \mathrm{SO}_4$. If we replace the cusp form on GL_2 by an Eisenstein series, we recover the Rankin-Selberg integrals in [7]. It is clear that Theorem 1.12 holds in this case and the unramfied computation in [7] proves Theorem 1.7 in this case.

For (1.5) when $n=11$, the associated quadruple Δ is

$$(\mathrm{GSp}_{10}, \mathrm{GL}_2, 0, \mathrm{GL}_5\times \mathrm{GL}_1) \tag{3.6}$$

and it is the model $(\mathrm{GSp}_{10}, \mathrm{GL}_2\ltimes U)$ studied in [44]. In this case $\Delta_{red}=((\mathrm{GL}_2)^3, \mathrm{GL}_2, 0, 1)$. It is clear that Theorem 1.12 holds in this case and the unramified computation in [44] proves Theorem 1.7 in this case.

For (1.5) when $n=12$, the associated quadruple Δ is

$$(\mathrm{GSO}_{12}, \mathrm{GL}_2, 0, \mathrm{GL}_6 \times \mathrm{GL}_1)$$

and it is the model $(\mathrm{GSO}_{12}, \mathrm{GL}_2 \ltimes U)$ studied in [44]. In this case $\Delta_{red} = ((\mathrm{GL}_2)^3, \mathrm{GL}_2, 0, 1)$. It is clear that Theorem 1.12 holds in this case and the unramified computation in [44] proves Theorem 1.7 in this case.

For (1.5) when $n=13$, the associated quadruple Δ is

$$(\mathrm{GSp}_{12}, \mathrm{GSp}_4, 0, \mathrm{GL}_3 \times \mathrm{GL}_3 \times \mathrm{GL}_1).$$

The nilpotent orbit induces a Bessel period for the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_4 \times \mathrm{GSp}_4$ whose stabilizer is $H = \mathrm{GSp}_4$. In this case $\Delta_{red} = (\mathrm{GSp}_4 \times \mathrm{GL}_4, \mathrm{GSp}_4, 0, 1)$ which is essentially the Gross-Prasad model for $\mathrm{SO}_6 \times \mathrm{SO}_5$. It is clear that Theorem 1.12 holds in this case. In this case the unramified computation can be done in a similar way as [44], which will give Theorem 1.7.

For (1.7), the associated quadruple Δ is

$$(\mathrm{GL}_6, \mathrm{GL}_2, 0, \mathrm{GL}_3 \times \mathrm{GL}_3)$$

and it is the Ginzburg-Rallis model $(\mathrm{GL}_6, \mathrm{GL}_2 \ltimes U)$ studied in [44]. In this case $\Delta_{red} = ((\mathrm{GL}_2)^3, \mathrm{GL}_2, 0, 1)$. It is clear that Theorem 1.12 holds in this case and the unramified computation in [44] proves Theorem 1.7 in this case.

For (1.11), the associated quadruple Δ is

$$(E_7, \mathrm{PGL}_2, 0, GE_6)$$

and it is the model $(E_7, \mathrm{PGL}_2 \ltimes U)$ studied in [44]. In this case $\Delta_{red} = ((\mathrm{PGL}_2)^3, \mathrm{PGL}_2, 0, 1)$. It is clear that Theorem 1.12 holds in this case and the unramified computation in [44] proves Theorem 1.7 in this case.

By the discussion above, the strongly tempered quadruple associated to Table 2 is given as follows. Here for ι, we only list the root type of the Levi subgroup L of G such that ι is principal in L.

(G, H, ρ_H)	ι	$\hat{\rho}$
$(\mathrm{SO}_{2m+1} \times \mathrm{SO}_{2n}, \mathrm{SO}_{2n}, 0)$	B_{m-n}	$std_{\mathrm{Sp}_{2m}} \otimes std_{\mathrm{SO}_{2n}}$
$(\mathrm{SO}_{2m+1} \times \mathrm{SO}_{2n}, \mathrm{SO}_{2m+1}, 0)$	D_{n-m}	$std_{\mathrm{Sp}_{2m}} \otimes std_{\mathrm{SO}_{2n}}$
$(\mathrm{GSp}_6 \times \mathrm{GL}_2, \mathrm{GL}_2, 0)$	A_2	$std_{\mathrm{GL}_2} \otimes \mathrm{Spin}_7$
$(\mathrm{GSpin}_{2m+1} \times \mathrm{GSp}_6, S(\mathrm{GSpin}_8 \times \mathrm{GSp}_6), std_{\mathrm{Sp}_6} \otimes \mathrm{HSpin}_8)$	B_{m-4}	$std_{\mathrm{Sp}_{2m}} \otimes \mathrm{Spin}_7$
$(\mathrm{GSp}_8 \times \mathrm{GL}_2, G(\mathrm{SL}_2 \times \mathrm{SL}_2), 0)$	A_2	$std_{\mathrm{GL}_2} \otimes \mathrm{Spin}_9$
$(\mathrm{GSp}_{10}, \mathrm{GL}_2, 0)$	A_4	Spin_{11}
$(\mathrm{GSO}_{12}, \mathrm{GL}_2, 0)$	A_5	HSpin_{12}
$(\mathrm{GSp}_{12}, \mathrm{GSp}_4, 0)$	$A_2 \times A_2$	Spin_{13}
$(\mathrm{GL}_6, \mathrm{GL}_2, 0)$	$A_2 \times A_2$	$\wedge^3$
$(E_7, \mathrm{PGL}_2, 0)$	E_6	std_{E_7}

TABLE 4. Dual quadruples of Table 2

4. Models in Table 2

In this section we will consider Table 2 of [28], this is for the case when $\hat{\rho} = T(\hat{\tau})$ is the direct sum of two irreducible representations of $\hat{G}$ that are dual to each other. All the representations in Table 2 of [28] are anomaly free, so we need to consider all of them. We still separate the cases based on whether $\hat{\mathfrak{l}}$ is abelian or not.

Number in [28]	$(\hat{G}, \hat{\rho})$	$\hat{W}_V$	$\hat{\mathfrak{l}}$
(2.1), m=n	$(\mathrm{GL}_n \times \mathrm{GL}_n, T(std_{\mathrm{GL}_n} \otimes std_{\mathrm{GL}_n}))$	A_{n-1}	0
(2.1), m=n+1 and (2.4), n=2	$(\mathrm{GL}_{n+1} \times \mathrm{GL}_n, T(std_{\mathrm{GL}_{n+1}} \otimes std_{\mathrm{GL}_n}))$	A_{n-1}	0
(2.3)	$(\mathrm{GL}_n, T(Sym^2))$	A_{n-1}	0
(2.6), m=n=2	$(\mathrm{Sp}_4 \times \mathrm{GL}_2, T(Std_{\mathrm{Sp}_4} \otimes Std_{\mathrm{GL}_2}))$	$A_1 \times A_1$	0
(2.6), m=2, n=3	$(\mathrm{Sp}_4 \times \mathrm{GL}_3, T(Std_{\mathrm{Sp}_4} \otimes Std_{\mathrm{GL}_3}))$	$C_2 \times A_2$	0
(2.6), m=2, n=4	$(\mathrm{Sp}_4 \times \mathrm{GL}_4, T(Std_{\mathrm{Sp}_4} \otimes Std_{\mathrm{GL}_4}))$	$C_2 \times A_3$	0
(2.6), m=2, n=5	$(\mathrm{Sp}_4 \times \mathrm{GL}_5, T(Std_{\mathrm{Sp}_4} \otimes Std_{\mathrm{SL}_5}))$	$C_2 \times A_3$	0
(2.6), m=n=3	$(\mathrm{Sp}_6 \times \mathrm{GL}_3), T(Std_{\mathrm{Sp}_6} \otimes Std_{\mathrm{GL}_3}))$	$A_3 \times A_2$	0

Table 5. Reductive models in Table 2 of [28]

Number in [28]	$(\hat{G}, \hat{\rho})$	$\hat{W}_V$	$\hat{\mathfrak{l}}$
(2.1), $m > n+1$, and (2.4), $n > 2$	$(\mathrm{GL}_m \times \mathrm{GL}_n, T(std_{\mathrm{GL}_m} \otimes std_{\mathrm{GL}_n}))$	A_{n-1}	A_{m-n-1}
(2.2), n=2m	$(\mathrm{GL}_{2m}, T(\wedge^2))$	A_{m-1}	$(A_1)^m$
(2.2),n=2m+1	$(\mathrm{GL}_{2m+1}, T(\wedge^2))$	A_{m-1}	$(A_1)^m$
(2.5)	$(\mathrm{Sp}_{2n}, T(std_{\mathrm{Sp}_{2n}})$	0	C_{m-1}
(2.6), $m > 2$, n=2	$(\mathrm{Sp}_{2m} \times \mathrm{SL}_2, T(Std_{\mathrm{Sp}_{2m}} \otimes Std_{\mathrm{SL}_2}))$	$A_1 \times A_1$	C_{m-2}
(2.6), m=2, $n > 5$	$(\mathrm{Sp}_4 \times \mathrm{SL}_m, T(Std_{\mathrm{Sp}_4} \otimes Std_{\mathrm{SL}_m}))$	$C_2 \times A_3$	A_{m-5}
(2.6), $m > 3$, n=3	$(\mathrm{Sp}_{2m} \times \mathrm{SL}_3, T(Std_{\mathrm{Sp}_{2m}} \otimes Std_{\mathrm{SL}_3}))$	$A_3 \times A_2$	C_{m-3}
(2.7), m=2k	$(\mathrm{SO}_{2k}, T(std_{\mathrm{SO}_{2k}}))$	A_1	D_{k-1}
(2.7), m=2k+1	$(\mathrm{SO}_{2k+1}, T(std_{\mathrm{SO}_{2k+1}}))$	A_1	B_{k-1}
(2.8), n=7	$(\mathrm{Spin}_7, T(\mathrm{Spin}_7))$	A_1	A_2
(2.8), n=9	$(\mathrm{Spin}_9, T(\mathrm{Spin}_9))$	$A_1 \times A_1$	A_2
(2.8), n=10	$(\mathrm{Spin}_{10}, T(\mathrm{HSpin}_{10}))$	A_1	A_3
(2.9)	$(G_2, T(std_{G_2}))$	A_1	A_1
(2.10)	$(E_6, T(std_{E_6}))$	A_2	D_4

Table 6. Non-reductive models in Table 2 of [28]

4.1. **The reductive case.** In this subsection we consider the reductive cases, i.e., the ones in Table 5.

For (2.1) with $m = n$, the associated quadruple Δ is given by

$$(G, H, \rho_H, \iota) = (\mathrm{GL}_n \times \mathrm{GL}_n, \mathrm{GL}_n, T(std_{\mathrm{GL}_n}), 1). \tag{4.1}$$

For (2.1) with $m = n+1$ and (2.4) with $n = 2$, the associated quadruple Δ is given by

$$(G, H, \rho_H, \iota) = (\mathrm{GL}_{n+1} \times \mathrm{GL}_n, \mathrm{GL}_n, 0, 1). \tag{4.2}$$

The period integrals in these two cases are exactly the Rankin-Selberg integral for $\mathrm{GL}_n \times \mathrm{GL}_n$ and $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$ in [26]. The result in loc. cit. proves Conjecture 1.1(1) and Theorem 1.7. For the dual side, Theorem 2.2 applied to the theta correspondence for $\mathrm{GL}_n \times \mathrm{GL}_{n+1}$ and $\mathrm{GL}_n \times \mathrm{GL}_n$ imply Conjecture 1.1(2) and this proves Theorem 1.9.

For (2.3), the generic stabilizer of $\hat{\rho}$ in $\hat{G}$ is not connected, hence it does not belong to the current framework of the BZSV duality. However, for this specific case, by the Rankin-Selberg integral in [8, 35, 40], we know that the dual integral should be the one in [8, 35, 40]. As the generic stabilizer is not connected, there are covering groups involved in the integral.

For (2.6) with $m = n = 2$, the associated quadruple Δ is given by

$$(G, H, \rho_H, \iota) = (\mathrm{GSp}_4 \times \mathrm{GL}_2, G(\mathrm{SL}_2 \times \mathrm{SL}_2), T(std_{\mathrm{GL}_{2,2},}), 1) \tag{4.3}$$

where the embedding of H into G is given by the canonical embedding from $\mathrm{GSpin}_4 = G(\mathrm{SL}_2 \times \mathrm{SL}_2)$ into $\mathrm{GSpin}_5 = \mathrm{GSp}_4$ and the projection of $G(\mathrm{SL}_2 \times \mathrm{SL}_2)$ into GL_2 via the first GL_2-copy. The representation ρ_H is the standard representation of the second GL_2-copy of H. This integral is essentially the Gross-Prasad model for $\mathrm{SO}_5 \times \mathrm{SO}_4$ except we replace the cusp form on one GL_2-copy by the theta series. The unramified computation in [25] proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.2 applied to the theta correspondence of $\mathrm{GL}_2 \times \mathrm{GL}_4$ and Gan-Gross-Prasad conjecture (Conjecture 9.11 of [13]) for non-tempered Arthur packet for the pair $(\mathrm{GL}_4 \times \mathrm{GSp}_4, \mathrm{GSp}_4)$ which is essentially the Gross-Prasad period for $\mathrm{SO}_6 \times \mathrm{SO}_5$. This proves Theorem 1.9.

For (2.6) with $m = 2, n = 3$, the associated quadruple Δ is given by

$$(G, H, \rho_H, \iota) = (\mathrm{GSp}_4 \times \mathrm{GL}_3, \mathrm{GSp}_4 \times \mathrm{GL}_3, T(std_{\mathrm{GSp}_4} \otimes std_{\mathrm{GL}_3}), 1).$$

By the theta correspondence for $\mathrm{GL}_3 \times \mathrm{GL}_4$ (note that the theta function constructed from $T(std_{\mathrm{GSp}_4} \otimes std_{\mathrm{GL}_3})$ is the restriction of the theta function from $T(std_{\mathrm{GL}_4} \otimes std_{\mathrm{GL}_3})$), the integral over GL_3 of a cusp form on GL_3 with the theta series associated to ρ_H produces an Eisenstein series of GL_4 induced from the cusp form on GL_3 and the trivial character of GL_1. Then the integral over GSp_4 is just the period integral for the pair $(\mathrm{GL}_4 \times \mathrm{GSp}_4, \mathrm{GSp}_4)$ which is essentially the Gross-Prasad period for $\mathrm{SO}_6 \times \mathrm{SO}_5$. The unramified computation in [25] and Theorem 2.4 applied to theta correspondence for $\mathrm{GL}_3 \times \mathrm{GL}_4$ proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.2 applied to the theta correspondence of $\mathrm{GL}_4 \times \mathrm{GL}_3$ and the global period integral conjecture for the pair $(\mathrm{GL}_4 \times \mathrm{GSp}_4, \mathrm{GSp}_4)$ (which is essentially the Gross-Prasad period for $\mathrm{SO}_6 \times \mathrm{SO}_5$) in [12]. This proves Theorem 1.9.

For (2.6) with $m = 2, n = 4$, the associated quadruple Δ is

$$(\mathrm{GSp}_4 \times \mathrm{GL}_4, S(\mathrm{GSp}_4 \times \mathrm{GL}_4), std_{\mathrm{Sp}_4} \otimes \wedge^2 \oplus T(std_{\mathrm{GL}_4})). \tag{4.4}$$

By the theta correspondence for $\mathrm{GSp}_4 \times \mathrm{GSO}_6$, the integral over Sp_4 of a cusp form on GSp_4 with the theta series associated to ρ_H produces an automorphic form of GL_4. Then the integral over GL_4 is just the Rankin-Selberg integral of $\mathrm{GL}_4 \times \mathrm{GL}_4$ as in [26] [7]. The Rankin-Selberg integral in [26] and Theorems 2.2 and 2.4 applied to theta correspondence for $\mathrm{GSp}_4 \times \mathrm{GSO}_6$ proves Conjecture 1.1(1) and Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.2 applied to the theta correspondence of $\mathrm{GL}_4 \times \mathrm{GL}_4$ and the global period integral conjecture for the pair $(\mathrm{GL}_4 \times \mathrm{GSp}_4, \mathrm{GSp}_4)$ (which is essentially the Gross-Prasad period for $\mathrm{SO}_6 \times \mathrm{SO}_5$) in [12]. This proves Theorem 1.9. This is a very interesting case because both Δ and $\hat{\Delta}$ are strongly tempered and they are not equal to each other.

For (2.6) with $m = 2, n = 5$, the associated quadruple Δ is

$$(\mathrm{GSp}_4 \times \mathrm{GL}_5, S(\mathrm{GSp}_4 \times \mathrm{GL}_4), std_{\mathrm{Sp}_4} \otimes \wedge^2). \tag{4.5}$$

By the theta correspondence for $\mathrm{GSp}_4 \times \mathrm{GSO}_6$, the integral over Sp_4 of a cusp form on GSp_4 with the theta series associated to ρ_H produces an automorphic form of GL_4. Then the integral over GL_4 is just the Rankin-Selberg integral of $\mathrm{GL}_5 \times \mathrm{GL}_4$. The Rankin-Selberg integral in [26] and Theorems 2.5 and 2.6 applied to theta correspondence $\mathrm{GSp}_4 \times \mathrm{GSO}_6$ proves Conjecture 1.1(1) and Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.2 applied to the theta correspondence of $\mathrm{GL}_4 \times \mathrm{GL}_5$ and the global period integral conjecture for the pair $(\mathrm{GL}_4 \times \mathrm{GSp}_4, \mathrm{GSp}_4)$ (which is essentially the Gross-Prasad period for $\mathrm{SO}_6 \times \mathrm{SO}_5$) in [12]. This proves Theorem 1.9.

For (2.6) with $m = n = 3$, the associated quadruple Δ is given by

$$(\mathrm{GSpin}_7 \times \mathrm{GL}_3, \mathrm{GSpin}_6 \times \mathrm{GL}_3, T(\mathrm{HSpin}_6 \otimes std_{\mathrm{GL}_3})). \tag{4.6}$$

By the theta correspondence for $\mathrm{GL}_3 \times \mathrm{GL}_4$ (note that GSpin_6 is essentially GL_4 up to some central isogeny which won't affect the unramified computation) the integral over GL_3 of a cusp form on GL_3 with the theta series associated to ρ_H produces an Eisenstein series of GSpin_6 induced from the cusp form on GL_3 and the trivial character of GL_1. Then the integral over GSpin_6 is just the period integral for the Gross-Prasad model of $\mathrm{GSpin}_7 \times \mathrm{GSpin}_6$. The unramified computation in [25] and Theorem 2.4 applied to theta correspondence for $\mathrm{GL}_3 \times \mathrm{GL}_4$ proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.5 applied to the theta correspondence of $\mathrm{GSp}_6 \times \mathrm{GSO}_6$ and the Rankin-Selberg integral of $\mathrm{GL}_4 \times \mathrm{GL}_3$. This proves Theorem 1.9.

By the discussion above, the strongly tempered quadruple associated to Table 5 is given as follows. Note that ι is trivial for all these cases.

