A tight bound on approximating arbitrary metrics by tree metric

Reference :[FRT2004]

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A simple but fundamental problem

Given : an undirected graph $G = (V, E)$.

compute a spanning tree $T \subseteq E$ which serves as *approximate* shortest path tree for all vertices.

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Is it possible ??

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Not possible ??

Counterexample : when G is a cycle

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Given : an undirected graph $G = (V, E)$.

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Known Results :

1. Elkin et al. [STOC 2005] : $O(\log^2 n \log \log n))$
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Lower bound : $\Omega(\log n)$

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we remove the restriction for T to be a subgraph of G

What is a metric ?

A metric (space) is an ordered pair (V, d) where $d : V \times V \to \mathcal{R}$ such that

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ if and only if $x = y$.
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(y, z)$

A useful view : a metric (V, d) as a complete graph with length of edge (u, v) equal to $d(u, v)$.

Tree Metric

A tree spanning the points V where the distance d between any two vertices is defined by the length of the path between them in the tree.

When does one metric dominate another metric ?

metric (V', d') is said to dominate another metric (V, d) if

1. $V \subseteq V'$
2. $d'(u, v) \geq d(u, v)$ for each $u, v \in V$

Ideally we would like $d'(u, v) \leq \alpha d(u, v)$.

α-probabilistically approximation for metric (V, d)

Let $\mathcal{S}$ be a collection of tree metrics over V and $\mathcal{D}$ be a probability distribution over them. Then $(\mathcal{S}, \mathcal{D})$ is said to α-probabilistically approximate (V, d) if

1. Each metric in $\mathcal{S}$ dominates (V, d)
2. $\mathbf{E}_{d' \in (\mathcal{S}, \mathcal{D})}[d'(u, v)] \leq \alpha d(u, v)$

α is usually called the distortion/stretch

Result [FRT 2004]

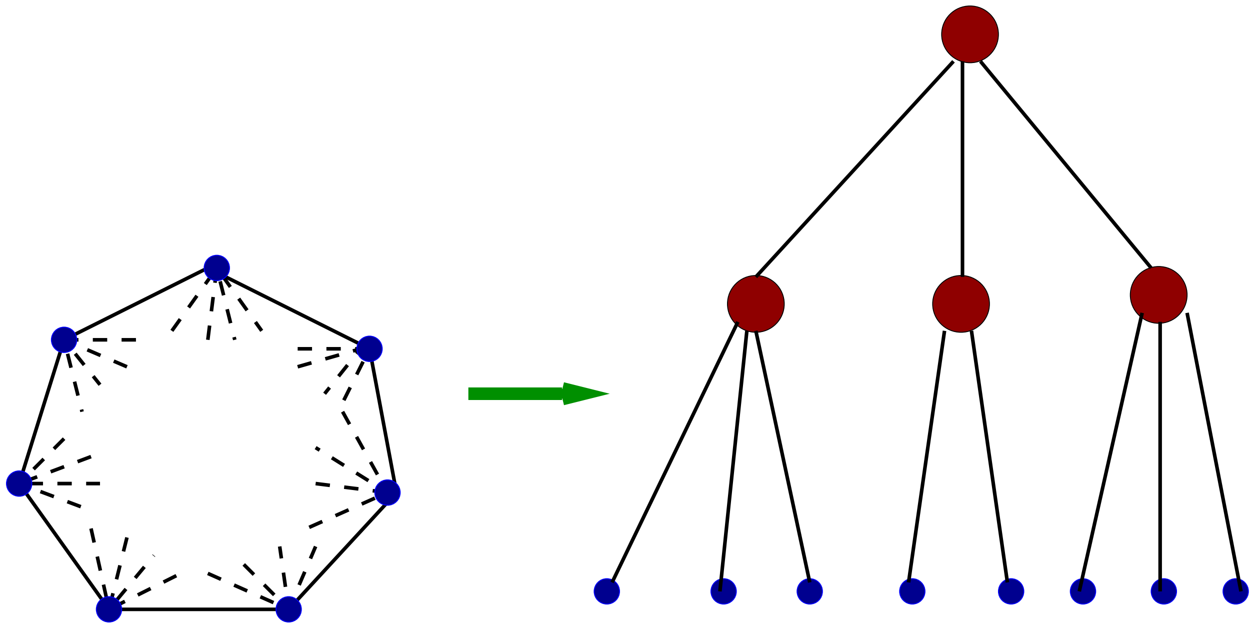
Given any metric (V, d), there exists a distribution over tree metrics which approximates (V, d) probabilistically with distortion $O(\log n)$.