(G, H, ρ_H)	$\hat{\rho}$
$(\mathrm{GL}_n \times \mathrm{GL}_n, \mathrm{GL}_n, T(std_{\mathrm{GL}_n}))$	$T(std_{\mathrm{GL}_n} \otimes std_{\mathrm{GL}_n})$
$(\mathrm{GL}_{n+1} \times \mathrm{GL}_n, \mathrm{GL}_n, 0)$	$T(std_{\mathrm{GL}_{n+1}} \otimes std_{\mathrm{GL}_n})$
$(\mathrm{GSp}_4 \times \mathrm{GL}_2, G(\mathrm{SL}_2 \times \mathrm{SL}_2), T(std_{\mathrm{GL}_{2,2}}))$	$T(Std_{\mathrm{GSp}_4} \otimes Std_{\mathrm{GL}_2})$
$(\mathrm{GSp}_4 \times \mathrm{GL}_3, H = G, T(std_{\mathrm{GSp}_4} \otimes std_{\mathrm{GL}_3}))$	$T(Std_{\mathrm{GSp}_4} \otimes Std_{\mathrm{GL}_3})$
$(\mathrm{GSp}_4 \times \mathrm{GL}_4, S(\mathrm{GSp}_4 \times \mathrm{GL}_4), std_{\mathrm{Sp}_4} \otimes \wedge^2 \oplus T(std_{\mathrm{GL}_4}))$	$T(Std_{\mathrm{GSp}_4} \otimes Std_{\mathrm{GL}_4})$
$(\mathrm{GSp}_4 \times \mathrm{GL}_5, S(\mathrm{GSp}_4 \times \mathrm{GL}_4), std_{\mathrm{Sp}_4} \otimes \wedge^2)$	$T(Std_{\mathrm{GSp}_4} \otimes Std_{\mathrm{GL}_5})$
$(\mathrm{GSpin}_7 \times \mathrm{GL}_3, \mathrm{GSpin}_6 \times \mathrm{GL}_3, T(\mathrm{HSpin}_6 \otimes std_{\mathrm{GL}_3}))$	$T(Std_{\mathrm{GSp}_6} \otimes Std_{\mathrm{GL}_3})$

TABLE 7. Dual quadruples of Table 5

[7] In this paper we will frequently use the fact that the theta series associated to $\rho \oplus \rho'$ is the product of the theta series associated to ρ and ρ'.

4.2. **The non-reductive case.** For (2.1) with $m > n+1$ and (2.4) with $n > 2$, the associated quadruple Δ is given by

$$(G, H, \rho_H, \iota) = (\mathrm{GL}_m \times \mathrm{GL}_n, \mathrm{GL}_n, 0, (\mathrm{GL}_1^n \times \mathrm{GL}_{m-n} \times T_{\mathrm{GL}_n}).$$

When $m-n$ is odd (resp. even), the nilpotent orbit induces a Bessel period (resp. Fourier-Jacobi period) for the unipotent radical of the parabolic subgroup $P = MU$ with $M = (\mathrm{GL}_1)^{m-n-1} \times \mathrm{GL}_{n+1} \times \mathrm{GL}_n$ (resp. $M = (\mathrm{GL}_1)^{m-n} \times \mathrm{GL}_n \times \mathrm{GL}_n$) whose stabilizer in M is $\mathrm{GL}_n \times \mathrm{GL}_n$. We can diagonally embed H into the stabilizer. In this case Δ_{red} is given by the quadruple (4.2) (resp. (4.1)). It is clear that Theorem 1.12 holds in this case. The period integral in this case is closely related to the Rankin-Selberg integral in [26]. However the difference is not negligible and we do not claim Theorem 1.7 for this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.2 applied to the theta correspondence for $\mathrm{GL}_n \times \mathrm{GL}_m$. This proves Theorem 1.9.

For (2.2) with $n = 2m$, the associated quadruple Δ is given by

$$(\mathrm{GL}_{2m}, \mathrm{GL}_m, T(std_{\mathrm{GL}_m}), (\mathrm{GL}_2)^m).$$

The nilpotent orbit induces a Bessel period for the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_m \times \mathrm{GL}_m$ whose stabilizer in M is $H = \mathrm{GL}_m$. In this case Δ_{red} is given by (4.1). It is clear that Theorem 1.12 holds in this case. The period integral in this case is exactly the Rankin-Selberg integral in [27]. The result in loc. cit. proves Conjecture 1.1(1) and Theorem 1.7.

For (2.2) with $n = 2m+1$, the associated quadruple Δ is given by

$$(\mathrm{GL}_{2m+1}, \mathrm{GL}_m, 0, (\mathrm{GL}_2)^m \times \mathrm{GL}_1).$$

The nilpotent orbit induces a Fourier-Jacobi period for the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_m \times \mathrm{GL}_1 \times \mathrm{GL}_m$ whose stabilizer in M is $\mathrm{GL}_n \times \mathrm{GL}_1$. We can naturally embed H into the stabilizer. In this case Δ_{red} is given by (4.1). It is clear that Theorem 1.12 holds in this case. The period integral in this case is exactly the Rankin-Selberg integral in [27]. The result in loc. cit. proves Conjecture 1.1(1) and Theorem 1.7.

For (2.5), the associated quadruple Δ is given by

$$(\mathrm{SO}_{2m+1}, \mathrm{SO}_2, 0, \mathrm{SO}_{2m-1} \times \mathrm{GL}_1).$$

It is the Gross-Prasad model of $\mathrm{SO}_{2m+1} \times \mathrm{SO}_2$ and Δ_{red} is given by (3.1) when $m = 1$. It is clear that Theorem 1.12 holds in this case. The unramified computation in [25] proves Theorem 1.7. For the dual side, Conjecture 1.1(2) follows from Theorem 2.5 applied to the theta correspondence for $\mathrm{Sp}_{2m} \times \mathrm{SO}_2$ and this proves Theorem 1.9.

For (2.6) with $m > 2, n = 2$, the associated quadruple Δ is given by

$$(G, H, \rho_H, \iota) = (\mathrm{GSpin}_{2m+1} \times \mathrm{GL}_2, G(\mathrm{SL}_2 \times \mathrm{SL}_2), T(std_{\mathrm{GL}_2}), (\mathrm{GL}_1)^2 \times \mathrm{GSpin}_{2m-3} \times T_{\mathrm{GL}_2,2}).$$

The nilpotent orbit ι induces a Bessel period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GSpin}_5 \times (\mathrm{GL}_1)^{m-2} \times \mathrm{GL}_2$ whose stabilizer in M is $\mathrm{GSpin}_4 \times \mathrm{GL}_2$. We then embeds $H = G(\mathrm{SL}_2 \times \mathrm{SL}_2)$ into $\mathrm{GSpin}_4 \times \mathrm{GL}_2$ via the identity map on GSpin_4 and the projection of $G(\mathrm{SL}_2 \times \mathrm{SL}_2)$ into GL_2 via the first GL_2-copy. The representation ρ_H is the standard representation of the second GL_2-copy of H. This integral is essentially the Gross-Prasad model for $\mathrm{GSpin}_{2m+1} \times \mathrm{GSpin}_4$ except we replace the cusp form on one GL_2-copy by theta series. In this case Δ_{red} is given by (4.3). It is clear that Theorem 1.12 holds in this case. The unramified computation in [25] proves Theorem 1.7. For the dual side, Conjecture 1.1(2) follows from Theorem 2.5 applied to the theta correspondence for $\mathrm{GSp}_{2n} \times \mathrm{GSO}_4$ and the Rankin-Selberg integral of $\mathrm{GL}_2 \times \mathrm{GL}_1$. This proves Theorem 1.9.

For (2.6) with $m = 2, n > 5$, the associated quadruple Δ is

$$(\mathrm{GSp}_4 \times \mathrm{GL}_n, S(\mathrm{GSp}_4 \times \mathrm{GL}_4), std_{\mathrm{Sp}_4} \otimes \wedge^2, T_{\mathrm{GSp}_4} \times (\mathrm{GL}_1)^4 \times \mathrm{GL}_{n-4}).$$

When n is odd (resp. even), the nilpotent orbit induces a Bessel period (resp. Fourier-Jacobi period) for the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GSp}_4 \times \mathrm{GL}_5 \times (\mathrm{GL}_1)^5$ (resp. $M = \mathrm{GSp}_4 \times \mathrm{GL}_4 \times (\mathrm{GL}_1)^4$) whose stabilizer in M is $\mathrm{GSp}_4 \times \mathrm{GL}_4$. We can naturally embed H into the stabilizer. In this case Δ_{red} is given by (4.5) (resp. (4.4)). It is clear that Theorem 1.12 holds in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.2 applied to the theta correspondence of $\mathrm{GL}_n \times \mathrm{GL}_4$ and the global period integral conjecture for the pair $(\mathrm{GL}_4 \times \mathrm{GSp}_4, \mathrm{GSp}_4)$ (which is essentially the Gross-Prasad period for $\mathrm{SO}_6 \times \mathrm{SO}_5$) in [12]. This proves Theorem 1.9.

For (2.6) with $m > 3, n = 3$, the associated quadruple Δ is given by

$$(\mathrm{GSpin}_{2m+1} \times \mathrm{GL}_3, \mathrm{GSpin}_6 \times \mathrm{GL}_3, T(\mathrm{HSpin}_6 \otimes std_{\mathrm{GL}_3}), (\mathrm{GL}_1)^3 \times \mathrm{GSpin}_{2m-5} \times T_{\mathrm{GL}_3}).$$

The nilpotent orbit ι induces a Bessel period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GSpin}_7 \times (\mathrm{GL}_1)^{m-3} \times \mathrm{GL}_3$ whose stabilizer in M is $H = \mathrm{GSpin}_6 \times \mathrm{GL}_3$. In this case Δ_{red} is given by (4.6). It is clear that Theorem 1.12 holds in this case. The unramified computation in [25] and Theorem 2.4 applied to theta correspondence for $\mathrm{GL}_4 \times \mathrm{GL}_3$ proves Theorem 1.7. For the dual side, Conjecture 1.1(2) follows from Theorem 2.5 applied to the theta correspondence of $\mathrm{GSp}_{2n} \times \mathrm{GSO}_6$ and the Rankin-Selberg period for $\mathrm{GL}_4 \times \mathrm{GL}_3$. This proves Theorem 1.9.

For (2.7) with $m = 2k$, the associated quadruple Δ is

$$(\mathrm{GSpin}_{2k}, \mathrm{GSpin}_3, T(\mathrm{Spin}_3), \mathrm{GL}_1 \times \mathrm{GSpin}_{2k-2}).$$

This is essentially the Gross-Prasad model for $\mathrm{GSpin}_{2k} \times \mathrm{GSpin}_3$ except we replace the cusp form on GSpin_3 by a theta series. In this case Δ_{red} is given by (4.1) when $n = 2$. It is clear that Theorem 1.12 holds in this case. The unramified computation in [25] proves Theorem 1.7.

For (2.7) with $m = 2k+1$, the generic stabilizer of $\hat{\rho}$ in $\hat{G}$ is not connected, hence it does not belong to the current framework of the BZSV duality. However, for this specific case, by the Rankin-Selberg integral in [23], we know that the dual integral should be the one in [23]. As the generic stabilizer is not connected, there are covering groups involved in the integral.

For (2.8) with $n = 7$, the associated quadruple Δ is given by

$$(\mathrm{GSp}_6, \mathrm{GL}_2, T(std_{\mathrm{GL}_2}), \mathrm{GL}_3 \times \mathrm{GL}_1).$$

This is essentially the same as the quadruple (3.4) except we replace the cusp form on GL_2 by theta series. The period integral in this case is exactly the Rankin-Selberg integral in [6] and Δ_{red} is given by (4.1) when $m = 2$. It is clear that Theorem 1.12 holds in this case. The unramfied computation in [6] and [44] proves Theorem 1.7.

For (2.8) with $n = 9$, the associated quadruple Δ is

$$(\mathrm{GSp}_8, G(\mathrm{SL}_2 \times \mathrm{SL}_2), T(std_{\mathrm{GL}_2,2}), \mathrm{GL}_3 \times \mathrm{GL}_1 \times \mathrm{GL}_1). \tag{4.7}$$

where $std_{\mathrm{GL}_2,2}$ is the standard representation of the second GL_2-copy. This is essentially the same as the quadruple (3.5) except we replace the cusp form on GL_2 by theta series and the period integral in this case is exactly the Rankin-Selberg integral in [7]. In this case Δ_{red} is given by (4.3). It is clear that Theorem 1.12 holds in this case. The unramfied computation in [7] proves Theorem 1.7.

For (2.8) with $n = 10$, the associated quadruple Δ is

$$(\mathrm{PGSO}_{10}, \mathrm{GL}_2, 0, \mathrm{GL}_4 \times \mathrm{GL}_1).$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{SO}_2$ whose stabilizer in M is $H = \mathrm{GL}_2$ (here the embedding is given by $h \mapsto (h, h, diag(\det(h), 1)))$. In this case Δ_{red} is given by (4.1) when $n = 2$. It is clear that Theorem 1.12 holds in this case. This integral is very close to the Rankin-Selberg integral in [16], though we again do not claim Theorem 1.7 in this case.

For (2.9), the generic stabilizer of $\hat{\rho}$ in $\hat{G}$ is not connected, hence it does not belong to the current framework of the BZSV duality. However, for this specific case, by the Rankin-Selberg integral in [15], we know that the dual integral should be the one in [15]. As the generic stabilizer is not connected, there are covering groups involved in the integral.

For (2.10), the associated quadruple Δ is

$$(GE_6, \mathrm{GL}_3, T(std_{\mathrm{GL}_3}), D_4).$$

In this case Δ_{red} is given by (4.1) when $n = 3$. The period integral associated to it is exactly the Rankin-Selberg integral in [14]. It is clear that Theorem 1.12 holds in this case. The unramified compuation in [14] proves Theorem 1.7.

By the discussion above, the strongly tempered quadruple associated to Table 6 is given as follows. Here for ι, we only list the root type of the Levi subgroup L of G such that ι is principal in L.

(G, H, ρ_H)	ι	$\hat{\rho}$
$(\mathrm{GL}_m \times \mathrm{GL}_n, \mathrm{GL}_n, 0)$	A_{m-n-1}	$T(std_{\mathrm{GL}_m} \otimes std_{\mathrm{GL}_n})$
$(\mathrm{GL}_{2m}, \mathrm{GL}_m, T(std_{\mathrm{GL}_m}))$	$(A_1)^m$	$T(\wedge^2)$
$(\mathrm{GL}_{2m+1}, \mathrm{GL}_m, 0)$	$(A_1)^m$	$T(\wedge^2)$
$(\mathrm{SO}_{2m+1}, \mathrm{SO}_2, 0)$	B_{m-1}	$T(std_{\mathrm{Sp}_{2n}})$
$(\mathrm{GSpin}_{2m+1} \times \mathrm{GL}_2, G(\mathrm{SL}_2 \times \mathrm{SL}_2), T(std_{\mathrm{GL}_2}))$	B_{m-2}	$T(Std_{\mathrm{GSp}_{2m}} \otimes Std_{\mathrm{GL}_2})$
$(\mathrm{GSp}_4 \times \mathrm{GL}_n, S(\mathrm{GSp}_4 \times \mathrm{GL}_4), std_{\mathrm{Sp}_4} \otimes \wedge^2, (\mathrm{GL}_1)^5)$	A_{n-5}	$T(Std_{\mathrm{Sp}_4} \otimes Std_{\mathrm{SL}_n})$
$(\mathrm{GSpin}_{2m+1} \times \mathrm{GL}_3, \mathrm{GSpin}_6 \times \mathrm{GL}_3, T(\mathrm{HSpin}_6 \otimes std_{\mathrm{GL}_3}))$	B_{m-3}	$T(Std_{\mathrm{Sp}_{2m}} \otimes Std_{\mathrm{SL}_3})$
$(\mathrm{GSpin}_{2k}, \mathrm{GSpin}_3, T(\mathrm{Spin}_3))$	D_{k-1}	$T(std_{\mathrm{SO}_{2k}})$
$(\mathrm{GSp}_6, \mathrm{GL}_2, T(std_{\mathrm{GL}_2}))$	A_2	$T(\mathrm{Spin}_7)$
$(\mathrm{GSp}_8, G(\mathrm{SL}_2 \times \mathrm{SL}_2), T(std_{\mathrm{GL}_2}))$	A_2	$T(\mathrm{Spin}_9)$
$(\mathrm{PGSO}_{10}, \mathrm{GL}_2, 0)$	A_3	$T(\mathrm{HSpin}_{10})$
$(GE_6, \mathrm{GL}_3, T(std_{\mathrm{GL}_3}))$	D_4	$T(std_{E_6})$

TABLE 8. Dual quadruples of Table 6

5. MODELS IN TABLE 11

In this section we will consider Table 11 of [28], this is for the case when $\hat{\rho}$ is the direct sum of two distinct irreducible symplectic representations of $\hat{G}$. It is easy to check that the representations in (11.5), (11.8), (11.13), (11.14), (11.15) of [28] are not anomaly free and the representation in (11.1) (resp. (11.11)) of [28] is only anomaly free when n is even (resp.

p odd). Hence it remains to consider the following cases. We still separate the cases based on whether $\hat{\mathfrak{l}}$ is abelian or not.

Number in [28]	$(\hat{G},\hat{\rho})$	$\hat{W}_V$	$\hat{\mathfrak{l}}$
(11.7)	$(\mathrm{Sp}_4\times\mathrm{Spin}_8\times\mathrm{SL}_2, std_{\mathrm{Sp}_4}\otimes std_{\mathrm{Spin}_8}\oplus\mathrm{HSpin}_8\otimes std_{\mathrm{SL}_2})$	$C_2\times D_4\times A_1$	0
(11.9)	$(\mathrm{SL}_2\times\mathrm{Spin}_7\times SL_2, std_{\mathrm{SL}_2}\otimes\mathrm{Spin}_7\oplus\mathrm{Spin}_7\otimes std_{\mathrm{SL}_2})$	$(A_1)^3\times B_2$	0
(11.10)	$(\mathrm{SL}_2\times\mathrm{SO}_6\times\mathrm{SL}_2, std_{\mathrm{SL}_2}\otimes std_{\mathrm{SO}_6}\oplus std_{\mathrm{SO}_6}\otimes std_{\mathrm{SL}_2})$	$A_1\times A_1\times B_2$	0
(11.11), p=2m+1	$(\mathrm{SO}_{2m+1}\times\mathrm{Sp}_{2m}, std_{\mathrm{SO}_{2m+1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}})$	$B_m\times C_m$	0
(11.11), p=2m-1	$(\mathrm{SO}_{2m-1}\times\mathrm{Sp}_{2m}, std_{\mathrm{SO}_{2m-1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}})$	$B_{m-1}\times D_m$	0

TABLE 9. Reductive models in Table 11 of [28]

Number in [28]	$(\hat{G},\hat{\rho})$	$\hat{W}_V$	$\hat{\mathfrak{l}}$
(11.1), n=2k	$(\mathrm{SL}_2\times\mathrm{SO}_{2k}\times\mathrm{SL}_2, std_{\mathrm{SL}_2}\otimes std_{\mathrm{SO}_{2k}}\oplus std_{\mathrm{SO}_{2k}}\otimes std_{\mathrm{SL}_2})$	$A_1\times A_1\times B_2$	D_{k-2}
(11.2)	$(\mathrm{Spin}_{12}, \mathrm{HSpin}_{12}^{+}\oplus\mathrm{HSpin}_{12}^{-})$	$(A_1)^2\times B_2$	$A_1\times A_1$
(11.3)	$(\mathrm{SL}_2\times\mathrm{Spin}_{12}, std_{\mathrm{SL}_2}\otimes std_{\mathrm{Spin}_{12}}\oplus\mathrm{HSpin}_{12})$	$(A_1)^3$	A_3
(11.4)	$(\mathrm{Sp}_4\times\mathrm{Spin}_{12}, std_{\mathrm{Sp}_4}\otimes std_{\mathrm{Spin}_{12}}\oplus\mathrm{HSpin}_{12})$	$C_2\times A_1\times D_4$	A_1
(11.6)	$(\mathrm{SL}_2\times\mathrm{Spin}_8\times SL_2, std_{\mathrm{SL}_2}\otimes std_{\mathrm{Spin}_8}\oplus\mathrm{HSpin}_8\otimes std_{\mathrm{SL}_2})$	$(A_1)^3$	A_1
(11.11), $p=2k+1>2m+1$	$(\mathrm{SO}_{2k+1}\times\mathrm{Sp}_{2m}, std_{\mathrm{SO}_{2k+1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}})$	$B_m\times C_m$	B_{k-m}
(11.11), $p=2n-1<2m-1$	$(\mathrm{SO}_{2n-1}\times\mathrm{Sp}_{2m}, std_{\mathrm{SO}_{2n-1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}})$	$B_{n-1}\times D_n$	C_{m-n}
(11.12)	$(\mathrm{Sp}_6, \wedge_0^3\oplus std_{\mathrm{Sp}_6})$	$A_1\times A_1$	A_1

TABLE 10. Non-reductive models in Table 11 of [28]

5.1. **The reductive case.** For (11.7), the associated quadruple Δ is

$$(\mathrm{GSp}_4\times\mathrm{GSpin}_8\times\mathrm{GL}_2, S(\mathrm{GSpin}_8\times G(\mathrm{Sp}_4\times\mathrm{SL}_2)), std_{\mathrm{Sp}_4}\otimes std_{\mathrm{Spin}_8}\oplus\mathrm{HSpin}_8\otimes std_{\mathrm{SL}_2}). \tag{5.1}$$

Note that when we take principal series on GSp_4 and GL_2, this period integral recovers the Rankin-Selberg integral in [17]. The unramified computation in loc. cit. proves Theorem 1.7 in this case. This quadruple is self-dual.

For (11.9), the associated quadruple Δ is given by

$$(\mathrm{GSp}_6\times\mathrm{GSO}_4, S(\mathrm{GSO}_4\times G(\mathrm{Sp}_4\times\mathrm{SL}_2)), std_{\mathrm{SO}_4}\times std_{\mathrm{Sp}_4}).$$

By the theta correspondence for $\mathrm{GSO}_4\times\mathrm{GSp}_4$, the integral over SO_4 of a cusp form on GSO_4 with the theta series associated to ρ_H produces an automorphic form on GSp_4. Then the integral over $G(\mathrm{Sp}_4\times\mathrm{SL}_2)$ is just the period integral for the pair $(\mathrm{GSp}_6\times\mathrm{GSp}_4, G(\mathrm{Sp}_4\times\mathrm{Sp}_2))$ in [44]. The unramified computation in [44] and Theorem 2.6 applied to theta correspondence for $\mathrm{GSO}_4\times\mathrm{GSp}_4$ proves Theorem 1.7 in this case.

For (11.10), the associated quadruple Δ is given by

$$(\mathrm{GL}_4\times\mathrm{GSO}_4, S(\mathrm{GSp}_4\times\mathrm{GSO}_4), std_{\mathrm{SO}_4}\times std_{\mathrm{Sp}_4}). \tag{5.2}$$

By the theta correspondence for $\mathrm{GSO}_4\times\mathrm{GSp}_4$, the integral over SO_4 of a cusp form on GSO_4 with the theta series associated to ρ_H produces an automorphic form on GSp_4. Then the integral over GSp_4 is just the period integral for the pair $(\mathrm{GL}_4\times\mathrm{GSp}_4, \mathrm{GSp}_4)$ which is essentially the Gross-Prasad model for $\mathrm{SO}_6\times\mathrm{SO}_5$. The unramified computation in [25] and Theorem 2.6 applied to theta correspondence for $\mathrm{GSO}_4\times\mathrm{GSp}_4$ proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1 follows from the theta correspondence for $\mathrm{SO}_6 \times \mathrm{Sp}_4$ (here we view $\mathrm{SL}_2 \times \mathrm{SL}_2$ as a subgroup of Sp_4) and the global period integral conjecture for the Gross-Prasad model $\mathrm{SO}_5 \times \mathrm{SO}_4$ in [12]. This proves Theorem 1.9.

For (11.11) when $p = 2m+1$, the associated quadruple Δ is given by

$$(\mathrm{SO}_{2m+1} \times \mathrm{Sp}_{2m}, H = G, std_{\mathrm{SO}_{2m+1}} \otimes std_{\mathrm{Sp}_{2m}} \oplus std_{\mathrm{Sp}_{2m}}). \tag{5.3}$$

By the theta correspondence for $\mathrm{SO}_{2m+2} \times \mathrm{Sp}_{2m}$, the integral over Sp_{2m} of a cusp form on Sp_{2m} with the theta series associated to ρ_H produces an automorphic form on SO_{2m+2}. Then the integral over SO_{2m+1} is just the period integral for the Gross-Prasad period for $\mathrm{SO}_{2m+2} \times \mathrm{SO}_{2m+1}$. The unramified computation in [25] and Theorem 2.6 applied to theta correspondence for $\mathrm{SO}_{2m+2} \times \mathrm{Sp}_{2m}$ proves Theorem 1.7 in this case. This quadruple is self-dual and it is clear that Conjecture 1.1 follows from the theta correspondence for $\mathrm{SO}_{2m+2} \times \mathrm{Sp}_{2m}$ and the global period integral conjecture for the Gross-Prasad model of $\mathrm{SO}_{2m+2} \times \mathrm{SO}_{2m+1}$ in [12]. This proves Theorem 1.9.

For (11.11) when $p = 2m-1$, the associated quadruple Δ is given by

$$(\mathrm{SO}_{2m+1} \times \mathrm{Sp}_{2m-2}, \mathrm{SO}_{2m} \times \mathrm{Sp}_{2m-2}, std_{\mathrm{SO}_{2m}} \otimes std_{\mathrm{Sp}_{2m-2}}). \tag{5.4}$$

By the theta correspondence for $\mathrm{SO}_{2m} \times \mathrm{Sp}_{2m-2}$, the integral over Sp_{2m-2} of a cusp form on Sp_{2m} with the theta series associated to ρ_H produces an automorphic form on SO_{2m}. Then the integral over SO_{2m} is just the Gross-Prasad period for $\mathrm{SO}_{2m+1} \times \mathrm{SO}_{2m}$. The unramified computation in [25] and Theorem 2.6 applied to theta correspondence for $\mathrm{SO}_{2m} \times \mathrm{Sp}_{2m-2}$ proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1 follows from the theta correspondence for $\mathrm{SO}_{2m} \times \mathrm{Sp}_{2m-2}$ and the global period integral conjecture for the Gross-Prasad model $\mathrm{SO}_{2m} \times \mathrm{SO}_{2m+1}$ in [12]. This proves Theorem 1.9.

By the discussion above, the strongly tempered quadruple associated to Table 9 is given as follows (note that ι is trivial for all these cases) where

$$* = (\mathrm{GSp}_4 \times \mathrm{GSpin}_8 \times \mathrm{GL}_2, S(\mathrm{GSpin}_8 \times G(\mathrm{Sp}_4 \times \mathrm{SL}_2)), std_{\mathrm{Sp}_4} \otimes std_{\mathrm{Spin}_8} \oplus \mathrm{HSpin}_8 \otimes std_{\mathrm{SL}_2})$$

(G, H, ρ_H)	$\hat{\rho}$
$*$	$std_{\mathrm{Sp}_4} \otimes std_{\mathrm{Spin}_8} \oplus \mathrm{HSpin}_8 \otimes std_{\mathrm{SL}_2}$
$(\mathrm{GSp}_6 \times \mathrm{GSO}_4, S(\mathrm{GSO}_4 \times G(\mathrm{Sp}_4 \times \mathrm{SL}_2)), std_{\mathrm{SO}_4} \times std_{\mathrm{Sp}_4})$	$std_{\mathrm{SL}_2} \otimes \mathrm{Spin}_7 \oplus \mathrm{Spin}_7 \otimes std_{\mathrm{SL}_2}$
$(\mathrm{GL}_4 \times \mathrm{GSO}_4, S(\mathrm{GSp}_4 \times \mathrm{GSO}_4), std_{\mathrm{SO}_4} \times std_{\mathrm{Sp}_4})$	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{SO}_6} \oplus std_{\mathrm{SO}_6} \otimes std_{\mathrm{SL}_2}$
$(\mathrm{SO}_{2m+1} \times \mathrm{Sp}_{2m}, H = G, std_{\mathrm{SO}_{2m+1}} \otimes std_{\mathrm{Sp}_{2m}} \oplus std_{\mathrm{Sp}_{2m}})$	$std_{\mathrm{SO}_{2m+1}} \otimes std_{\mathrm{Sp}_{2m}} \oplus std_{\mathrm{Sp}_{2m}}$
$(\mathrm{SO}_{2m+1} \times \mathrm{Sp}_{2m-2}, \mathrm{SO}_{2m} \times \mathrm{Sp}_{2m-2}, std_{\mathrm{SO}_{2m}} \otimes std_{\mathrm{Sp}_{2m-2}})$	$std_{\mathrm{SO}_{2m-1}} \otimes std_{\mathrm{Sp}_{2m}} \oplus std_{\mathrm{Sp}_{2m}}$

TABLE 11. Dual quadruples of Table 9

5.2. **The non-reductive case.**

For (11.1) when $n = 2k$, the associated quadruple Δ is

$$(\mathrm{GSpin}_{2k} \times \mathrm{GSO}_4, S(\mathrm{GSp}_4 \times \mathrm{GSO}_4), std_{\mathrm{SO}_4} \times std_{\mathrm{Sp}_4}, \mathrm{GSpin}_{2k-4} \times (\mathrm{GL}_1)^2 \times T_{\mathrm{GSO}_4}).$$

The nilpotent orbit ι induces a Bessel period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GSpin}_6 \times (\mathrm{GL}_1)^{k-3} \times \mathrm{GSO}_4$ whose stabilizer in M is $\mathrm{GSpin}_5 \times \mathrm{GSO}_4$. We can embed H into the stabilizer as in (5.2) and Δ_{red} is given by (5.2). It is clear that Theorem 1.12 holds in this case. The unramified computation in [25] and Theorem 2.6 applied to theta correspondence for $\mathrm{GSO}_4 \times \mathrm{GSp}_4$ proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1 follows from the theta correspondence for $\mathrm{SO}_{2k} \times \mathrm{Sp}_4$ (here we view $\mathrm{SL}_2 \times \mathrm{SL}_2$ as a subgroup of Sp_4) and the global period integral conjecture for the Gross-Prasad model $\mathrm{SO}_5 \times \mathrm{SO}_4$ in [12]. This proves Theorem 1.9.

For (11.2), the associated quadruple Δ is

$$(\mathrm{GSO}_{12}, S(\mathrm{GSp}_4 \times \mathrm{GSO}_4), 0, \mathrm{GL}_2 \times \mathrm{GL}_2 \times (\mathrm{GL}_1)^3).$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_4 \times \mathrm{GSO}_4$ whose stabilizer in M is H. In this case Δ_{red} is given by (5.2). It is clear that Theorem 1.12 holds in this case.

For (11.3), we first introduce a reductive quadruple which belongs to Table S of [28]. Let $G = (\mathrm{GL}_2)^5$ and $H = S(\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2)$ where the embedding $H \to G$ is given by mapping the first GL_2-copy into the first GL_2-copy, and mapping the second (resp. third) GL_2-copy diagonally into the second and third (resp. fourth and fifth) GL_2-copy. Let $\rho_H = std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_2}$ be the triple product representation and ι be trivial. The quadruple

$$\Delta_0 = (G, H, \rho_H, \iota) = ((\mathrm{GL}_2)^5, S(\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2), std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_2}, 1) \tag{5.5}$$

will be used to explain several models in this paper. This quadruple comes from Table S of [28], it is obtained by combining two copies of Model (S.3) with $n = 4$. We claim the dual quadruple is given by

$$\hat{\Delta}_0 = (\hat{G}, \widehat{G/Z_\Delta}, \hat{\rho}, 1),\ \hat{\rho} = std_{\mathrm{GL}_2,1} \otimes std_{\mathrm{GL}_2,2} \otimes std_{\mathrm{GL}_2,3} \oplus std_{\mathrm{GL}_2,1} \otimes std_{\mathrm{GL}_2,4} \otimes std_{\mathrm{GL}_2,5}$$

where $std_{\mathrm{GL}_2,i}$ represents the standard representation of the i-th GL_2-copy. To justify the duality, we will prove Theorem 1.7 and Theorem 1.9 for this case.

We start with Theorem 1.7. By the theta correspondence for $\mathrm{GSp}_2 \times \mathrm{GSO}_4$, the integral of a cusp form on the first GL_2-copy with the theta series produces cusp forms on the other two GL_2-copies of H. Then the period integral over the remaining two copies of GL_2 are just the period for two trilinear GL_2-models (i.e., the first, second, third GL_2-copies and the first, fourth, fifth GL_2-copies). Then Theorem 1.7 follows from the unramified computation in [25]. In fact, in this case, Conjecture 1.1(1) follows from the result in [24] and Theorem 2.6 applied to theta correspondence for $\mathrm{GSp}_2 \times \mathrm{GSO}_4$. For the dual side, Conjecture 1.1(2) in this case is also a direct consequence of the result in [24] and Theorem 2.5 applied to theta correspondence for $\mathrm{GSp}_2 \times \mathrm{GSO}_4$. This proves Theorem 1.9. Later in Section 9, we will use a similar argument to prove Theorem 1.15 for most of the cases.

For (11.3) the associated quadruple Δ is

$$(\mathrm{GSO}_{12} \times \mathrm{PGL}_2, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), 0, \mathrm{GL}_4 \times (\mathrm{GL}_1)^3 \times T_{\mathrm{PGL}_2}). \tag{5.6}$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GSO}_4 \times \mathrm{PGL}_2$ whose stabilizer in M is $S(\mathrm{GL}_2 \times \mathrm{GSO}_4) \times \mathrm{GL}_2$. We can embed H into the stabilizer by mapping the GL_2-copy of H into the GL_2-copy of the stabilizer and by mapping $\mathrm{GSO}_4 = \mathrm{GL}_2 \times \mathrm{GL}_2/\mathrm{GL}_1^{diag}$ into $\mathrm{GSO}_4 \times \mathrm{PGL}_2$ via the idenity map on GSO_4 and the projection map $\mathrm{GSO}_4 = \mathrm{GL}_2 \times \mathrm{GL}_2/\mathrm{GL}_1^{diag} \to \mathrm{PGL}_2$ via the firts GL_2-copy of GSO_4. It is clear that the induced embedding from H into M is the same as (5.5). In this case Δ_{red} is given by (5.5). It is clear that Theorem 1.12 holds in this case.

For (11.4), the associated quadruple Δ is

$$(\mathrm{GSp}_4 \times \mathrm{GSpin}_{12}, S(\mathrm{GSpin}_8 \times G(\mathrm{Sp}_4 \times \mathrm{SL}_2)), std_{\mathrm{Sp}_4} \otimes std_{\mathrm{Spin}_8}, T_{\mathrm{GSp}_4} \times \mathrm{GL}_2 \times (\mathrm{GL}_1)^5).$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GSp}_4 \times \mathrm{GL}_2 \times \mathrm{GSpin}_8$ whose stabilizer in M is $\mathrm{GSpin}_4 \times S(\mathrm{GL}_2 \times \mathrm{GSpin}_8)$ and we can naturally embed H into the stabilizer. In this case Δ_{red} is given by (5.1). It is clear that Theorem 1.12 holds in this case.

For (11.6), the associated quadruple Δ is

$$(\mathrm{GSO}_8 \times \mathrm{GSO}_4, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), 0, \mathrm{GL}_2 \times (\mathrm{GL}_1)^3 \times T_{\mathrm{GSO}_4}). \tag{5.7}$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GSO}_4 \times \mathrm{GL}_2 \times \mathrm{GSO}_4$ whose stabilizer in M is $S(\mathrm{GSO}_4 \times \mathrm{GL}_2) \times \mathrm{GSO}_4$. We can embed H into the stabilizer by making the GL_2-copy of H into the GL_2-copy of the stabilizer and by mapping the GSO_4-copy of H diagonally into the two GSO_4-copies of the stabilizer. It is clear that the induced embedding from H into M is the same as (5.5). In this case Δ_{red} is given by (5.5). It is clear that Theorem 1.12 holds in this case.

For (11.11) when $p = 2k+1 > 2m+1$, the associated quadruple Δ is

$$(\mathrm{SO}_{2m+1} \times \mathrm{Sp}_{2k}, \mathrm{SO}_{2m+1} \times \mathrm{Sp}_{2m}, std_{\mathrm{SO}_{2m+1}} \otimes std_{\mathrm{Sp}_{2m}}, T_{\mathrm{SO}_{2m+1}} \times \mathrm{Sp}_{2k-2m} \times (\mathrm{GL}_1)^m).$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{Sp}_{2m} \times (\mathrm{GL}_1)^{k-m} \times \mathrm{SO}_{2m+1}$ whose stabilizer in M is H. In this case Δ_{red} is given by (5.3). It is clear that Theorem 1.12 holds in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.5 applied to the theta correspondence for $\mathrm{Sp}_{2m} \times \mathrm{SO}_{2k+2}$ and the Gan-Gross-Prasad conjecture (Conjecture 9.11 of [13]) for non-tempered Arthur packet of the Gross-Prasad model of $\mathrm{SO}_{2k+2} \times \mathrm{SO}_{2k+1}$. This proves Theorem 1.9.

For (11.11) when $p = 2n-1 < 2m-1$, the associated quadruple Δ is

$$(\mathrm{SO}_{2m+1} \times \mathrm{Sp}_{2n-2}, \mathrm{SO}_{2n} \times \mathrm{Sp}_{2n-2}, std_{\mathrm{SO}_{2n}} \otimes std_{\mathrm{Sp}_{2n-2}}, \mathrm{SO}_{2m-2n+1} \times (\mathrm{GL}_1)^n \times T_{\mathrm{Sp}_{2n-2}}). \tag{5.8}$$

In this case Δ_{red} is given by (5.4). It is clear that Theorem 1.12 holds in this case. By the theta correspondence for $\mathrm{SO}_{2n} \times \mathrm{Sp}_{2n-2}$, the integral over Sp_{2n-2} of a cusp form on Sp_{2n} with the theta series associated to ρ_H produces an automorphic form on SO_{2n}. Then the integral over SO_{2n} is just the Gross-Prasad period for $\mathrm{SO}_{2m+1} \times \mathrm{SO}_{2n}$. The unramified computation in [25] and Theorem 2.6 applied to theta correspondence for $\mathrm{SO}_{2n} \times \mathrm{Sp}_{2n-2}$ proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from the theta correspondence for $\mathrm{Sp}_{2m} \times \mathrm{SO}_{2n}$ and the global period integral conjecture for the Gross-Prasad period of $\mathrm{SO}_{2n} \times \mathrm{SO}_{2n-1}$ in [12]. This proves Theorem 1.9.

For (11.12), the associated quadruple Δ is

$$(\mathrm{GSpin}_7, \mathrm{GL}_2, S(\mathrm{GL}_2 \times \mathrm{GL}_2), std_{\mathrm{GL}_2}, \mathrm{GL}_2 \times (\mathrm{GL}_1)^2).$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GSpin}_3 \times \mathrm{GL}_2$ whose stabilizer in M is H. The representation ρ_H is the standard representation on the first GL_2-copy. In this case Δ_{red} is given by (5.3) when $m = 1$. It is clear that Theorem 1.12 holds in this case.