Outline of the algorithm

1. A deterministic construction of a tree metric which dominates (V, d).
2. Randomization is added to ensure expected stretch $O(\log n)$ for each edge.

Construction of a tree metric which dominates (V, d)

Let smallest distance =1, Let the diameter Δ is equal to 2^δ for some $\delta > 0$.



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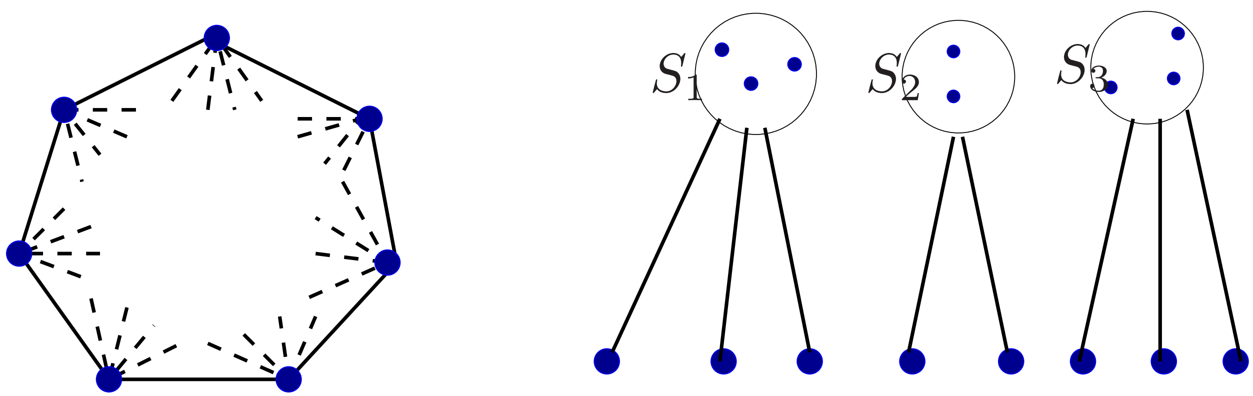

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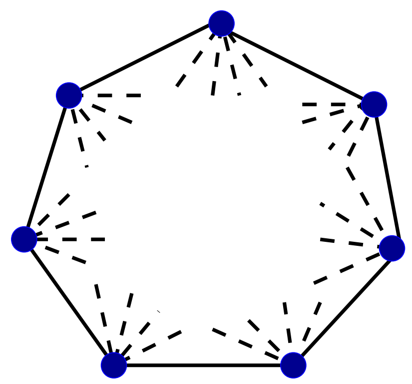

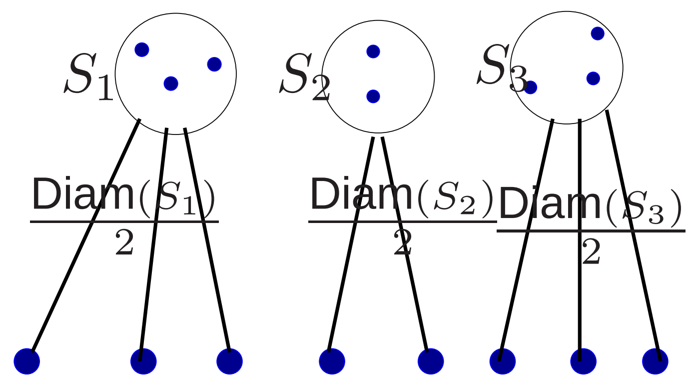

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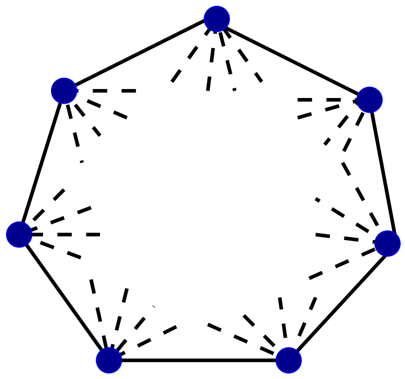
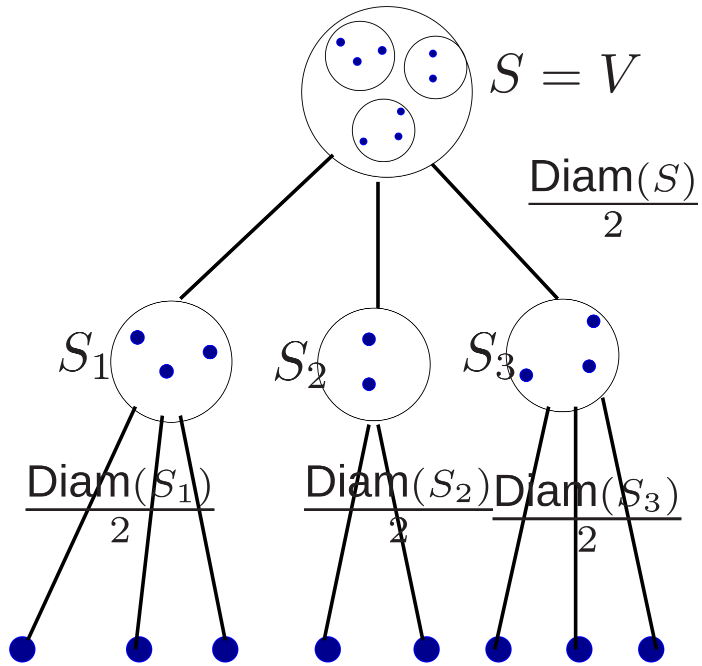
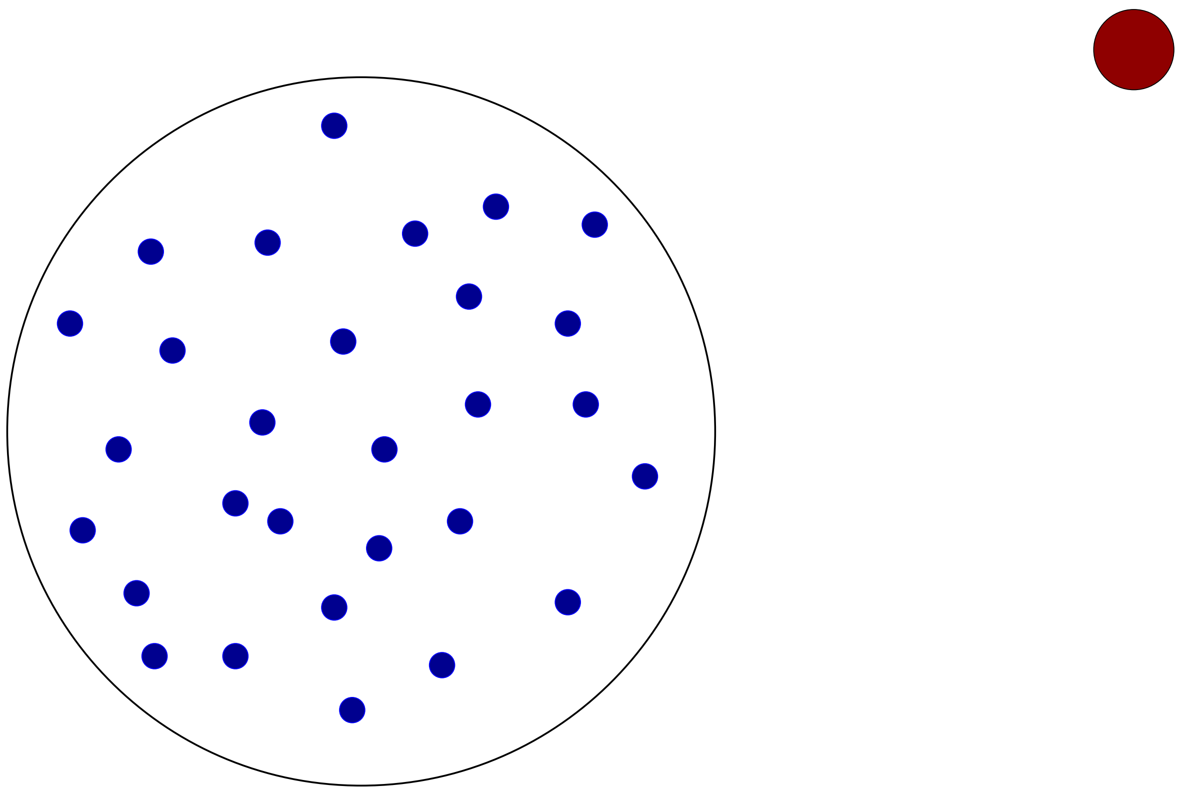