By the discussion above, the strongly tempered quadruple associated to Table 10 is given as follows. Here for ι, we only list the root type of the Levi subgroup L of G such that ι is principal in L and

$$* = (\mathrm{GSpin}_4 \times \mathrm{GSpin}_{12}, S(\mathrm{GSpin}_8 \times G(\mathrm{Sp}_4 \times \mathrm{SL}_2)), std_{\mathrm{Sp}_4} \otimes std_{\mathrm{Spin}_8}).$$

(G, H, ρ_H)	ι	$\hat{\rho}$
$(\mathrm{GSpin}_{2k} \times \mathrm{GSO}_4, S(\mathrm{GSp}_4 \times \mathrm{GSO}_4), std_{\mathrm{SO}_4} \times std_{\mathrm{Sp}_4})$	D_{k-2}	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{SO}_{2k}} \oplus std_{\mathrm{SO}_{2k}} \otimes std_{\mathrm{SL}_2}$
$(\mathrm{GSO}_{12}, S(\mathrm{GSp}_4 \times \mathrm{GSO}_4), 0)$	$A_1 \times A_1$	$\mathrm{HSpin}_{12}^{+} \oplus \mathrm{HSpin}_{12}^{-}$
$(\mathrm{GSO}_{12} \times \mathrm{PGL}_2, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), 0)$	A_3	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{Spin}_{12}} \oplus \mathrm{HSpin}_{12}$
$*$	A_1	$std_{\mathrm{Sp}_4} \otimes std_{\mathrm{Spin}_{12}} \oplus \mathrm{HSpin}_{12}$
$(\mathrm{GSO}_8 \times \mathrm{GSO}_4, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), 0)$	A_1	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{Spin}_8} \oplus \mathrm{HSpin}_8 \otimes std_{\mathrm{SL}_2}$
$(\mathrm{SO}_{2m+1} \times \mathrm{Sp}_{2k}, \mathrm{SO}_{2m+1} \times \mathrm{Sp}_{2m}, std_{\mathrm{SO}_{2m+1}} \otimes std_{\mathrm{Sp}_{2m}})$	C_{k-m}	$std_{\mathrm{SO}_{2k+1}} \otimes std_{\mathrm{Sp}_{2m}} \oplus std_{\mathrm{Sp}_{2m}}$
$(\mathrm{SO}_{2m+1} \times \mathrm{Sp}_{2n-2}, \mathrm{SO}_{2n} \times \mathrm{Sp}_{2n-2}, std_{\mathrm{SO}_{2n}} \otimes std_{\mathrm{Sp}_{2n-2}})$	B_{m-n}	$std_{\mathrm{SO}_{2n-1}} \otimes std_{\mathrm{Sp}_{2m}} \oplus std_{\mathrm{Sp}_{2m}}$
$(\mathrm{GSpin}_7, S(\mathrm{GL}_2 \times \mathrm{GL}_2), std_{\mathrm{GL}_2})$	A_1	$\wedge^3 \oplus std_{\mathrm{Sp}_6}$

TABLE 12. Dual quadruples of Table 10

6. MODELS IN TABLE 12

In this section we will consider Table 12 of [28], this is for the case when $\hat{\rho}$ is the direct sum of three irreducible representations of $\hat{G}$ with two of them dual to each other (i.e. $\hat{\rho} = \hat{\rho}_0 \oplus T(\hat{\tau})$). It is easy to check that the representations in (12.4), (12.9), (12.10), (12.11), (12.11) of [28] are not anomaly free. Hence it remains to consider the following cases. We still separate the cases based on whether $\hat{\mathfrak{l}}$ is abelian or not.

Number in [28]	$(\hat{G}, \hat{\rho})$	$\hat{W}_V$	$\hat{\mathfrak{l}}$
(12.5)	$(\mathrm{SL}_6 \times \mathrm{SL}_2, \wedge^3 \oplus T(std_{\mathrm{SL}_6} \otimes std_{\mathrm{SL}_2}))$	$A_1 \times A_1 \times A_3$	0
(12.7), m=1	$(\mathrm{SL}_2 \times \mathrm{SL}_4, std_{\mathrm{SL}_2} \otimes \wedge^2 \oplus T(std_{\mathrm{SL}_4}))$	$A_1 \times A_1$	0
(12.7), m=2	$(\mathrm{Sp}_4 \times \mathrm{SL}_4, std_{\mathrm{Sp}_4} \otimes \wedge^2 \oplus T(std_{\mathrm{SL}_4}))$	$C_2 \times A_3$	0
(12.7), m=3	$(\mathrm{Sp}_6 \times \mathrm{Spin}_6, std_{\mathrm{Sp}_6} \otimes std_{\mathrm{Spin}_6} \oplus T(\mathrm{HSpin}_6))$	$A_3 \times A_3$	0
(12.8)	$(\mathrm{SL}_2 \times \mathrm{SL}_4 \times \mathrm{SL}_2, std_{\mathrm{SL}_2} \otimes \wedge^2 \oplus T(std_{\mathrm{SL}_4} \otimes std_{\mathrm{SL}_2}))$	$A_1 \times A_1 \times A_3$	0

TABLE 13. Reductive models in Table 12 of [28]

Number in [28]	$(\hat{G}, \hat{\rho})$	$\hat{W}_V$	$\hat{\mathfrak{l}}$
(12.1)	$(\mathrm{Spin}_{12}, \mathrm{HSpin}_{12} \oplus T(std_{\mathrm{Spin}_{12}}))$	$A_1 \times A_1 \times A_1$	A_3
(12.2)	$(\mathrm{SL}_2 \times \mathrm{Spin}_{10}, std_{\mathrm{SL}_2} \otimes std_{\mathrm{Spin}_{10}} \oplus T(std_{\mathrm{Spin}_{10}}))$	$A_1 \times A_1 \times A_3$	A_1
(12.3)	$(\mathrm{SL}_2 \times \mathrm{Spin}_8, std_{\mathrm{SL}_2} \otimes std_{\mathrm{Spin}_8} \oplus T(std_{\mathrm{Spin}_8}))$	$A_1 \times A_1 \times A_1$	A_1
(12.6)	$(\mathrm{SL}_6, \wedge^3 \oplus T(std_{\mathrm{SL}_6}))$	$A_1 \times A_1$	$A_1 \times A_1$
(12.7), $m > 3$	$(\mathrm{Sp}_{2m} \times \mathrm{SO}_6, std_{\mathrm{Sp}_{2m}} \otimes std_{\mathrm{SO}_6} \oplus T(\mathrm{HSpin}_6))$	$A_3 \times A_3$	C_{m-3}

TABLE 14. Non-reductive models in Table 12 of [28]

6.1. **The reductive case.** For (12.5), the associated quadruple Δ is

$$(\mathrm{GL}_6 \times \mathrm{GL}_2, \mathrm{GL}_2 \times S(\mathrm{GL}_4 \times \mathrm{GL}_2), \wedge^2 \otimes std_{\mathrm{GL}_2}).$$

At this moment we do not have much evidence that the above is the dual quadruple other than the fact that $\wedge^2 \otimes std_{\mathrm{GL}_2}$ is the only feasible choice of symplectic representation. We believe an unramified computation similar to [25] and [44] can confirm the duality in this case.

For (12.7) with $m=1$, the associated quadruple Δ is

$$(\mathrm{GL}_4\times \mathrm{GL}_2, \mathrm{GL}_2\times \mathrm{GL}_2, 0).$$

This is the model $(\mathrm{GL}_4\times \mathrm{GL}_2, \mathrm{GL}_2\times \mathrm{GL}_2)$ studied in [44] and the unramified computation in [44] proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.5 applied to the theta correspondence of $\mathrm{GSp}_2\times \mathrm{GSO}_6$ and Gan-Gross-Prasad conjecture (Conjecture 9.11 of [13]) for non-tempered Arthur packet of the Rankin-Selberg integral of $\mathrm{GL}_4\times \mathrm{GL}_4$. This proves Theorem 1.9.

For (12.7) with $m=2$, the associated quadruple Δ is

$$(\mathrm{GL}_4\times \mathrm{GSp}_4, \mathrm{GL}_4\times \mathrm{GSp}_4, T(std_{\mathrm{GL}_4}\otimes std_{\mathrm{GSp}_4})).$$

Observe that this is the dual to the quadruple in (4.4), thus both Theorems 1.7 and 1.9 have been proved there.

For (12.7) with $m=3$, the associated quadruple Δ is

$$(\mathrm{GSpin}_7\times \mathrm{GSpin}_6, \mathrm{GSpin}_6\times \mathrm{GSpin}_6, T(\mathrm{HSpin}_6\otimes \mathrm{HSpin}_6)). \tag{6.1}$$

By the theta correspondence for $\mathrm{GL}_4\times \mathrm{GL}_4$, the integral over the second GSpin_6-copy of a cusp form on GSpin_6 with the theta series associated to ρ_H produces the same cusp form with an extra central value of the Spin L-function. Then the integral over the other copy of GSpin_6 is just the period integral for the Gross-Prasad model $\mathrm{GSpin}_7\times \mathrm{GSpin}_6$. The unramified computation in [25] and Theorem 2.4 applied to theta correspondence for $\mathrm{GL}_4\times \mathrm{GL}_4$ proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from the theta correspondence for $\mathrm{GSp}_6\times \mathrm{GSO}_6$ and the Rankin-Selberg integral of $\mathrm{GL}_4\times \mathrm{GL}_4$. This proves Theorem 1.9.

For (12.8), the associated quadruple Δ is

$$(\mathrm{GL}_2\times \mathrm{GL}_4\times \mathrm{GL}_2, S(\mathrm{GL}_2\times \mathrm{GL}_4)\times \mathrm{GL}_2, std_{\mathrm{GL}_2}\otimes \wedge^2\oplus T(std_{\mathrm{GL}_4}\times std_{\mathrm{GL}_2})). \tag{6.2}$$

Note that when we put principal series on both GL_2 copies, this period integral recovers the Rankin-Selberg integral in [36]. The unramified computation in [36] proves Theorem 1.7 in this case. This quadruple is self-dual.

By the discussion above, the strongly tempered quadruple associated to Table 13 is given as follows (ι is trivial for all these cases) where

$$* = (\mathrm{GL}_2\times \mathrm{GL}_4\times \mathrm{GL}_2, S(\mathrm{GL}_2\times \mathrm{GL}_4)\times \mathrm{GL}_2, std_{\mathrm{GL}_2}\otimes \wedge^2\oplus T(std_{\mathrm{GL}_4}\times std_{\mathrm{GL}_2})).$$

(G,H,ρ_H)	$\hat{\rho}$
$(\mathrm{GL}_6\times \mathrm{GL}_2, \mathrm{GL}_2\times S(\mathrm{GL}_4\times \mathrm{GL}_2), \wedge^2\otimes std_{\mathrm{GL}_2})$	$\wedge^3\oplus T(std_{\mathrm{SL}_6}\otimes std_{\mathrm{SL}_2})$
$(\mathrm{GL}_4\times \mathrm{GL}_2, \mathrm{GL}_2\times \mathrm{GL}_2, 0)$	$std_{\mathrm{SL}_2}\otimes \wedge^2\oplus T(std_{\mathrm{SL}_4})$
$(\mathrm{GL}_4\times \mathrm{GSp}_4, \mathrm{GL}_4\times \mathrm{GSp}_4, T(std_{\mathrm{GL}_4}\otimes std_{\mathrm{GSp}_4}))$	$std_{\mathrm{Sp}_4}\otimes \wedge^2\oplus T(std_{\mathrm{SL}_4})$
$(\mathrm{GSpin}_7\times \mathrm{GSpin}_6, \mathrm{GSpin}_6\times \mathrm{GSpin}_6, T(\mathrm{HSpin}_6\otimes \mathrm{HSpin}_6))$	$std_{\mathrm{Sp}_6}\otimes std_{\mathrm{Spin}_6}\oplus T(\mathrm{HSpin}_6)$
$*$	$std_{\mathrm{SL}_2}\otimes \wedge^2\oplus T(std_{\mathrm{SL}_4}\otimes std_{\mathrm{SL}_2})$

TABLE 15. Dual quadruples of Table 13

6.2. **The non-reductive case.** For (12.1), we first introduce a reductive quadruple which belongs to Table S of [28]. Let $G = (\mathrm{GL}_2)^4$ and $H = S(\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2)$ where the embedding $H \to G$ is given by mapping the first two GL_2-copies into the first two GL_2-copy, and mapping the last GL_2-copy diagonally into the third and fourth GL_2-copy. Let $\rho_H = std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_2} \oplus T(std_{\mathrm{GL}_2,2})$ where $std_{\mathrm{GL}_2,i}$ represents the standard representation of the i-th GL_2-copy and ι be trivial. This quadruple

$$\Delta_0 = (G, H, \rho_H, \iota) = ((\mathrm{GL}_2)^4, S(\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2), std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_2} \oplus T(std_{\mathrm{GL}_2,2}), 1) \tag{6.3}$$

is almost the same as (5.5) except we replace the cusp form on one GL_2-copy by theta series. It is obtained by combining Model (S.3) and (S.11) in Table S of [28] with $n = 4$ and $m = 2$. We claim the dual quadruple is given by

$$\hat{\Delta}_0 = (\hat{G}, \widehat{G/Z_\Delta}, \hat{\rho}, 1),\ \hat{\rho} = T(std_{\mathrm{GL}_2,1} \otimes std_{\mathrm{GL}_2,2}) \oplus std_{\mathrm{GL}_2,1} \otimes std_{\mathrm{GL}_2,3} \otimes std_{\mathrm{GL}_2,4}.$$

We can use the same argument as in (5.5) to prove Theorem 1.7 and Theorem 1.9 for this case.

For (12.1), the associated quadruple Δ is

$$(\mathrm{GSO}_{12}, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), T(std_{\mathrm{GL}_2}), \mathrm{GL}_4 \times (\mathrm{GL}_1)^3).$$

The attached period integral is the same as model in (5.6) except we replace the cusp form on GL_2 by theta series. In this case Δ_{red} is given by (6.3) and it is clear that Theorem 1.12 holds in this case.

For (12.2), the associated quadruple Δ is

$$(\mathrm{GSpin}_{10} \times \mathrm{GL}_2, S(\mathrm{GL}_2 \times \mathrm{GSpin}_6) \times \mathrm{GL}_2, T(\mathrm{HSpin}_6 \otimes std_{\mathrm{GL}_2}), \mathrm{GL}_2 \times (\mathrm{GL}_1)^4 \times T_{\mathrm{GL}_2})$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_2 \times \mathrm{GSpin}_6 \times \mathrm{GL}_2$ whose stabilizer in M is H. In this case Δ_{red} is given by (6.2). It is clear that Theorem 1.12 holds in this case.

For (12.3), the associated quadruple Δ is

$$(\mathrm{GSO}_8 \times \mathrm{GL}_2, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), T(std_{\mathrm{GL}_2}), \mathrm{GL}_2 \times (\mathrm{GL}_1)^3 \times T_{\mathrm{GL}_2}).$$

The attached period integral is the same as the model (5.7) except we replace the cusp form on one GL_2-copy by theta series. In this case Δ_{red} is given by (6.3) and it is clear that Theorem 1.12 holds in this case.

For (12.6), we first introduce a reductive quadruple from Table S of [28] (it is obtained by combining Model (S.10) and Model (S.3) with $n = 4$)

$$(G, H, \rho_H, \iota) = (\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2, \mathrm{GL}_2 \times \mathrm{GL}_2, T(std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_2}), 1) \tag{6.4}$$

where H embeds into G by mapping the first GL_2-copy into the first GL_2-copy and mapping the second GL_2-copy diagonally into the second and third GL_2-copy. We claim the dual quadruple is given by

$$(\hat{G}, \widehat{G/Z_\Delta}, \hat{\rho}, 1),\ \hat{\rho} = T(std_{\mathrm{GL}_2,1}) \oplus std_{\mathrm{GL}_2,1} \otimes std_{\mathrm{GL}_2,2} \otimes std_{\mathrm{GL}_2,3}$$

where $std_{\mathrm{GL}_2,i}$ is the standard representation of the i-th GL_2-copy. To justify the duality, we will prove Theorem 1.7 and Theorem 1.9 for this case.

We start with Theorem 1.7. By the theta correspondence for $\mathrm{GL}_2 \times \mathrm{GL}_2$, the integral over the first GL_2-copy of a cusp form in π with the theta series gives a cusp form on GL_2 (in the same space π, note though Theorem 2.2 applied to the correspondence does introduce the central value of the standard L-function). Then the integral over the other GL_2-copy is just the period integral for the trilinear GL_2-model. As a result, Conjecture 1.1(1) and Theorem 1.7 follow from the theta correspondence for $\mathrm{GL}_2\times\mathrm{GL}_2$ and the result in [24]. For the dual side, Conjecture 1.1(2) follows from the theta correspondence for $\mathrm{GSp}_2\times\mathrm{GSO}_4$ and the Rankin-Selberg integral of $\mathrm{GL}_2\times\mathrm{GL}_2$. This proves Theorem 1.9 in this case. Later in Section 9, we will use a similar argument to prove Theorem 1.15 for some of the cases (more precisely for those cases containing model (S.10) of [28]).

Now we can write down the associated quadruple Δ of (12.6). It is given by

$$(\mathrm{GL}_6,\mathrm{GL}_2\times\mathrm{GL}_2,0,\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_1\times\mathrm{GL}_1).$$

The nilpotent orbit ι induces a Fourier-Jacobi period on the unipotent radical of the parabolic subgroup $P=MU$ with $M=\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2$ whose stabilizer in M is H. In this case Δ_{red} is given by (6.4). It is clear that Theorem 1.12 holds in this case.

For (12.7) when $m>3$, the associated quadruple Δ is

$$(\mathrm{GSpin}_{2m+1}\times\mathrm{GSpin}_6,\mathrm{GSpin}_6\times\mathrm{GSpin}_6,T(\mathrm{HSpin}_6\otimes\mathrm{HSpin}_6),\mathrm{GSpin}_{2m-5}\times(\mathrm{GL}_1)^3\times(\mathrm{GL}_1)^4).$$

The nilpotent orbit ι induces a Bessel period on the unipotent radical of the parabolic subgroup $P=MU$ with $M=\mathrm{GL}_1^{m-3}\times\mathrm{GSpin}_7\times\mathrm{GSpin}_6$ whose stabilizer in M is H. In this case Δ_{red} is given by (6.1). It is clear that Theorem 1.12 holds in this case. The unramified computation in [25] and Theorem 2.4 applied to theta correspondence for $\mathrm{GL}_4\times\mathrm{GL}_4$ proves Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from the theta correspondence for $\mathrm{GSp}_{2m}\times\mathrm{GSO}_6$ and the Rankin-Selberg integral of $\mathrm{GL}_4\times\mathrm{GL}_4$. This proves Theorem 1.9.

By the discussion above, the strongly tempered quadruple associated to Table 14 is given as follows. Here for ι, we only list the root type of the Levi subgroup L of G such that ι is principal in L and

$$*=(\mathrm{GSpin}_{10}\times\mathrm{GL}_2,S(\mathrm{GL}_2\times\mathrm{GSpin}_6)\times\mathrm{GL}_2,T(\mathrm{HSpin}_6\otimes std_{\mathrm{GL}_2})).$$

(G,H,ρ_H)	ι	$\hat{\rho}$
$(\mathrm{GSO}_{12},S(\mathrm{GL}_2\times\mathrm{GSO}_4),T(std_{\mathrm{GL}_2}))$	A_3	$\mathrm{HSpin}_{12}\oplus T(std_{\mathrm{Spin}_{12}})$
$*$	A_1	$std_{\mathrm{SL}_2}\otimes std_{\mathrm{Spin}_{10}}\oplus T(std_{\mathrm{Spin}_{10}})$
$(\mathrm{GSO}_8\times\mathrm{GL}_2,S(\mathrm{GL}_2\times\mathrm{GSO}_4),T(std_{\mathrm{GL}_2}))$	A_1	$std_{\mathrm{SL}_2}\otimes std_{\mathrm{Spin}_8}\oplus T(std_{\mathrm{Spin}_8})$
$(\mathrm{GL}_6,\mathrm{GL}_2\times\mathrm{GL}_2,0)$	$A_1\times A_1$	$\wedge^3\oplus T(std_{\mathrm{SL}_6})$
$(\mathrm{GSpin}_{2m+1}\times\mathrm{GSpin}_6,\mathrm{GSpin}_6\times\mathrm{GSpin}_6,T(\mathrm{HSpin}_6\otimes\mathrm{HSpin}_6))$	B_{m-3}	$std_{\mathrm{Sp}_{2m}}\otimes std_{\mathrm{SO}_6}\oplus T(\mathrm{HSpin}_6)$

Table 16. Dual quadruples of Table 14

7. Models in Table 22

In this section we will consider Table 22 of [28], this is for the case when $\hat{\rho}$ is the direct sum of four irreducible representations of $\hat{G}$ of the form $T(\rho_1)\oplus T(\rho_2)$. All the representations in Table 22 of [28] are anomaly free, so we need to consider all of them. We still separate the cases based on whether $\hat{\mathfrak{l}}$ is abelian or not.