Construction of level $\delta - 1$

$\pi = x_1, x_2, ...x_n$



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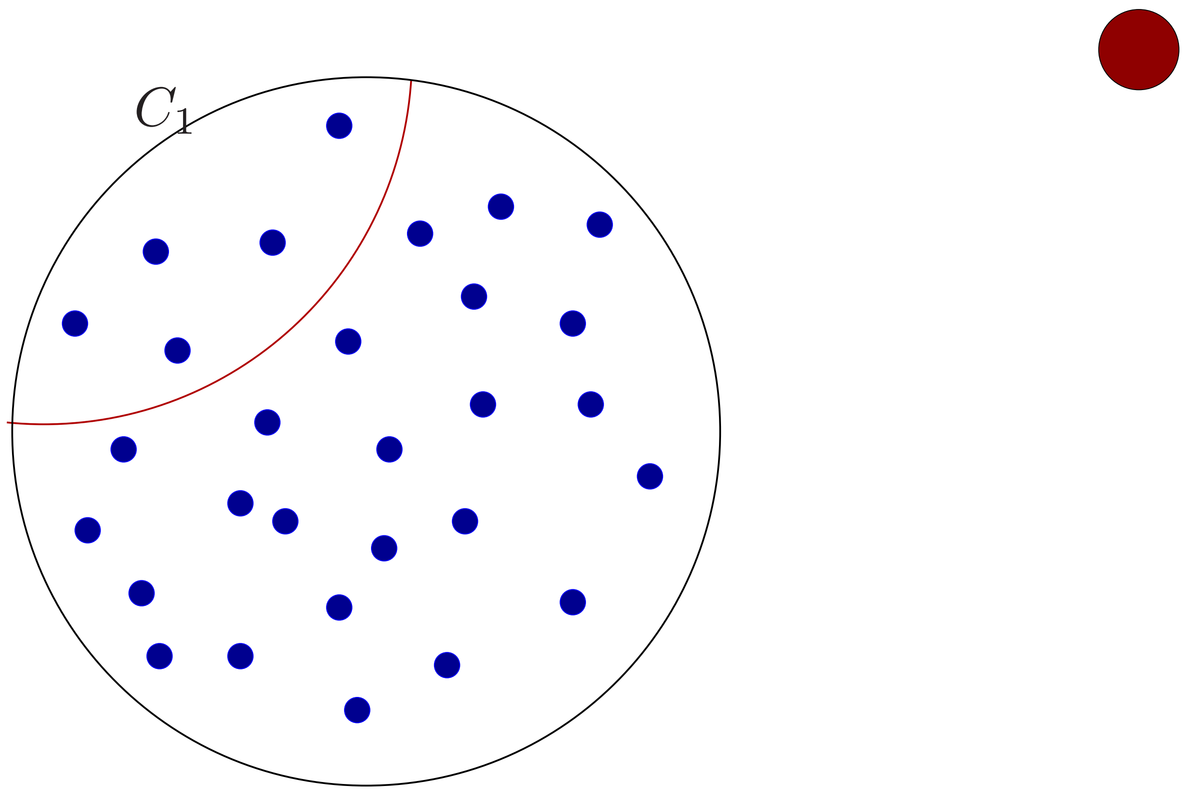

C_1 : vertices within distance $2^{\delta-2}$ from x_1

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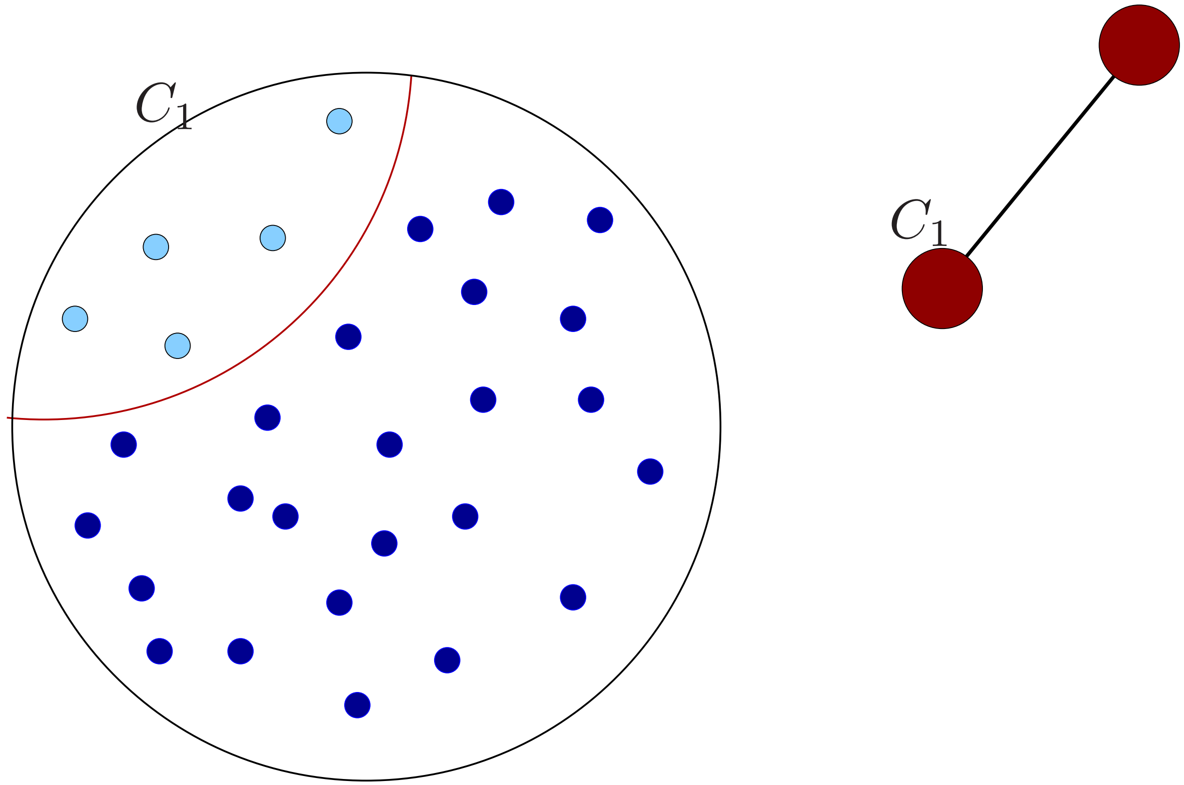
$\pi = \mathbf{x_1}, x_2, ...x_n$