Number in [28]	$(\hat{G},\hat{\rho})$	$\hat{W}_V$	$\hat{\mathfrak{l}}$
(22.2), n=2m	$(\mathrm{SL}_n, T(\wedge^2)\oplus T(std_{\mathrm{SL}_n}))$	$A_{m-1}\times A_{m-1}$	0
(22.2), n=2m+1	$(\mathrm{SL}_n, T(\wedge^2)\oplus T(std_{\mathrm{SL}_n}))$	$A_m\times A_{m-1}$	0
(22.3), m=n	$(\mathrm{SL}_m\times \mathrm{SL}_n, T(std_{\mathrm{SL}_m}\otimes std_{\mathrm{SL}_n})\oplus T(std_{\mathrm{SL}_n}))$	$A_{n-1}\times A_{n-1}$	0
(22.3), m=n+1	$(\mathrm{SL}_m\times \mathrm{SL}_n, T(std_{\mathrm{SL}_m}\otimes std_{\mathrm{SL}_n})\oplus T(std_{\mathrm{SL}_n}))$	$A_{n-1}\times A_{n-1}$	0
(22.3), m=n-1	$(\mathrm{SL}_m\times \mathrm{SL}_n, T(std_{\mathrm{SL}_m}\otimes std_{\mathrm{SL}_n})\oplus T(std_{\mathrm{SL}_n}))$	$A_m\times A_{m-1}$	0
(22.3), m=n-2	$(\mathrm{SL}_m\times \mathrm{SL}_n, T(std_{\mathrm{SL}_m}\otimes std_{\mathrm{SL}_n})\oplus T(std_{\mathrm{SL}_n}))$	$A_m\times A_{m-1}$	0
(22.4), n=3	$(\mathrm{SL}_3, T(std_{\mathrm{SL}_3})\oplus T(std_{\mathrm{SL}_3}))$	A_1	0
(22.5), m=2	$(\mathrm{Sp}_4, T(std_{\mathrm{Sp}_4})\oplus T(std_{\mathrm{Sp}_4}))$	$A_1\times A_1$	0

TABLE 17. Reductive models in Table 22 of [28]

Number in [28]	$(\hat{G},\hat{\rho})$	$\hat{W}_V$	$\hat{\mathfrak{l}}$
(22.1)	$(\mathrm{Spin}_8, T(std_{\mathrm{Spin}_8})\oplus T(\mathrm{HSpin}_8))$	$A_1\times A_1\times A_1$	A_1
(22.3), $m>n+1$	$(\mathrm{SL}_m\times \mathrm{SL}_n, T(std_{\mathrm{SL}_m}\otimes std_{\mathrm{SL}_n})\oplus T(std_{\mathrm{SL}_n}))$	$A_{n-1}\times A_{n-1}$	A_{m-n+1}
(22.3), $m<n-2$	$(\mathrm{SL}_m\times \mathrm{SL}_n, T(std_{\mathrm{SL}_m}\otimes std_{\mathrm{SL}_n})\oplus T(std_{\mathrm{SL}_n}))$	$A_m\times A_{m-1}$	A_{n-m-2}
(22.4), $n>3$	$(\mathrm{SL}_n, T(std_{\mathrm{SL}_n})\oplus T(std_{\mathrm{SL}_n}))$	A_1	A_{n-3}
(22.5), $m>2$	$(\mathrm{Sp}_{2m}, T(std_{\mathrm{Sp}_{2m}})\oplus T(std_{\mathrm{Sp}_{2m}}))$	$A_1\times A_1$	C_{m-2}

TABLE 18. Non-reductive models in Table 22 of [28]

7.1. **The reductive case.** For (22.2) with $n=2m$, the associated quadruple Δ is

$$(\mathrm{GL}_{2m}, \mathrm{GL}_m\times \mathrm{GL}_m, T(std_{\mathrm{GL}_m})).$$

The period integral in this case is exactly the Rankin-Selberg integral in [5]. The result in loc. cit. proves Conjecture 1.1(1) and Theorem 1.7 in this case.

For (22.2) with $n=2m+1$, the associated quadruple Δ is

$$(\mathrm{GL}_{2m+1}, \mathrm{GL}_{m+1}\times \mathrm{GL}_m, T(std_{\mathrm{GL}_{m+1}})).$$

The period integral in this case is exactly the Rankin-Selberg integral in [5]. The unramified computation in loc. cit. proves Conjecture 1.1(1) and Theorem 1.7 in this case.

For (22.3) with $m=n$, the associated quadruple Δ is

$$(\mathrm{GL}_n\times \mathrm{GL}_n, \mathrm{GL}_n\times \mathrm{GL}_n, T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n}\oplus std_{\mathrm{GL}_n})). \tag{7.1}$$

By the theta correspondence for $\mathrm{GL}_n\times\mathrm{GL}_n$, the integral over GL_n of a cusp form on GL_n with the theta series associated to $T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n})$ produces a cusp form on GL_n. Then the integral over the other GL_n-copy is just the Rankin-Selberg integral of $\mathrm{GL}_n\times\mathrm{GL}_n$. This quadruple is self-dual. The Rankin-Selberg integral of $\mathrm{GL}_n\times\mathrm{GL}_n$ and Theorems 2.2 and 2.4 applied to the theta correspondence for $\mathrm{GL}_n\times\mathrm{GL}_n$ proves Conjecture 1.1, Theorem 1.7 and Theorem 1.9. Notice that the theta correspondence introduces an extra central value of the standard L-function in this case.

For (22.3) with $m=n+1$, the associated quadruple Δ is

$$(\mathrm{GL}_{n+1}\times \mathrm{GL}_n, \mathrm{GL}_n\times \mathrm{GL}_n, T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n})). \tag{7.2}$$

By the theta correspondence for $\mathrm{GL}_n\times\mathrm{GL}_n$, the integral over GL_n of a cusp form on GL_n with the theta series associated to ρ_H produces another cusp form on GL_n. Then the integral over the other GL_n-copy is just the Rankin-Selberg integral of $\mathrm{GL}_{n+1}\times\mathrm{GL}_n$. The Rankin-Selberg integral of $\mathrm{GL}_{n+1}\times\mathrm{GL}_n$ in [26] and Theorems 2.2 and 2.4 applied to the theta correspondence for $\mathrm{GL}_n\times\mathrm{GL}_n$ proves Conjecture 1.1(1) and Theorem 1.7 in this case. Again notice that the theta correspondence introduces an extra central value of the standard L-function. For the dual side, Conjecture 1.1(2) follows from the theta correspondence of $\mathrm{GL}_{n+1}\times\mathrm{GL}_n$ with the Rankin-Selberg integral of $\mathrm{GL}_n\times\mathrm{GL}_n$. This proves Theorem 1.9.

For (22.3) with $m=n-1$, the associated quadruple Δ is

$$(\mathrm{GL}_n\times\mathrm{GL}_{n-1},\mathrm{GL}_n\times\mathrm{GL}_{n-1},T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_{n-1}}\oplus std_{\mathrm{GL}_n})). \tag{7.3}$$

By the theta correspondence for $\mathrm{GL}_n\times\mathrm{GL}_n$, the integral over GL_n of a cusp form on GL_n with the theta series associated to ρ_H produces another cusp form on GL_n. Then the integral over GL_{n-1} is just the Rankin-Selberg integral of $\mathrm{GL}_n\times\mathrm{GL}_{n-1}$. This quadruple is self-dual. The Rankin-Selberg integral of $\mathrm{GL}_n\times\mathrm{GL}_{n-1}$ and Theorems 2.2 and 2.4 applied to the theta correspondence for $\mathrm{GL}_n\times\mathrm{GL}_n$ proves Conjecture 1.1, Theorem 1.7 and Theorem 1.9 in this case. As before, the theta correspondence introduces an extra central value of the standard L-function.

For (22.3) with $m=n-2$, the associated quadruple Δ is

$$(\mathrm{GL}_n\times\mathrm{GL}_{n-2},\mathrm{GL}_{n-1}\times\mathrm{GL}_{n-2},T(std_{\mathrm{GL}_{n-1}}\otimes std_{\mathrm{GL}_{n-2}})). \tag{7.4}$$

By the theta correspondence for $\mathrm{GL}_{n-1}\times\mathrm{GL}_{n-2}$, the integral over GL_{n-2} of a cusp form on GL_{n-2} with the theta series associated to ρ_H produces an Eisenstein series on GL_{n-1} which is induced from the cuspidal automorphic representation on GL_{n-2} and the trivial character. Then the integral over GL_{n-1} is just the Rankin-Selberg integral of $\mathrm{GL}_n\times\mathrm{GL}_{n-1}$. The Rankin-Selberg integral of $\mathrm{GL}_{n-1}\times\mathrm{GL}_n$ in [26] and Theorems 2.2 and 2.4 applied to the theta correspondence for $\mathrm{GL}_{n-1}\times\mathrm{GL}_{n-2}$ proves Conjecture 1.1(1) and Theorem 1.7 in this case. For the dual side, Conjecture 1.1(2) follows from the theta correspondence of $\mathrm{GL}_{n-1}\times\mathrm{GL}_n$ with the Rankin-Selberg integral of $\mathrm{GL}_{n-1}\times\mathrm{GL}_{n-2}$. This proves Theorem 1.9.

For (22.4) with $n=3$, the associated quadruple Δ is

$$(\mathrm{GL}_3,\mathrm{GL}_2\times\mathrm{GL}_1,T(std_{\mathrm{GL}_2})). \tag{7.5}$$

The period integral is essentially the Rankin-Selberg integral of $\mathrm{GL}_3\times\mathrm{GL}_2$ except that we replace the cusp form on GL_2 by theta series. The result in [26] proves Conjecture 1.1(1) and Theorem 1.7 in this case.

For (22.5) with $m=2$, the associated quadruple Δ is

$$(\mathrm{GSpin}_5\times\mathrm{GL}_1,\mathrm{GSpin}_4\times\mathrm{GL}_1,T(\mathrm{HSpin}_4^+\oplus\mathrm{HSpin}_4^-\otimes std_{\mathrm{GL}_1})). \tag{7.6}$$

The period integral is essentially the Gross-Prasad period for $\mathrm{GSpin}_5\times\mathrm{GSpin}_4$ except that we replace the cusp form on GSpin_4 by theta series. The unramified computation in [25] proves Theorem 1.7 in this case.

By the discussion above, the strongly tempered quadruple associated to Table 13 is given as follows (ι is trivial for all these cases).

7.2. **The non-reductive case.** For (22.1), we first introduce a reductive quadruple which belongs to Table S of [28]. Let $G=(\mathrm{GL}_2)^3$, $H=S(\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2)$ and $\rho_H=std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}\oplus T(std_{\mathrm{GL}_2,2})\oplus T(std_{\mathrm{GL}_2,3})$ where $std_{\mathrm{GL}_2,i}$ represents the standard representation of the i-th GL_2-copy and ι be trivial. This quadruple

$$\Delta_0=(G,H,\rho_H,\iota)=((\mathrm{GL}_2)^3,S(\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2),std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}\oplus T(std_{\mathrm{GL}_2,2}\oplus T(std_{\mathrm{GL}_2,3}),1) \tag{7.7}$$

(G, H, ρ_H)	$\hat{\rho}$
$(\mathrm{GL}_{2m}, \mathrm{GL}_m \times \mathrm{GL}_m, T(std_{\mathrm{GL}_m}))$	$T(\wedge^2) \oplus T(std_{\mathrm{GL}_{2m}})$
$(\mathrm{GL}_{2m+1}, \mathrm{GL}_{m+1} \times \mathrm{GL}_m, T(std_{\mathrm{GL}_{m+1}}))$	$T(\wedge^2) \oplus T(std_{\mathrm{GL}_{2m+1}})$
$(\mathrm{GL}_n \times \mathrm{GL}_n, \mathrm{GL}_n \times \mathrm{GL}_n, T(std_{\mathrm{GL}_n} \otimes std_{\mathrm{GL}_n} \oplus std_{\mathrm{GL}_n}))$	$T(std_{\mathrm{GL}_n} \otimes std_{\mathrm{GL}_n}) \oplus T(std_{\mathrm{GL}_n})$
$(\mathrm{GL}_{n+1} \times \mathrm{GL}_n, \mathrm{GL}_n \times \mathrm{GL}_n, T(std_{\mathrm{GL}_n} \otimes std_{\mathrm{GL}_n}))$	$T(std_{\mathrm{GL}_{n+1}} \otimes std_{\mathrm{GL}_n}) \oplus T(std_{\mathrm{GL}_n})$
$(\mathrm{GL}_n \times \mathrm{GL}_{n-1}, \mathrm{GL}_n \times \mathrm{GL}_{n-1}, T(std_{\mathrm{GL}_n} \otimes std_{\mathrm{GL}_{n-1}} \oplus std_{\mathrm{GL}_n}))$	$T(std_{\mathrm{GL}_n} \otimes std_{\mathrm{GL}_{n-1}}) \oplus T(std_{\mathrm{GL}_n})$
$(\mathrm{GL}_n \times \mathrm{GL}_{n-2}, \mathrm{GL}_{n-1} \times \mathrm{GL}_{n-2}, T(std_{\mathrm{GL}_{n-1}} \otimes std_{\mathrm{GL}_{n-2}}))$	$T(std_{\mathrm{GL}_n} \otimes std_{\mathrm{GL}_{n-2}}) \oplus T(std_{\mathrm{GL}_n})$
$(\mathrm{GL}_3, \mathrm{GL}_2 \times \mathrm{GL}_1, T(std_{\mathrm{GL}_2}))$	$T(std_{\mathrm{SL}_3}) \oplus T(std_{\mathrm{SL}_3})$
$(\mathrm{GSpin}_5 \times \mathrm{GL}_1, \mathrm{GSpin}_4 \times \mathrm{GL}_1, T(\mathrm{HSpin}_4^+ \oplus \mathrm{HSpin}_4^- \otimes std_{\mathrm{GL}_1}))$	$T(std_{\mathrm{Sp}_4}) \oplus T(std_{\mathrm{Sp}_4})$

TABLE 19. Dual quadruples of Table 17

is almost the same as (5.5) except we replace the cusp form on two GL_2-copies by theta series. It is obtained by combining two copies of Model (S.11) in Table S of [28] with $m = 2$. We claim the dual quadruple is given by

$$\hat{\Delta}_0 = (\hat{G}, \widehat{G/Z_\Delta}, \hat{\rho}, 1),\ \hat{\rho} = T(std_{\mathrm{GL}_{2,1}} \otimes std_{\mathrm{GL}_{2,2}}) \oplus T(std_{\mathrm{GL}_{2,1}} \otimes std_{\mathrm{GL}_{2,3}}).$$

We can use the same argument as in (5.5) to prove Theorem 1.7 and Theorem 1.9 for this case.

For (22.1), the associated quadruple Δ is

$$(\mathrm{GSO}_8, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), T(std_{\mathrm{GL}_2} \oplus std_{\mathrm{GL}_2}), \mathrm{GL}_2 \times (\mathrm{GL}_1)^3).$$

The period integral is the same as (5.7) except we replace the cusp form on both GL_2-copies by theta series. In this case Δ_{red} is given by (7.7) and it is clear that Theorem 1.12 holds in this case.

For (22.3) when $m > n + 1$, the associated quadruple Δ is

$$(\mathrm{GL}_m \times \mathrm{GL}_n, \mathrm{GL}_n \times \mathrm{GL}_n, T(std_{\mathrm{GL}_n} \otimes std_{\mathrm{GL}_n}), (\mathrm{GL}_1)^n \times \mathrm{GL}_{m-n} \times T_{\mathrm{GL}_n}).$$

When $n-m$ is odd (resp. even), the nilpotent orbit ι induces a Bessel period (resp. Fourier-Jacobi period) on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_1^{m-n-1} \times \mathrm{GL}_{n+1} \times \mathrm{GL}_n$ (resp. $M = \mathrm{GL}_1^{m-n} \times \mathrm{GL}_n \times \mathrm{GL}_n$) whose stabilizer in M is H. In this case Δ_{red} is given by (7.2) (resp. (7.1)). It is clear that Theorem 1.12 holds in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.2 applied to the theta correspondence of $\mathrm{GL}_n \times \mathrm{GL}_{m+1}$ and Gan-Gross-Prasad conjecture (Conjecture 9.11 of [13]) for non-tempered Arthur packet of the Rankin-Selberg integral of $\mathrm{GL}_{m+1} \times \mathrm{GL}_m$. This proves Theorem 1.9.

For (22.3) when $m < n - 2$, the associated quadruple Δ is

$$(\mathrm{GL}_m \times \mathrm{GL}_n, \mathrm{GL}_m \times \mathrm{GL}_{m+1}, T(std_{\mathrm{GL}_m} \otimes std_{\mathrm{GL}_{m+1}}), T_{\mathrm{GL}_m} \times (\mathrm{GL}_1)^{m-1} \times \mathrm{GL}_{n-m-1}).$$

When $n - m - 1$ is odd (resp. even), the nilpotent orbit ι induces a Bessel period (resp. Fourier-Jacobi period) on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_1^{n-m-2} \times \mathrm{GL}_{m+2} \times \mathrm{GL}_m$ (resp. $M = \mathrm{GL}_1^{n-m-1} \times \mathrm{GL}_{m+1} \times \mathrm{GL}_m$) whose stabilizer in M is H. In this case Δ_{red} is given by (7.4) (resp. (7.3)). It is clear that Theorem 1.12 holds in this case. For the dual side, Conjecture 1.1(2) follows from Theorem 2.2 applied to the theta correspondence of $\mathrm{GL}_n \times \mathrm{GL}_{m+1}$ and the Rankin-Selberg integral of $\mathrm{GL}_{m+1} \times \mathrm{GL}_m$. This proves Theorem 1.9.

For (22.4) when $n > 3$, we need to introduce another reductive quadruple from Table S of [28] (it is obtained by combining two copies of Model (S.10))

$$(G, H, \rho_H, \iota) = (\mathrm{GL}_2 \times \mathrm{GL}_1, \mathrm{GL}_2 \times \mathrm{GL}_1, T(std_{\mathrm{GL}_2} \oplus std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_1}), 1). \tag{7.8}$$

We claim that the dual quadruple is given by

$$(\hat{G}, \hat{G}, \hat{\rho}, 1),\ \hat{\rho} = T(std_{\mathrm{GL}_2} \oplus std_{\mathrm{GL}_2} \otimes std_{\mathrm{GL}_1}),$$

i.e., it is self-dual. We can use the same argument as in (6.4) to prove Theorem 1.7 and Theorem 1.9 for this case.

The associated quadruple Δ for (22.4) with $n > 3$ is given by

$$(\mathrm{GL}_n, \mathrm{GL}_2, T(std_{\mathrm{GL}_2}), \mathrm{GL}_{n-2} \times \mathrm{GL}_1 \times \mathrm{GL}_1).$$

When $n-2$ is odd (resp. even), the nilpotent orbit ι induces a Bessel period (resp. Fourier-Jacobi period) on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_1^{n-3} \times \mathrm{GL}_3$ (resp. $M = \mathrm{GL}_1^{n-2} \times \mathrm{GL}_2$) whose stabilizer in M is H. In this case Δ_{red} is given by (7.5) (resp. (7.8)). It is clear that Theorem 1.12 holds in this case.