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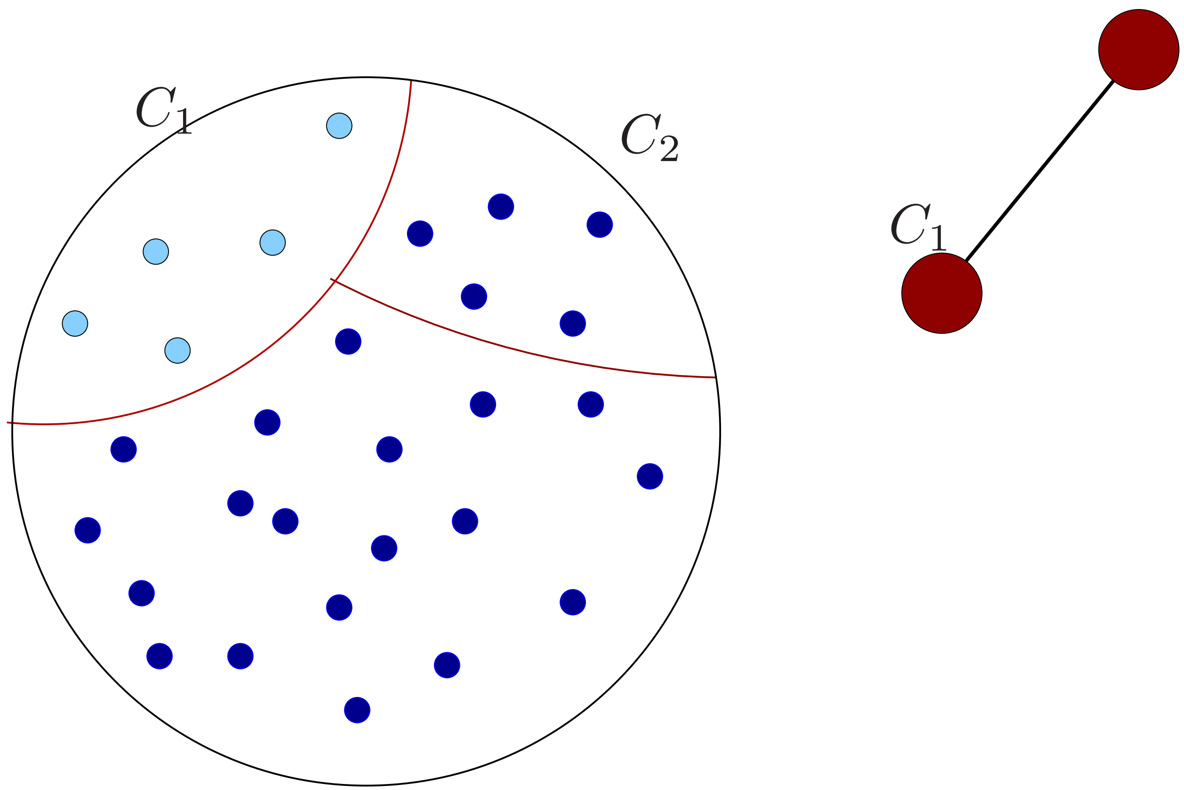
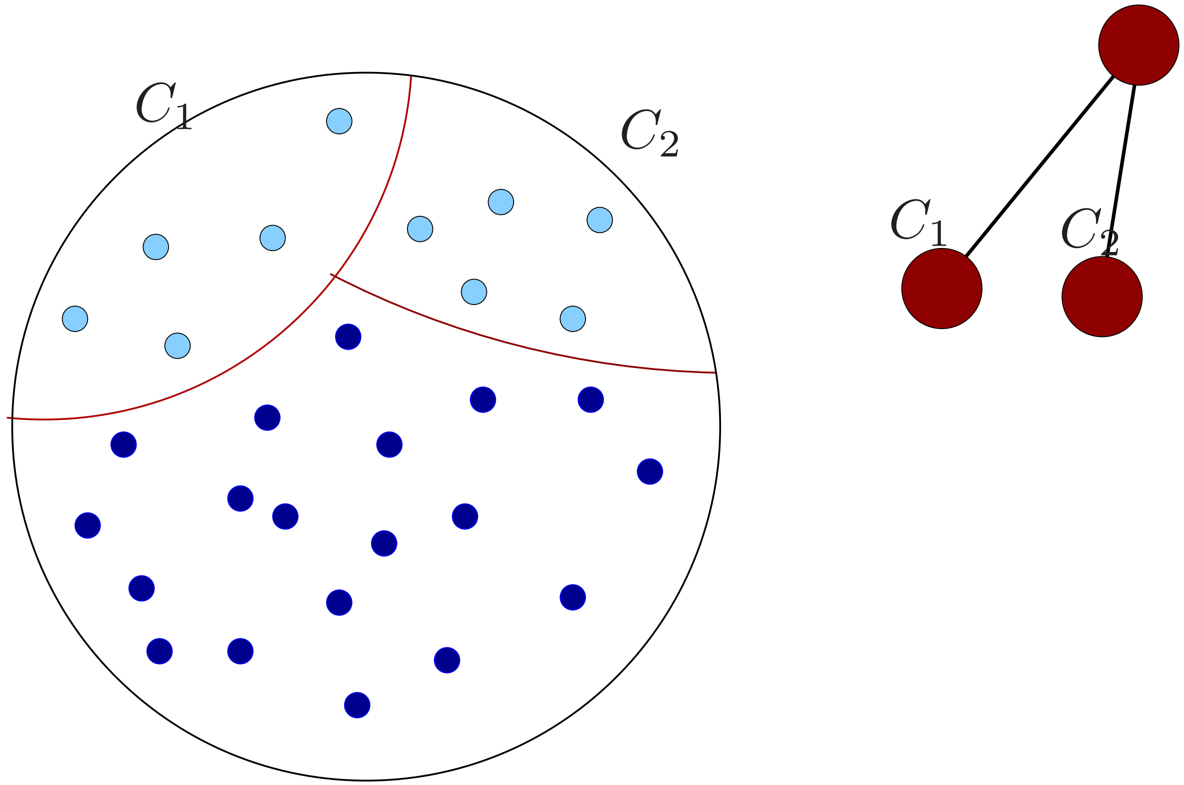


C_2 : vertices within distance $2^{\delta-2}$ from x_2

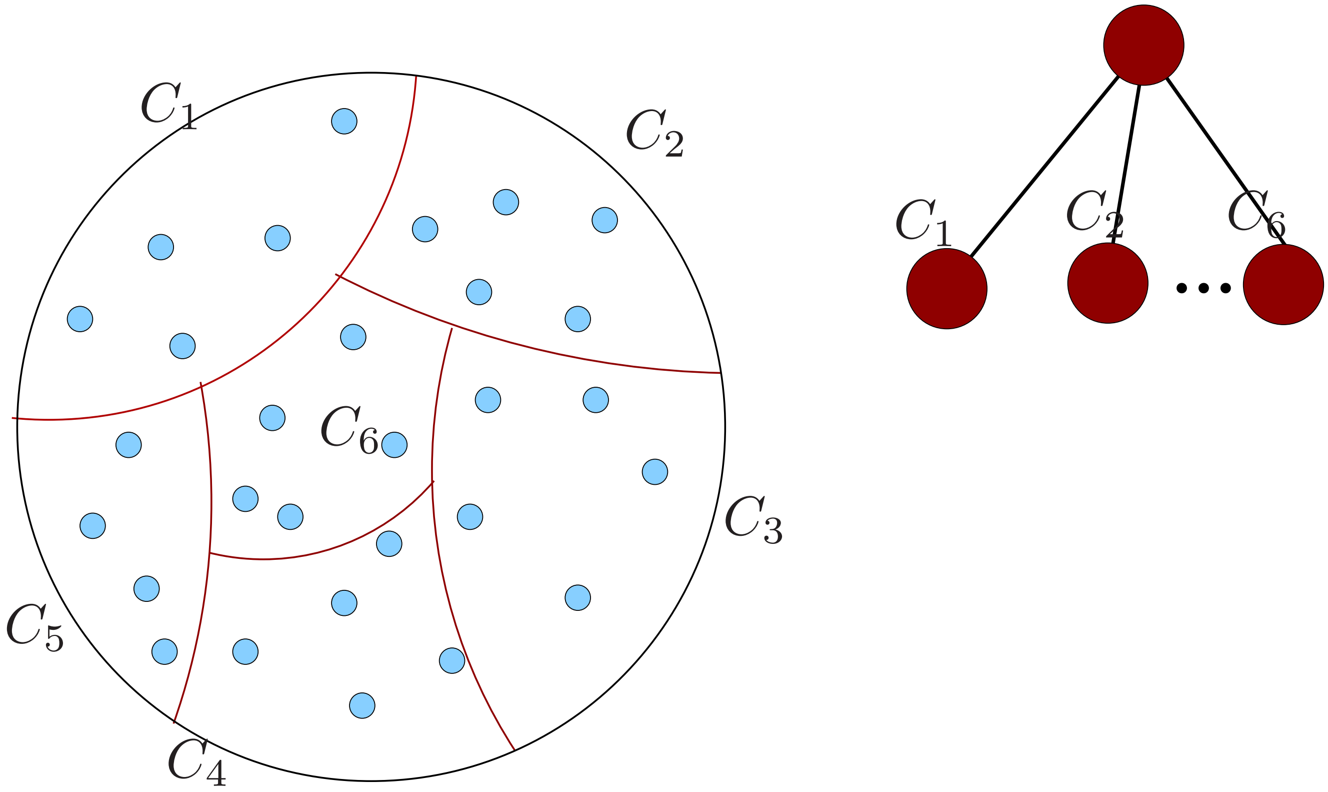
Construction of level $\delta - 1$

$\pi = x_1, \mathbf{x_2}, ..., x_n$



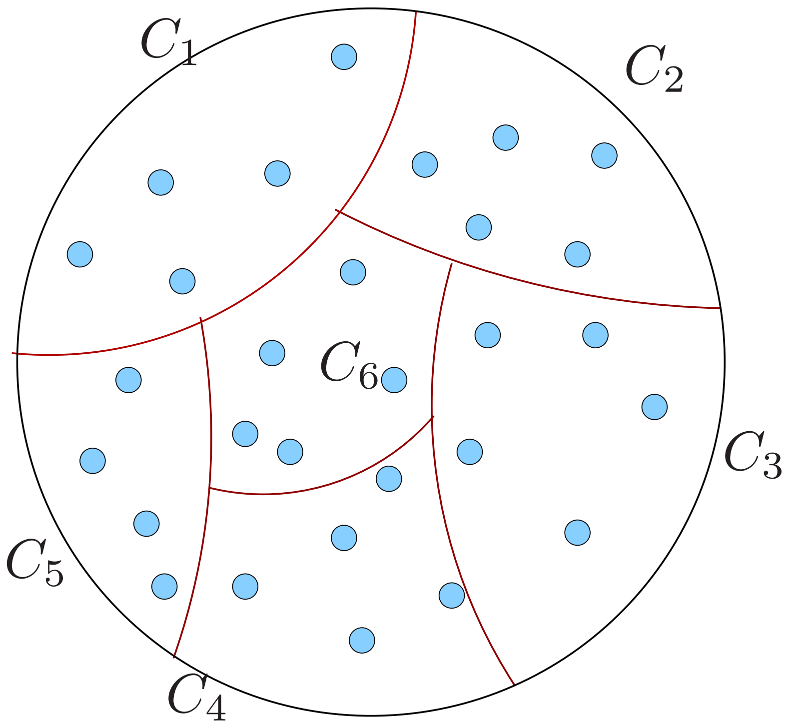
Construction of level $\delta - 1$

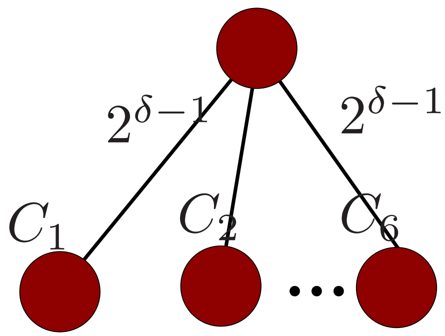
$\pi = x_1, x_2, ..., \mathbf{x_n}$



Construction of level $\delta - 1$ is complete now

$\pi = x_1, x_2, ..., x_n$





Construction of level $\delta - 1$

The highest level is δ, and consists of cluster $S = \{V\}$.