For (22.5) when $m > 2$, the associated quadruple Δ is

$$(\mathrm{GSpin}_{2m+1} \times \mathrm{GL}_1, \mathrm{GSpin}_4 \times \mathrm{GL}_1, T(\mathrm{HSpin}_4^+ \oplus \mathrm{HSpin}_4^- \otimes std_{\mathrm{GL}_1}), \mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GSpin}_{2m-3}).$$

The nilpotent orbit ι induces a Bessel period on the unipotent radical of the parabolic subgroup $P = MU$ with $M = \mathrm{GL}_1^{m-2} \times \mathrm{GSpin}_5$ whose stabilizer in M is H. In this case Δ_{red} is given by (7.6). It is clear that Theorem 1.12 holds in this case. The period integral is essentially the Gross-Prasad period for $\mathrm{GSpin}_{2m+1} \times \mathrm{GSpin}_4$ except that we replace the cusp form on GSpin_4 by theta series. The unramified computation in [25] proves Theorem 1.7.

By the discussion above, the strongly tempered quadruple associated to Table 18 is given as follows. Here for ι, we only list the root type of the Levi subgroup L of G such that ι is principal in L and

$$* = (\mathrm{GSpin}_{2m+1} \times \mathrm{GL}_1, \mathrm{GSpin}_4 \times \mathrm{GL}_1, T(\mathrm{HSpin}_4^+ \oplus \mathrm{HSpin}_4^- \otimes std_{\mathrm{GL}_1})).$$

(G, H, ρ_H)	ι	$\hat{\rho}$
$(\mathrm{GSO}_8, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), T(std_{\mathrm{GL}_2} \oplus std_{\mathrm{GL}_2}))$	A_1	$T(std_{\mathrm{Spin}_8}) \oplus T(\mathrm{HSpin}_8)$
$(\mathrm{GL}_m \times \mathrm{GL}_n, \mathrm{GL}_n \times \mathrm{GL}_n, T(std_{\mathrm{GL}_n} \otimes std_{\mathrm{GL}_n}))$	A_{m-n+1}	$T(std_{\mathrm{SL}_m} \otimes std_{\mathrm{SL}_n}) \oplus T(std_{\mathrm{SL}_n})$
$(\mathrm{GL}_m \times \mathrm{GL}_n, \mathrm{GL}_m \times \mathrm{GL}_{m+1}, T(std_{\mathrm{GL}_m} \otimes std_{\mathrm{GL}_{m+1}}))$	A_{n-m-2}	$T(std_{\mathrm{SL}_m} \otimes std_{\mathrm{SL}_n}) \oplus T(std_{\mathrm{SL}_n})$
$(\mathrm{GL}_n, \mathrm{GL}_2, T(std_{\mathrm{GL}_2})$	A_{n-3}	$T(std_{\mathrm{SL}_n}) \oplus T(std_{\mathrm{SL}_n})$
$*$	B_{m-2}	$T(std_{\mathrm{Sp}_{2m}}) \oplus T(std_{\mathrm{Sp}_{2m}})$

TABLE 20. Dual quadruples of Table 18

8. Summary

We summarize our findings in this paper into the following 6 tables.

- Table 21 contains reductive strongly tempered quadruples for which we have provided evidence for Conjecture 1.1(1) and (2) (i.e., Theorem 1.7 and 1.9).
- Table 22 contains the remaining reductive strongly tempered quadruples. For all of them except $(\mathrm{GL}_6 \times \mathrm{GL}_2, \mathrm{GL}_2 \times S(\mathrm{GL}_4 \times \mathrm{GL}_2), \wedge^2 \otimes std_{\mathrm{GL}_2})$, we have provided evidence for Conjecture 1.1(1) (i.e. Theorem 1.7).
- Table 23 contains non-reductive strongly tempered quadruples for which we have provided evidence for Conjecture 1.1(1) and (2) (i.e., Theorem 1.7, 1.9 and 1.12).
- Table 24 contains non-reductive strongly tempered quadruples for which we have provided evidence only for Conjecture 1.1(1) (i.e., Theorem 1.7 and 1.12).
- Table 25 contains non-reductive strongly tempered quadruples for which we have provided evidence for Conjecture 1.1(1) by assuming Conjecture 2.8 and we have provided evidence for Conjecture 1.1 (2) (i.e. Theorem 1.9 and 1.12).
- Table 26 contains the remaining non-reductive strongly tempered quadruples. For each of them, we have only provided evidence for Conjecture 1.1(1) by assuming Conjecture 2.8 (i.e., Theorem 1.12).

For quadruples (G, H, ρ_H, ι) in Table 21 and 22, the nilpotent orbit ι is trivial. For all the quadruples $\Delta = (G, H, \rho_H, , \iota)$ in Table 21–26, the dual quadruple is given by $(\hat{G}, \widehat{G/Z_\Delta}, \hat{\rho}, 1)$ where $\hat{\rho}$ is given in the tables and $Z_\Delta = Z_G \cap ker(\rho_H)$.

№	(G, H, ρ_H)	$\hat{\rho}$
1	$(\mathrm{SO}_{2m+1}\times\mathrm{SO}_{2m},\ \mathrm{SO}_{2m},\ 0)$	$std_{\mathrm{Sp}_{2m}}\otimes std_{\mathrm{SO}_{2m}}$
2	$(\mathrm{SO}_{2m+2}\times\mathrm{SO}_{2m+1},\ \mathrm{SO}_{2m+1},\ 0)$	$std_{\mathrm{Sp}_{2m}}\otimes std_{\mathrm{SO}_{2m+2}}$
3	$(\mathrm{GSp}_6\times\mathrm{GSpin}_7,\ S(\mathrm{GSp}_6\times\mathrm{GSpin}_7),\ std_{\mathrm{Sp}_6}\otimes\mathrm{Spin}_7)$	$std_{\mathrm{Sp}_6}\otimes\mathrm{Spin}_7$
4	$(\mathrm{GSp}_6\times\mathrm{GSpin}_9, S(\mathrm{GSp}_6\times\mathrm{GSpin}_8), std_{\mathrm{Sp}_6}\otimes\mathrm{HSpin}_8)$	$std_{\mathrm{Sp}_8}\otimes\mathrm{Spin}_7$
5	$(\mathrm{GL}_n\times\mathrm{GL}_n,\mathrm{GL}_n,T(std_{\mathrm{GL}_n}))$	$T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n})$
6	$(\mathrm{GL}_{n+1}\times\mathrm{GL}_n,\mathrm{GL}_n,0)$	$T(std_{\mathrm{GL}_{n+1}}\otimes std_{\mathrm{GL}_n})$
7	$(\mathrm{GSp}_4\times\mathrm{GL}_2,G(\mathrm{SL}_2\times\mathrm{SL}_2),T(std_{\mathrm{GL}_{2,2}}))$	$T(Std_{\mathrm{GSp}_4}\otimes Std_{\mathrm{GL}_2})$
8	$(\mathrm{GSp}_4\times\mathrm{GL}_3,H=G,T(std_{\mathrm{GSp}_4}\otimes std_{\mathrm{GL}_3}))$	$T(Std_{\mathrm{GSp}_4}\otimes Std_{\mathrm{GL}_3})$
9	$(\mathrm{GSp}_4\times\mathrm{GL}_4,S(\mathrm{GSp}_4\times\mathrm{GL}_4),std_{\mathrm{Sp}_4}\otimes\wedge^2\oplus T(std_{\mathrm{GL}_4}))$	$T(Std_{\mathrm{GSp}_4}\otimes Std_{\mathrm{GL}_4})$
10	$(\mathrm{GSp}_4\times\mathrm{GL}_5,S(\mathrm{GSp}_4\times\mathrm{GL}_4),std_{\mathrm{Sp}_4}\otimes\wedge^2)$	$T(Std_{\mathrm{GSp}_4}\otimes Std_{\mathrm{GL}_5})$
11	$(\mathrm{GSpin}_7\times\mathrm{GL}_3,\mathrm{GSpin}_6\times\mathrm{GL}_3,T(\mathrm{HSpin}_6\otimes std_{\mathrm{GL}_3}))$	$T(Std_{\mathrm{GSp}_6}\otimes Std_{\mathrm{GL}_3})$
12	$(\mathrm{SO}_{2m+1}\times\mathrm{Sp}_{2m},H=G,std_{\mathrm{SO}_{2m+1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}})$	$std_{\mathrm{SO}_{2m+1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}}$
13	$(\mathrm{SO}_{2m+1}\times\mathrm{Sp}_{2m-2},\mathrm{SO}_{2m}\times\mathrm{Sp}_{2m-2},std_{\mathrm{SO}_{2m}}\otimes std_{\mathrm{Sp}_{2m-2}})$	$std_{\mathrm{SO}_{2m-1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}}$
14	$(\mathrm{GL}_4\times\mathrm{GSO}_4,S(\mathrm{GSp}_4\times\mathrm{GSO}_4),std_{\mathrm{SO}_4}\times std_{\mathrm{Sp}_4})$	$std_{\mathrm{SL}_2}\otimes std_{\mathrm{SO}_6}\oplus std_{\mathrm{SO}_6}\otimes std_{\mathrm{SL}_2}$
15	$(\mathrm{GL}_4\times\mathrm{GL}_2,\mathrm{GL}_2\times\mathrm{GL}_2,0)$	$std_{\mathrm{SL}_2}\otimes\wedge^2\oplus T(std_{\mathrm{SL}_4})$
16	$(\mathrm{GL}_4\times\mathrm{GSp}_4,\mathrm{GL}_4\times\mathrm{GSp}_4,T(std_{\mathrm{GL}_4}\otimes std_{\mathrm{GSp}_4}))$	$std_{\mathrm{Sp}_4}\otimes\wedge^2\oplus T(std_{\mathrm{SL}_4})$
17	$(\mathrm{GSpin}_7\times\mathrm{GSpin}_6,\mathrm{GSpin}_6\times\mathrm{GSpin}_6,T(\mathrm{HSpin}_6\otimes\mathrm{HSpin}_6))$	$std_{\mathrm{Sp}_6}\otimes std_{\mathrm{Spin}_6}\oplus T(\mathrm{HSpin}_6)$
18	$(\mathrm{GL}_n\times\mathrm{GL}_n,\mathrm{GL}_n\times\mathrm{GL}_n,T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n}\oplus std_{\mathrm{GL}_n}))$	$T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n})\oplus T(std_{\mathrm{GL}_n})$
19	$(\mathrm{GL}_{n+1}\times\mathrm{GL}_n,\mathrm{GL}_n\times\mathrm{GL}_n,T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n}))$	$T(std_{\mathrm{GL}_{n+1}}\otimes std_{\mathrm{GL}_n})\oplus T(std_{\mathrm{GL}_n})$
20	$(\mathrm{GL}_n\times\mathrm{GL}_{n-1},\mathrm{GL}_n\times\mathrm{GL}_{n-1},T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_{n-1}}\oplus std_{\mathrm{GL}_n}))$	$T(std_{\mathrm{GL}_{n-1}}\otimes std_{\mathrm{GL}_n})\oplus T(std_{\mathrm{GL}_n})$
21	$(\mathrm{GL}_n\times\mathrm{GL}_{n-2},\mathrm{GL}_{n-1}\times\mathrm{GL}_{n-2},T(std_{\mathrm{GL}_{n-1}}\otimes std_{\mathrm{GL}_{n-2}}))$	$T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_{n-2}})\oplus T(std_{\mathrm{GL}_n})$
22	$((\mathrm{GL}_2)^5,S(\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2),std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2})$	$*$
23	$\sharp$	$**$
24	$(\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2,\mathrm{GL}_2\times\mathrm{GL}_2,T(std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}))$	$***$
25	$\sharp\sharp$	$****$
26	$(\mathrm{GL}_2\times\mathrm{GL}_1,\mathrm{GL}_2\times\mathrm{GL}_1,T(std_{\mathrm{GL}_2}\oplus std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_1}))$	$T(std_{\mathrm{GL}_2}\oplus std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_1})$

TABLE 21. Reductive strongly tempered quadruples 1

$$\sharp=((\mathrm{GL}_2)^4,S(\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2),std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}\oplus T(std_{\mathrm{GL}_{2,2}})).$$

$$\sharp\sharp=((\mathrm{GL}_2)^3,S(\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2),std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}\otimes std_{\mathrm{GL}_2}\oplus T(std_{\mathrm{GL}_{2,2}}\oplus T(std_{\mathrm{GL}_{2,3}})).$$

$$*=std_{\mathrm{GL}_{2,1}}\otimes std_{\mathrm{GL}_{2,2}}\otimes std_{\mathrm{GL}_{2,3}}\oplus std_{\mathrm{GL}_{2,1}}\otimes std_{\mathrm{GL}_{2,4}}\otimes std_{\mathrm{GL}_{2,5}}.$$

$$**=T(std_{\mathrm{GL}_{2,1}}\otimes std_{\mathrm{GL}_{2,2}})\oplus std_{\mathrm{GL}_{2,1}}\otimes std_{\mathrm{GL}_{2,3}}\otimes std_{\mathrm{GL}_{2,4}}.$$

$$***=T(std_{\mathrm{GL}_{2,1}})\oplus std_{\mathrm{GL}_{2,1}}\otimes std_{\mathrm{GL}_{2,2}}\otimes std_{\mathrm{GL}_{2,3}}.$$

$$****=T(std_{\mathrm{GL}_{2,1}}\otimes std_{\mathrm{GL}_{2,2}})\oplus T(std_{\mathrm{GL}_{2,1}}\otimes std_{\mathrm{GL}_{2,3}}).$$

№	(G, H, ρ_H)	$\hat{\rho}$
1	$(\mathrm{GSp}_6 \times \mathrm{GSp}_4,\ G(\mathrm{Sp}_4 \times \mathrm{Sp}_2), 0)$	$\mathrm{Spin}_5 \otimes \mathrm{Spin}_7$
2	$(\mathrm{GSp}_6 \times \mathrm{GSO}_4, S(\mathrm{GSO}_4 \times G(\mathrm{Sp}_4 \times \mathrm{SL}_2)), std_{\mathrm{SO}_4} \times std_{\mathrm{Sp}_4})$	$std_{\mathrm{SL}_2} \otimes \mathrm{Spin}_7 \oplus \mathrm{Spin}_7 \otimes std_{\mathrm{SL}_2}$
3	$*$	$std_{\mathrm{Sp}_4} \otimes std_{\mathrm{Spin}_8} \oplus \mathrm{HSpin}_8 \otimes std_{\mathrm{SL}_2}$
4	$(\mathrm{GL}_6 \times \mathrm{GL}_2, \mathrm{GL}_2 \times S(\mathrm{GL}_4 \times \mathrm{GL}_2), \wedge^2 \otimes std_{\mathrm{GL}_2})$	$\wedge^3 \oplus T(std_{\mathrm{SL}_6} \otimes std_{\mathrm{SL}_2})$
5	$**$	$std_{\mathrm{SL}_2} \otimes \wedge^2 \oplus T(std_{\mathrm{SL}_4} \otimes std_{\mathrm{SL}_2})$
6	$(\mathrm{GL}_{2m}, \mathrm{GL}_m \times \mathrm{GL}_m, T(std_{\mathrm{GL}_m}))$	$T(\wedge^2) \oplus T(std_{\mathrm{GL}_{2m}})$
7	$(\mathrm{GL}_{2m+1}, \mathrm{GL}_{m+1} \times \mathrm{GL}_m, T(std_{\mathrm{GL}_{m+1}}))$	$T(\wedge^2) \oplus T(std_{\mathrm{GL}_{2m+1}})$
8	$(\mathrm{GL}_3, \mathrm{GL}_2 \times \mathrm{GL}_1, T(std_{\mathrm{GL}_2}))$	$T(std_{\mathrm{SL}_3}) \oplus T(std_{\mathrm{SL}_3})$
9	$(\mathrm{GSpin}_5 \times \mathrm{GL}_1, \mathrm{GSpin}_4 \times \mathrm{GL}_1, T(\mathrm{HSpin}_4^+ \oplus \mathrm{HSpin}_4^- \otimes std_{\mathrm{GL}_1}))$	$T(std_{\mathrm{Sp}_4}) \oplus T(std_{\mathrm{Sp}_4})$

Table 22. Reductive strongly tempered quadruples 2

$$* = (\mathrm{GSp}_4 \times \mathrm{GSpin}_8 \times \mathrm{GL}_2, S(\mathrm{GSpin}_8 \times G(\mathrm{Sp}_4 \times \mathrm{SL}_2)), std_{\mathrm{Sp}_4} \otimes std_{\mathrm{Spin}_8} \oplus \mathrm{HSpin}_8 \otimes std_{\mathrm{SL}_2}).$$

$$** = (\mathrm{GL}_2 \times \mathrm{GL}_4 \times \mathrm{GL}_2, S(\mathrm{GL}_2 \times \mathrm{GL}_4) \times \mathrm{GL}_2, std_{\mathrm{GL}_2} \otimes \wedge^2 \oplus T(std_{\mathrm{GL}_4} \times std_{\mathrm{GL}_2})).$$

№	(G, H, ρ_H)	ι	$\hat{\rho}$
1	$(\mathrm{SO}_{2m+1} \times \mathrm{SO}_{2n}, \mathrm{SO}_{2n}, 0)$	B_{m-n}	$std_{\mathrm{Sp}_{2m}} \otimes std_{\mathrm{SO}_{2n}}$
2	$(\mathrm{SO}_{2m+1} \times \mathrm{SO}_{2n}, \mathrm{SO}_{2m+1}, 0)$	D_{n-m}	$std_{\mathrm{Sp}_{2m}} \otimes std_{\mathrm{SO}_{2n}}$
3	$(\mathrm{GSpin}_{2m+1} \times \mathrm{GSp}_6, S(\mathrm{GSpin}_8 \times \mathrm{GSp}_6), std_{\mathrm{Sp}_6} \otimes \mathrm{HSpin}_8)$	B_{m-4}	$std_{\mathrm{Sp}_{2m}} \otimes \mathrm{Spin}_7$
4	$(\mathrm{SO}_{2m+1}, \mathrm{SO}_2, 0)$	B_{m-1}	$T(std_{\mathrm{Sp}_{2n}})$
5	$(\mathrm{GSpin}_{2m+1} \times \mathrm{GL}_2, G(\mathrm{SL}_2 \times \mathrm{SL}_2), T(std_{\mathrm{GL}_2}))$	B_{m-2}	$T(Std_{\mathrm{GSp}_{2m}} \otimes Std_{\mathrm{GL}_2})$
6	$(\mathrm{GSpin}_{2m+1} \times \mathrm{GL}_3, \mathrm{GSpin}_6 \times \mathrm{GL}_3, T(\mathrm{HSpin}_6 \otimes std_{\mathrm{GL}_3}))$	B_{m-3}	$T(Std_{\mathrm{Sp}_{2m}} \otimes Std_{\mathrm{SL}_3})$
7	$(\mathrm{SO}_{2m+1} \times \mathrm{Sp}_{2n-2}, \mathrm{SO}_{2n} \times \mathrm{Sp}_{2n-2}, std_{\mathrm{SO}_{2n}} \otimes std_{\mathrm{Sp}_{2n-2}})$	B_{m-n}	$std_{\mathrm{SO}_{2n-1}} \otimes std_{\mathrm{Sp}_{2m}} \oplus std_{\mathrm{Sp}_{2m}}$
8	$(\mathrm{GSpin}_{2k} \times \mathrm{GSO}_4, S(\mathrm{GSp}_4 \times \mathrm{GSO}_4), std_{\mathrm{SO}_4} \times std_{\mathrm{Sp}_4})$	D_{k-2}	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{SO}_{2k}} \oplus std_{\mathrm{SO}_{2k}} \otimes std_{\mathrm{SL}_2}$
9	$(\mathrm{GSpin}_{2m+1} \times \mathrm{GSpin}_6, \mathrm{GSpin}_6 \times \mathrm{GSpin}_6, T(\mathrm{HSpin}_6 \otimes \mathrm{HSpin}_6))$	B_{m-3}	$std_{\mathrm{Sp}_{2m}} \otimes std_{\mathrm{SO}_6} \oplus T(\mathrm{HSpin}_6)$

Table 23. Non-reductive strongly tempered quadruples 1

№	(G,H,ρ_H)	ι	$\hat{\rho}$
1	$(\mathrm{GSp}_6\times\mathrm{GL}_2,\mathrm{GL}_2,0)$	A_2	$std_{\mathrm{GL}_2}\otimes\mathrm{Spin}_7$
2	$(\mathrm{GSp}_8\times\mathrm{GL}_2,G(\mathrm{SL}_2\times\mathrm{SL}_2),0)$	A_2	$std_{\mathrm{GL}_2}\otimes\mathrm{Spin}_9$
3	$(\mathrm{GSp}_{10},\mathrm{GL}_2,0)$	A_4	Spin_{11}
4	$(\mathrm{GSO}_{12},\mathrm{GL}_2,0)$	A_5	HSpin_{12}
5	$(\mathrm{GL}_6,\mathrm{GL}_2,0)$	$A_2\times A_2$	$\wedge^3$
6	$(E_7,\mathrm{PGL}_2,0)$	E_6	std_{E_7}
7	$(\mathrm{GL}_{2m},\mathrm{GL}_m,T(std_{\mathrm{GL}_m}))$	$(A_1)^m$	$T(\wedge^2)$
8	$(\mathrm{GL}_{2m+1},\mathrm{GL}_m,0)$	$(A_1)^m$	$T(\wedge^2)$
9	$(\mathrm{GSpin}_{2k},\mathrm{GSpin}_3,T(\mathrm{Spin}_3))$	D_{k-1}	$T(std_{\mathrm{SO}_{2k}})$
10	$(\mathrm{GSp}_6,\mathrm{GL}_2,T(std_{\mathrm{GL}_2}))$	A_2	$T(\mathrm{Spin}_7)$
11	$(\mathrm{GSp}_8,G(\mathrm{SL}_2\times\mathrm{SL}_2),T(std_{\mathrm{GL}_2}))$	A_2	$T(\mathrm{Spin}_9)$
12	$(GE_6,\mathrm{GL}_3,T(std_{\mathrm{GL}_3}))$	D_4	$T(std_{E_6})$
13	$*$	B_{m-2}	$T(std_{\mathrm{Sp}_{2m}})\oplus T(std_{\mathrm{Sp}_{2m}})$

TABLE 24. Non-reductive strongly tempered quadruples 2

$$*=(\mathrm{GSpin}_{2m+1}\times\mathrm{GL}_1,\mathrm{GSpin}_4\times\mathrm{GL}_1,T(\mathrm{HSpin}_4^+\oplus\mathrm{HSpin}_4^-\otimes std_{\mathrm{GL}_1})).$$

№	(G,H,ρ_H)	ι	$\hat{\rho}$
1	$(\mathrm{GL}_m\times\mathrm{GL}_n,\mathrm{GL}_n,0)$	A_{m-n-1}	$T(std_{\mathrm{GL}_m}\otimes std_{\mathrm{GL}_n})$
2	$(\mathrm{GSp}_4\times\mathrm{GL}_n,S(\mathrm{GSp}_4\times\mathrm{GL}_4),std_{\mathrm{Sp}_4}\otimes\wedge^2)$	A_{n-5}	$T(Std_{\mathrm{Sp}_4}\otimes Std_{\mathrm{SL}_m})$
3	$(\mathrm{SO}_{2m+1}\times\mathrm{Sp}_{2k},\mathrm{SO}_{2m+1}\times\mathrm{Sp}_{2m},std_{\mathrm{SO}_{2m+1}}\otimes std_{\mathrm{Sp}_{2m}})$	C_{k-m}	$std_{\mathrm{SO}_{2k+1}}\otimes std_{\mathrm{Sp}_{2m}}\oplus std_{\mathrm{Sp}_{2m}}$
4	$(\mathrm{GL}_m\times\mathrm{GL}_n,\mathrm{GL}_n\times\mathrm{GL}_n,T(std_{\mathrm{GL}_n}\otimes std_{\mathrm{GL}_n}))$	A_{m-n+1}	$T(std_{\mathrm{SL}_m}\otimes std_{\mathrm{SL}_n})\oplus T(std_{\mathrm{SL}_n})$
5	$(\mathrm{GL}_m\times\mathrm{GL}_n,\mathrm{GL}_m\times\mathrm{GL}_{m+1},T(std_{\mathrm{GL}_m}\otimes std_{\mathrm{GL}_{m+1}}))$	A_{n-m-2}	$T(std_{\mathrm{SL}_m}\otimes std_{\mathrm{SL}_n})\oplus T(std_{\mathrm{SL}_n})$

TABLE 25. Non-reductive strongly tempered quadruples 3

№	(G, H, ρ_H)	ι	$\hat{\rho}$
1	$(\mathrm{GSp}_{12}, \mathrm{GSp}_4, 0)$	$A_2 \times A_2$	Spin_{13}
2	$(\mathrm{PGSO}_{10}, \mathrm{GL}_2, 0)$	A_3	$T(\mathrm{HSpin}_{10})$
3	$(\mathrm{GSO}_{12}, S(\mathrm{GSp}_4 \times \mathrm{GSO}_4), 0)$	$A_1 \times A_1$	$\mathrm{HSpin}_{12}^+ \oplus \mathrm{HSpin}_{12}^-$
4	$(\mathrm{GSO}_{12} \times \mathrm{PGL}_2, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), 0)$	A_3	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{Spin}_{12}} \oplus \mathrm{HSpin}_{12}$
5	$*$	A_1	$std_{\mathrm{Sp}_4} \otimes std_{\mathrm{Spin}_{12}} \oplus \mathrm{HSpin}_{12}$
6	$(\mathrm{GSO}_8 \times \mathrm{GSO}_4, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), 0)$	A_1	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{Spin}_8} \oplus \mathrm{HSpin}_8 \otimes std_{\mathrm{SL}_2}$
7	$(\mathrm{GSpin}_7, S(\mathrm{GL}_2 \times \mathrm{GL}_2), std_{\mathrm{GL}_2})$	A_1	$\wedge^3 \oplus std_{\mathrm{Sp}_6}$
8	$(\mathrm{GSO}_{12}, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), T(std_{\mathrm{GL}_2}))$	A_3	$\mathrm{HSpin}_{12} \oplus T(std_{\mathrm{Spin}_{12}})$
9	$**$	A_1	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{Spin}_{10}} \oplus T(std_{\mathrm{Spin}_{10}})$
10	$(\mathrm{GSO}_8 \times \mathrm{GL}_2, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), T(std_{\mathrm{GL}_2}))$	A_1	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{Spin}_8} \oplus T(std_{\mathrm{Spin}_8})$
11	$(\mathrm{GL}_6, \mathrm{GL}_2 \times \mathrm{GL}_2, 0)$	$A_1 \times A_1$	$\wedge^3 \oplus T(std_{\mathrm{SL}_6})$
12	$(\mathrm{GSO}_8, S(\mathrm{GL}_2 \times \mathrm{GSO}_4), T(std_{\mathrm{GL}_2} \oplus std_{\mathrm{GL}_2}))$	A_1	$T(std_{\mathrm{Spin}_8}) \oplus T(\mathrm{HSpin}_8)$
13	$(\mathrm{GL}_n, \mathrm{GL}_2, T(std_{\mathrm{GL}_2})$	A_{n-3}	$T(std_{\mathrm{SL}_n}) \oplus T(std_{\mathrm{SL}_n})$

Table 26. Non-reductive strongly tempered quadruples 4

$$* = (\mathrm{GSpin}_4 \times \mathrm{GSpin}_{12}, S(\mathrm{GSpin}_8 \times G(\mathrm{Sp}_4 \times \mathrm{SL}_2)), std_{\mathrm{Sp}_4} \otimes std_{\mathrm{Spin}_8}).$$

$$** = (\mathrm{GSpin}_{10} \times \mathrm{GL}_2, S(\mathrm{GL}_2 \times \mathrm{GSpin}_6) \times \mathrm{GL}_2, T(\mathrm{HSpin}_6 \otimes std_{\mathrm{GL}_2})).$$

9. Table S of [28]

In this section we will discuss Table S of [28]. This table contains several models from Table 1, 2, 11, 12, 22 of [28] with some A_1 components (i.e. $\hat{G} = \hat{G}_1 \times \hat{G}_2$ where $\hat{G}_1$ is of Type A_1). Then one can obtain more multiplicity free representations by gluing those representations together via the A_1-components. Two of the models in Table S contain two A_1-components (i.e. (S.1) and (S.2)) and hence it can be used to create infinite many multiplicity free representations. We refer the reader to [28] for the details of the gluing process. In this section we will discuss how to write down the dual of those models by defining a gluing process for the dual of models in Table S.

We first recall Table S from [28]. Note that underlined section of the SL_2 part is where we can glue the representations. Model (S.1) and (S.2) contains two underlined SL_2 and we can use it glue representations with any arbitrary length.

№in [28]	$\hat{G}$	$\hat{\rho}$
(S.1)	$\underline{\mathrm{SL}_2} \times \mathrm{Sp}_{2m} \times \underline{\mathrm{SL}_2}$	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{Sp}_{2m}} \otimes std_{\mathrm{SL}_2}$
(S.2)	$\underline{\mathrm{SL}_2} \times \mathrm{Spin}_8 \times \underline{\mathrm{SL}_2}$	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{Spin}_8} \oplus \mathrm{HSpin}_8 \otimes std_{\mathrm{SL}_2}$
(S.3)	$\mathrm{SO}_n \times \underline{\mathrm{SL}_2}$	$std_{\mathrm{SO}_n} \otimes std_{\mathrm{SL}_2}$
(S.4)	$\mathrm{Spin}_{12} \times \underline{\mathrm{SL}_2}$	$\mathrm{HSpin}_{12} \oplus std_{\mathrm{Spin}_{12}} \otimes std_{\mathrm{SL}_2}$
(S.5)	$\mathrm{Spin}_9 \times \underline{\mathrm{SL}_2}$	$\mathrm{Spin}_9 \otimes std_{\mathrm{SL}_2}$
(S.6)	$\mathrm{Spin}_8 \times \underline{\mathrm{SL}_2}$	$T(std_{\mathrm{Spin}_8}) \oplus \mathrm{HSpin}_8 \otimes std_{\mathrm{SL}_2}$
(S.7)	$\mathrm{Spin}_7 \times \underline{\mathrm{SL}_2}$	$\mathrm{Spin}_7 \otimes std_{\mathrm{SL}_2}$
(S.8)	$\mathrm{SL}_2 \times \mathrm{Spin}_7 \times \underline{\mathrm{SL}_2}$	$std_{\mathrm{SL}_2} \otimes std_{\mathrm{Spin}_7} \oplus \mathrm{Spin}_7 \otimes std_{\mathrm{SL}_2}$
(S.9)	$\underline{\mathrm{SL}_2}$	std_{SL_2}
(S.10)	$\underline{\mathrm{SL}_2}$	$T(std_{\mathrm{SL}_2})$
(S.11)	$\mathrm{SL}_m \times \underline{\mathrm{SL}_2}$	$T(std_{\mathrm{SL}_m} \otimes std_{\mathrm{SL}_2})$
(S.12)	$\mathrm{SL}_4 \times \underline{\mathrm{SL}_2}$	$T(std_{\mathrm{SL}_4}) \oplus \wedge^2 \otimes std_{\mathrm{SL}_2}$
(S.13)	$\mathrm{Sp}_{2m} \times \underline{\mathrm{SL}_2}$	$std_{\mathrm{Sp}_{2m}} \otimes \mathrm{Sym}^2$
(S.14)	$\mathrm{Sp}_{2m} \times \underline{\mathrm{SL}_2}$	$T(std_{\mathrm{Sp}_{2m}} \otimes std_{\mathrm{SL}_2})$
(S.15)	$\mathrm{Spin}_5 \times \underline{\mathrm{SL}_2}$	$\mathrm{Spin}_5 \oplus std_{\mathrm{Spin}_5} \otimes std_{\mathrm{SL}_2}$
(S.16)	$G_2 \times \underline{\mathrm{SL}_2}$	$std_{G_2} \otimes std_{\mathrm{SL}_2}$

Table 27. Table S of [28]

It is clear that if one glue some anomaly-free representations with some non anomaly-free representations in Table S, one will get a non anomaly-free representation. Hence we can consider them separately. We first consider the anomaly-free representations in Table S, this corresponds to Model (S.1), (S.2), (S.3) when n is even, (S.4)-(S.7), (S.10)-(S.12) and (S.14). For each of them we have already write down its dual quadruple in the previous sections. We just need to describe how to glue the dual together [8].

Let $\hat{\Delta} = (\hat{G}, \hat{G}, \hat{\rho}, 1)$ be one of such model and let $\Delta = (G, H, \rho_H, \iota)$ be its dual. If the model is not (S.1), (S.2) or (S.10), then $\hat{G} = \hat{G}_1 \times \hat{G}_2$ and $G = G_1 \times G_2$ with $\hat{G}_1, G_1$ being of Type A_1. Moreover, by our description of Δ in the previous section, we know that

[8](S.1) is Model 1 of Table 23 with n=4, (S.2) is Model 6 of Table 26, (S.3) when n is even is Model 2 of Table 23 with $m = 1$, (S.4) is Model 4 of Table 26, (S.5) is Model 2 of Table 24, (S.6) is Model 10 of Table 26, (S.7) is Model 1 of Table 24, (S.10) is Model 6 of Table 21 with $n = 1$, (S.11) is Model 1 of Table 25 with $n = 2$, (S.12) is Model 15 of Table 21, and (S.14) is Model 5 of Table 23.

the projection map $H \to G_1$ is surjective and we can write the group HG_1 (i.e. the group generated by H and G_1) as $G_1 \times H_1 \times H_2$ with $H_2 = H \cap G_2$ and $H_1 \simeq G_1$ is in the centralizer of H_2 in $G_1H \cap G_2$. Moreover the image of the diagonal embedding from H_1 into $G_1 \times H_1$ belongs to H. In particular, the representation ρ_H induces a representation (still denoted by ρ_H) on $H_1 \times H_2$ (on H_2-part this is given by restriction and on H_1 part it is given by restriction and the diagonal embedding from H_1 into $G_1 \times H_1$). Finally, the nilpotent orbit ι is the product of some nilpotent oribt of G_2 with the trivial nilpotent orbit of G_1. For example, consider Model (S.3) when $n = 4$, the dual is the trilinear GL_2 model

$$(G, H, \rho_H, \iota) = (\mathrm{PGL}_2^3, \mathrm{PGL}_2, 0, 1)$$

and in this case

$$G_1 = \{(1, 1, h)|\; h \in \mathrm{PGL}_2\},\; H = \{(h, h, h)|\; h \in \mathrm{PGL}_2\},\; H_1 = \{(h, h, 1)|\; h \in \mathrm{PGL}_2\},\; H_2 = \{1\}.$$

If the Model is (S.1) or (S.2), then $\hat{G} = \hat{G}_{11} \times \hat{G}_{12} \times \hat{G}_2$ and $G = G_{11} \times G_{12} \times G_2$ with $\hat{G}_{11}, \hat{G}_{12}, G_{11}, G_{12}$ being of Type A_1. Moreover, by our description of Δ in the previous section, we know that the projection map $H \to G_{11} \times G_{12}$ is surjective and we can write the group $HG_{11}G_{12}$ as $G_{11} \times G_{12} \times H_{11} \times H_{12} \times H_2$ with $H_2 \subset H$, $H_{1i} \simeq G_{1i}$, and the image of the diagonal embedding from H_{1i} into $G_{1i} \times H_{1i}$ belongs to H for $i = 1, 2$. Also the representation ρ_H would be 0 in this case. Finally, the nilpotent orbit ι is the product of some nilpotent oribt of G_2 with the trivial nilpotent orbit of $G_{11} \times G_{12}$.

Lastly, if the model is $(S.10)$, then $\hat{\Delta} = (\mathrm{SL}_2, \mathrm{SL}_2, T(std), 1)$ and $\Delta = (\mathrm{PGL}_2, \mathrm{GL}_1, 0, 1)$. In particular $\hat{G} = \hat{G}_1$ and $G = G_1$ are of Type A_1.

Now we can describe the gluing process on the dual side. Suppose we are gluing two representations $(\hat{G}, \hat{G}, \hat{\rho})$ and $(\hat{G}', \hat{G}', \hat{\rho}')$. In particular we can write

$$\hat{G} = \hat{G}_1 \times \hat{G}_2,\; \hat{G}' = \hat{G}_1' \times \hat{G}_2'$$

and we are gluing $\hat{G}_1$ with $\hat{G}_1'$. The goal is to write down the dual of

$$\hat{\Delta}_{glue} = (\hat{G}_2 \times \hat{G}_1 \times \hat{G}_2', \hat{G}_2 \times \hat{G}_1 \times \hat{G}_2', \hat{\rho} \oplus \hat{\rho}', 1).$$

Here we consider $\hat{\rho}$ (or $\hat{\rho}'$) as a representation of $\hat{G}_2 \times \hat{G}_1 \times \hat{G}_2'$ where the $\hat{G}_2'$ (or $\hat{G}_2$) component acts trivially. Let $\Delta = (G, H, \rho_H, \iota)$ and $\Delta' = (G', H', \rho_H', \iota')$ be the dual of $\hat{\Delta}$ and $\hat{\Delta}'$.

There are two cases. First we consider the case when both representations are not (S.10). In this case, by our discussion above, we have the decomposition [9]

$$G_1H = G_1 \times H_1 \times H_2,\; G_1'H' = G_1' \times H_1' \times H_2'.$$

Then the dual would be given by

$$\Delta_{glue} = (G_2 \times G_1 \times G_2', H_2 \times H_1 \times G_1 \times H_1' \times H_2', \rho_H \oplus \rho_H' \oplus \rho', \iota \times \iota')$$

where ρ' is the tensor product representation of $H_1 \times G_1 \times H_1'$. Note that when the model is not (S.1), (S.2) or (S.10), we have explained how to view ρ_H (resp. ρ_H') as a representation of $H_1 \times H_2$ (resp. $H_1' \times H_2'$). In the cases of $(S.1)$ or $(S.2)$ the representation ρ_H (resp. ρ_H') is just 0. Also note that ι (resp. ι') is the product of some nilpotent orbit of G_2 (resp. G_2') with the trivial nilpotent orbit of $G_1 = G_1'$ and hence we can view $\iota \times \iota'$ as a nilpotent orbit of $G_1 \times G_1 \times G_2'$. Model 5.5 and 6.3 are examples of this case.

[9]If we are in the case of (S.1) or (S.2), then G_1 would be one of the G_{1i} and H_1 would be the corresponding H_{1i}

Now let's prove Theorem 1.15 in this case. The idea is similar to the proof of Conjecture 1.1 for the model 5.5. Roughly speaking, we will show that the period integral associated to Δ_{glue} (resp. $\hat{\Delta}_{glue}$) is a combination of the period integrals associated to $\Delta, \Delta', \Delta_1$ (resp. $\hat{\Delta}, \hat{\Delta}', \hat{\Delta}_1$) [10] where

$$\Delta_1 = (\mathrm{SL}_2^3, \mathrm{SL}_2^3, std \otimes std \otimes std, 1), \hat{\Delta}_1 = (\mathrm{PGL}_2^3, \mathrm{PGL}_2, 0, 1).$$

In particular, Conjecture 1.1 for $(\Delta_{glue}, \hat{\Delta}_{glue})$ will follow from Conjecture 1.1 for $(\Delta, \hat{\Delta})$, $(\Delta', \hat{\Delta}')$ and $(\Delta_1, \hat{\Delta}_1)$. As Conjecture 1.1 is know for $(\Delta_1, \hat{\Delta}_1)$ (by the Rallis inner product formula and the work of Harris-Kudla [24] for the triple product period), we know that Conjecture 1.1 for $(\Delta_{glue}, \hat{\Delta}_{glue})$ will follow from Conjecture 1.1 for $(\Delta, \hat{\Delta})$ and $(\Delta', \hat{\Delta}')$. This proves Theorem 1.15.

It remains to explain why the the period integral associated to Δ_{glue} (resp. $\hat{\Delta}_{glue}$) is a combination of the period integrals associated to $\Delta, \Delta', \Delta_1$ (resp. $\hat{\Delta}, \hat{\Delta}', \hat{\Delta}_1$). We start with the period integral associated to Δ_{glue}. In these case we start with an automorphic form $\phi = \phi_{G_2}\phi_{G_1}\phi_{G_2'}$ on $G_2 \times G_1 \times G_2'$. We first integrate over G_1 (note that the projection of nilpotent orbit $\iota \times \iota'$ to G_1 is the trivial orbit, so the unipotent integral associated to $\iota \times \iota'$ commutes with the integral over G_1). Since the symplectic representation ρ_H (resp. ρ_H') is on the group $H_1 \times H_2$ (resp. $H_1' \times H_2'$), the integral over G_1 is just the integral of ϕ_{G_1} with theta function associated to ρ' (recall that ρ' is the tensor product representation of $H_1 \times G_1 \times H_1'$). By the theta correspondence of $\mathrm{Sp}_2 \times \mathrm{SO}_4$, the integral

$$\int_{G_1(k)\backslash G_1(\mathbb{A})} \phi_{G_1}(g_1)\Theta_{\rho'}(h_1, g_1, h_1')dg_1$$

gives an automorphic form $\phi_{H_1\times H_1'}(h_1, h_1')$ in the irreducible space $\pi \otimes \pi$ on $H_1 \times H_1'$ (assuming $\phi \in \pi$ of $G_1 \simeq H_1 \simeq H_2$). We may as well assume $\phi_{H_1\times H_1'}(h_1, h_1')$ has the form $\phi_{H_1}(h_1)\phi_{H_1'}(h_1')$. Note that by Rallis inner product formula $\|\phi_{G_1}\|$ " = " $\|\phi_{H_1}\|\|\phi_{H_1'}\|$. Then the remaining integrals (i.e. the unipotent integral associated to $\iota \times \iota'$ and integral over $H_2 \times H_1 \times H_1' \times H_2'$) become the product of the period integrals of the automorphic forms $\phi_{G_2} \times \phi_{H_1}$ and $\phi_{H_1'} \times \phi_{G_2'}$ associated to the quadruples $(G_2 \times H_1, H_2 \times H_1, \rho_H, \iota)$ and $(H_1' \times G_2', H_1' \times H_2', \rho_H', \iota')$ respectively [11]. But these two quadruples are just Δ and Δ' via the isomorphism $H_1 \simeq G_1 \simeq G_1' \simeq H_1'$. As a result, Conjecture 1.1(1) for Δ_{glue} would follow from Conjecture 1.1(1) for Δ and Δ'.

For the other direction, the period integral associated to $\hat{\Delta}_{glue}$ is given by

$$\int_{\hat{G}_1(k)\backslash\hat{G}_1(\mathbb{A})} \int_{\hat{G}_2(k)\backslash\hat{G}_2(\mathbb{A})} \int_{\hat{G}_2'(k)\backslash\hat{G}_2'(\mathbb{A})} \phi_{\hat{G}_1}(g_1)\phi_{\hat{G}_2}(g_2)\phi_{\hat{G}_2'}(g_2')\Theta_{\hat{\rho}}(g_1, g_2)\Theta_{\hat{\rho}'}(g_1, g_2')dg_2'dg_2dg_1. \tag{9.1}$$

By Conjecture 1.1(2) for Δ and Δ', we know that

- the integral

$$\int_{\hat{G}_2(k)\backslash\hat{G}_2(\mathbb{A})} \phi_{\hat{G}_2}(g_2)\Theta_{\hat{\rho}}(g_1, g_2)dg_2$$

[10] $(\Delta_1, \hat{\Delta}_1)$ is just Model 2 in Table 21 when $m = 1$, the period integral associated to Δ_1 is the theta correspondence of $\mathrm{Sp}_2 \times \mathrm{SO}_4$ while the period integral associated to $\hat{\Delta}_1$ is the triple product integral

[11] note here for the embedding of $H_1 \times H_2$ (respectively $H_1' \times H_2'$), the component H_1 (resp. H_1') diagonally embeds into $G_2 \times H_1$ (resp. $H_1' \times G_2'$)

is non-vanishing only if the Arthur parameter of $\phi_{\hat{G}_2}$ factors through $H_1 \times H_2 \to G_2$, i.e. it is the lifting of an Arthur packet $\Pi_{\hat{H}_1} \otimes \Pi_{\hat{H}_2}$ of $\hat{H}_1(\mathbb{A}) \times \hat{H}_2(\mathbb{A})$. Moreover, if this is the case and the packet $\Pi_{\hat{H}_1} \otimes \Pi_{\hat{H}_2}$ is tempered, then the automorphic function

$$\phi_1(g_1) := \int_{\hat{G}_2(k)\backslash \hat{G}_2(\mathbb{A})} \phi_{\hat{G}_2}(g_2)\Theta_{\hat{\rho}}(g_1, g_2)dg_2,\ g_1 \in \hat{G}_1(\mathbb{A})$$

belongs to the packet $\Pi_{\hat{H}_1}$ (as $H_1 \simeq G_1$ we can view $\Pi_{\hat{H}_1}$ as a packet for $\hat{G}_1$);

- the integral

$$\int_{\hat{G}_2'(k)\backslash \hat{G}_2'(\mathbb{A})} \phi_{\hat{G}_2'}(g_2')\Theta_{\hat{\rho}'}(g_1, g_2')dg_2'$$

is non-vanishing only if the Arthur parameter of $\phi_{\hat{G}_2'}$ factors through $H_1' \times H_2' \to G_2'$, i.e. it is the lifting of an Arthur packet $\Pi_{\hat{H}_1'} \otimes \Pi_{\hat{H}_2'}$ of $\hat{H}_1'(\mathbb{A}) \times \hat{H}_2'(\mathbb{A})$. Moreover, if this is the case and the packet $\Pi_{\hat{H}_1'} \otimes \Pi_{\hat{H}_2'}$ is tempered, then the automorphic function

$$\phi_1'(g_1) := \int_{\hat{G}_2'(k)\backslash \hat{G}_2'(\mathbb{A})} \phi_{\hat{G}_2'}(g_2')\Theta_{\hat{\rho}'}(g_1, g_2')dg_2',\ g_1 \in \hat{G}_1(\mathbb{A})$$

belongs to the packet $\Pi_{\hat{H}_1'}$ (as $H_1' \simeq G_1$ we can view $\Pi_{\hat{H}_1'}$ as a packet for $\hat{G}_1$).

By the above two facts, the integral (9.1) becomes

$$\int_{\hat{G}_1(k)\backslash \hat{G}_1(\mathbb{A})} \phi_{\hat{G}_1}(g_1)\phi_1(g_1)\phi_1'(g_1)dg_1$$

which is exactly a triple product integral on $\hat{H}_1 \times \hat{G}_1 \times \hat{H}_1'$. In particular, Conjecture 1.1(2) for Δ_{glue} follows from the work of Harris-Kudla [24] for the triple product period and Conjecture 1.1(2) for Δ and Δ'. This finishes the proof of Theorem 1.15 for this case.

Next we consider the case when at least one of the representation is (S.10). We may assume that $(\hat{G}', \hat{G}', \hat{\rho}')$ is $(S.10)$. Then we have the decomposition

$$G_1H = G_1 \times H_1 \times H_2,\ G' \simeq G_1.$$

The dual of

$$\hat{\Delta}_{glue} = (\hat{G}_2 \times \hat{G}_1, \hat{G}_2 \times \hat{G}_1, \hat{\rho} \oplus T(std_{G_1}), 1).$$

would be given by

$$\Delta_{glue} = (G_2 \times G_1, H_2 \times H_1 \times G_1, \rho_H \oplus T(\rho'), \iota)$$

where ρ' is the tensor product representation of $H_1 \times G_1$. Model 6.4 and 7.8 are examples of this case.

Now let's prove Theorem 1.15 in this case. The idea is similar to the proof of Conjecture 1.1 for the model 6.4. Roughly speaking, we will show that the period integral associated to Δ_{glue} (resp. $\hat{\Delta}_{glue}$) is a combination of the period integrals associated to Δ, Δ_2 (resp. $\hat{\Delta}, \hat{\Delta}_2$) where [12]

$$\Delta_2 = (\mathrm{GL}_2 \times \mathrm{GL}_2, \mathrm{GL}_2 \times \mathrm{GL}_2, T(std \otimes std), 1),\ \hat{\Delta}_2 = (\mathrm{GL}_2 \times \mathrm{GL}_2, \mathrm{GL}_2, T(std), 1).$$

[12] $(\Delta_2, \hat{\Delta}_2)$ is just Model 5 in Table 21 when $n = 2$, the period integral associated to Δ_2 is the theta correspondence of $\mathrm{GL}_2 \times \mathrm{GL}_2$ while the period integral associated to $\hat{\Delta}_2$ is the Rankin-Selberg integral of $\mathrm{GL}_2 \times \mathrm{GL}_2$

In particular, Conjecture 1.1 for $(\Delta_{glue}, \hat{\Delta}_{glue})$ will follow from Conjecture 1.1 for $(\Delta, \hat{\Delta})$, and $(\Delta_2, \hat{\Delta}_2)$. As Conjecture 1.1 is know for $(\Delta_2, \hat{\Delta}_2)$ (by the theory of Rankin-Selberg integral and the Rallis inner product formula), we know that Conjecture 1.1 for $(\Delta_{glue}, \hat{\Delta}_{glue})$ will follow from Conjecture 1.1 for $(\Delta, \hat{\Delta})$. This proves Theorem 1.15.

It remains to explain why the period integral associated to Δ_{glue} (resp. $\hat{\Delta}_{glue}$) is a combination of the period integrals associated to Δ, Δ_2 (resp. $\hat{\Delta}, \hat{\Delta}_2$). The argument is very similar to the previous case as well as the case of the Model 6.4, we will skip it here.

This completes the description of the dual BZSV quadruples associated to representations glued from anomaly-free representations in Table S of [28], as well as the proof of Theorem 1.15 for those cases.

It remains to consider the non anomaly-free representations in Table S of [28], which are (S.3) when n is odd, (S.8), (S.9), (S.13), (S.15) and (S.16). It is easy to see that if we glue the model (S.8), (S.13) or (S.15) with another model, then the representation we get is not anomaly-free. Hence we just need to consider (S.3) when n is odd, (S.9) and (S.16). There are 6 different cases.

If we glue (S.3) when n is odd with (S.9), we get the model (11.11) of Table 11 with $m = 1$, this has already been considered in Section 5. If we glue (S.9) with itself, the representation we get is just $T(std)$ of SL_2 which is model (S.10). For the remaining four cases, the generic stabilizer of the representation is not connected [13].

References

[1] D. Ben-Zvi, Y. Sakellaridis and A. Venkatesh, *Relative Langlands duality.* preprint

[2] R. Beuzart-Plessis, *A local trace formula for the Gan-Gross-Prasad conjecture for unitary groups: the archimedean case.* Astérisque no. 418 (2020).

[3] A. Braverman, M. Finkelberg, V. Ginzburg, and R. Travkin *Mirabolic Satake equivalence and supergroups.* Compos. Math.157(2021), no.8, 1724–1765.

[4] A. Braverman, M. Finkelberg, and R. Travkin *Gaiotto conjecture for* $\mathrm{Rep_q}(\mathrm{GL(N-1|N)})$. arXiv:2107.02653, 2021.

[5] D. Bump and S. Friedberg, The exterior square automorphic L-functions on $\mathrm{GL}(n)$. In *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II (Ramat Aviv, 1989)*, volume 3 of *Israel Math. Conf. Proc.*, pages 47–65. Weizmann, Jerusalem, 1990.

[6] D. Bump and D. Ginzburg, *Spin L-functions on symplectic groups.* Internat. Math. Res. Notices, (8):153–160, 1992.

[7] D. Bump and D. Ginzburg, *Spin L-functions on* GSp_8 *and* GSp_{10}. Trans. Amer. Math. Soc., 352(2):875–899, 2000.

[8] D. Bump and D. Ginzburg, *Symmetric square L-functions on* $\mathrm{GL}(r)$. Ann. of Math. (2), 136(1):137–205, 1992.

[9] Sanath Devalapurkar, *ku-theoretic spectral decompositions for spheres and projective spaces.* arXiv:2402.03995

[10] M. Finkelberg, R. Travkin, and R. Yang *Gaiotto Conjecture for* $\mathrm{Rep_q}(\mathrm{F(4)})$. arXiv:2406.05521, 2024.

[11] Tony Feng and Jonathan Wang. *Geometric Langlands duality for periods*, 2024. arXiv:2402.00180

[12] W. Gan, B. Gross and D. Prasad, *Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups.* Sur les conjectures de Gross et Prasad. I. Astérisque No. 346 (2012), 1–109. ISBN: 978-2-85629-348-5

[13] W. Gan, B. Gross and D. Prasad, *Branching laws for classical groups: the non-tempered case.* Compositio Mathematica. 2020;156(11):2298-2367

[14] D. Ginzburg, *On standard L-functions for* E_6 *and* E_7. J. Reine Angew. Math., 465:101-131, 1995.

[13] while in this paper we suggested the form of dual quadruples for some representations with non-connected stabilizer, it is not clear to us what the dual should be for the cases here

[15] D. Ginzburg, *On the standard L-function for G_2.* Duke Math. J., 69(2):315–333, 1993.
[16] D. Ginzburg, *On spin L-functions for orthogonal groups.* Duke Math. J., 77(3):753–798, 1995.
[17] D. Ginzburg and J. Hundley, *Multivariable Rankin-Selberg integrals for orthogonal groups.* Int. Math. Res. Not., (58):3097–3119, 2004.
[18] W. Gan and B. Jun, *Generalized Whittaker models as instances of relative Langlands duality.* arXiv:2309.08874
[19] W. T. Gan, Y. Qiu, and S. Takeda, *The regularized Siegel-Weil formula (the second term identity) and the Rallis inner product formula* Invent. Math. 198 (2014), no. 3, 739–831.
[20] S. Gelbart and H. Jacquet, *A relation between automorphic representations of GL(2) and GL(3),* Ann. Sci. Ecole Normale Sup., 4e serie, 11 (1978), 471-552.
[21] D. Ginzburg, D. Jiang, and S. Rallis, *Nonvanishing of the central critical value of the third symmetric power L-functions.* Forum Math., 13(1):109-132, 2001
[22] B. Gross, *On the motive of a reductive group,* Invent.Math.130(2), 287–313.
[23] D. Ginzburg, S. Rallis, D. Soudry, *L-functions for symplectic groups.* Bulletin de la S. M. F., tome 126, no 2 (1998), p. 181-244
[24] M. Harris and S. Kudla, *On a conjecture of Jacquet.* Contributions to automorphic forms, geometry, and number theory. Johns Hopkins Univ. Press, 2004, 355–371
[25] A. Ichino and T. Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross–Prasad conjecture.* Geometric and Functional Analysis 19 (2010), no. 5, 1378-1425.
[26] H. Jacquet, I. I. Piatetskii-Shapiro and J. A. Shalika, *Rankin-Selberg Convolutions*, American Journal of Mathematics Vol. 105, No. 2 (Apr., 1983), pp. 367-464.
[27] H. Jacquet and J. Shalika. Exterior square L-functions. In *Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988)*, volume 11 of *Perspect. Math.*, pages 143–226. Academic Press, Boston, MA, 1990.
[28] F. Knop, *Classification of multiplicity free symplectic representations.* Journal of Algebra Volume 301, Issue 2, 531-553.
[29] F. Knop and B. Schalke, *The dual group of a spherical variety.* Trans. Moscow Math. Soc. 2017, 187-216.
[30] S. Kudla and S. Rallis, *A regularized Siegel–Weil formula: the first term identity.* Ann. Math. 140, 1–80 (1994)
[31] E. Lapid and Z. Mao, *A conjecture on Whittaker–Fourier coefficients of cusp forms*, Journal of Number Theory 146, 448-505.
[32] J.-S. Li, *Nonvanishing theorems for the cohomology of certain arithmetic quotients* J. Reine Angew. Math. 428 (1992), 177–217.
[33] I. Losev, *Coisotropic representations of reductive groups.* Trudy Mosc. Mat. Ob-va, 66(2005), p. 157-181 (in Russian). English translation in: Trans. Moscow Math. Soc. (2005), 143-168.
[34] Z. Mao, C. Wan and L. Zhang, *BZSV duality for some strongly tempered spherical varieties.* arXiv:2310.17837
[35] S. J. Patterson and I. I. Piatetski-Shapiro, *The symmetric-square L-function attached to a cuspidal automorphic representation of* GL_3. Math. Ann., 283(4):551–572, 1989.
[36] A. Pollack and S. Shah, *Multivariate Rankin-Selberg integrals on* GL_4 *and* $\mathrm{GU}_{2,2}$. Canadian Mathematical Bulletin, Vol. 61 (4), 2018, 822-835
[37] Y. Sakellaridis, *Functorial transfer between relative trace formulas in rank one.* Duke Math. J., 170(2):279-364, 2021.
[38] Y. Sakellaridis, *Spherical functions on spherical varieties.* Amer. J. Math., 135(5):1291-1381, 2013.
[39] Y. Sakellaridis and A. Venkatesh, *Periods and harmonic analysis on spherical varieties*, Astérisque (2017), no. 396, viii+360.
[40] S. Takeda, *The twisted symmetric square L-function of* $\mathrm{GL}(r)$. Duke Math. J., 163(1):175–266, 2014.
[41] R. Travkin and R. Yang *Twisted Gaiotto equivalence for* GL(M|N). arXiv:2306.09556, 2023
[42] J.-L. Waldspurger, *Une formule intégrale reliée à la conjecture locale de Gross-Prasad, 2e partie : extension aux représentations tempérées.* in "Sur les conjectures de Gross et Prasad. I" Astérisque No. 346 (2012), 171-312
[43] C. Wan, *Multiplicity One Theorem for the Ginzburg-Rallis Model: the tempered case.* Trans. Amer. Math. Soc. 371 (2019), 7949-7994.

[44] C. Wan and L. Zhang, *Periods of automorphic forms associated to strongly tempered spherical varieties.* Accepted by Memoir of AMS. arxiv 2102.03695, 109 pages.

[45] S. Yamana, *L-functions and theta correspondence for classical groups.* Invent. Math. 196 (2014), 651–732.

Department of Mathematics & Computer Science, Rutgers University – Newark, Newark, NJ 07102, USA
Email address: zmao@rutgers.edu

Department of Mathematics & Computer Science, Rutgers University – Newark, Newark, NJ 07102, USA
Email address: chen.wan@rutgers.edu

Department of Mathematics, National University of Singapore, Singapore
Email address: matzhlei@nus.edu.sg