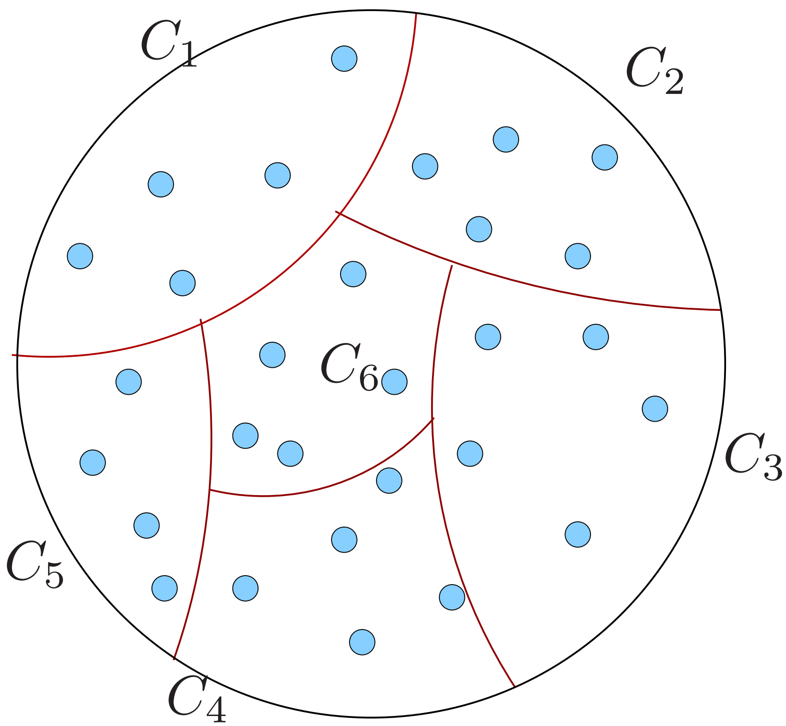
$\pi = x_1, x_2, ..., x_n$.

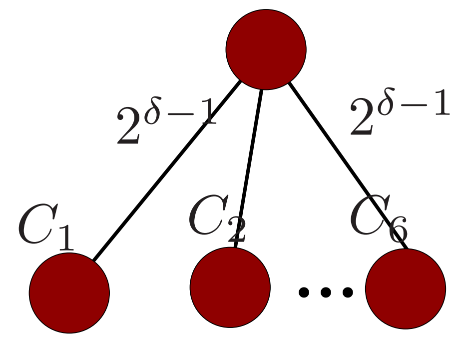
For $j = 1$ to n do

1. Create a new cluster consisting of all unassigned vertices of S which are at distance $\leq 2^{\delta-2}$ from vertex x_j.
2. Assign length $2^{\delta-1}$ to the edge.

Construction of level $\delta - 2$

$\pi = x_1, x_2, ..., x_n$





Construction of level $\delta - 2$

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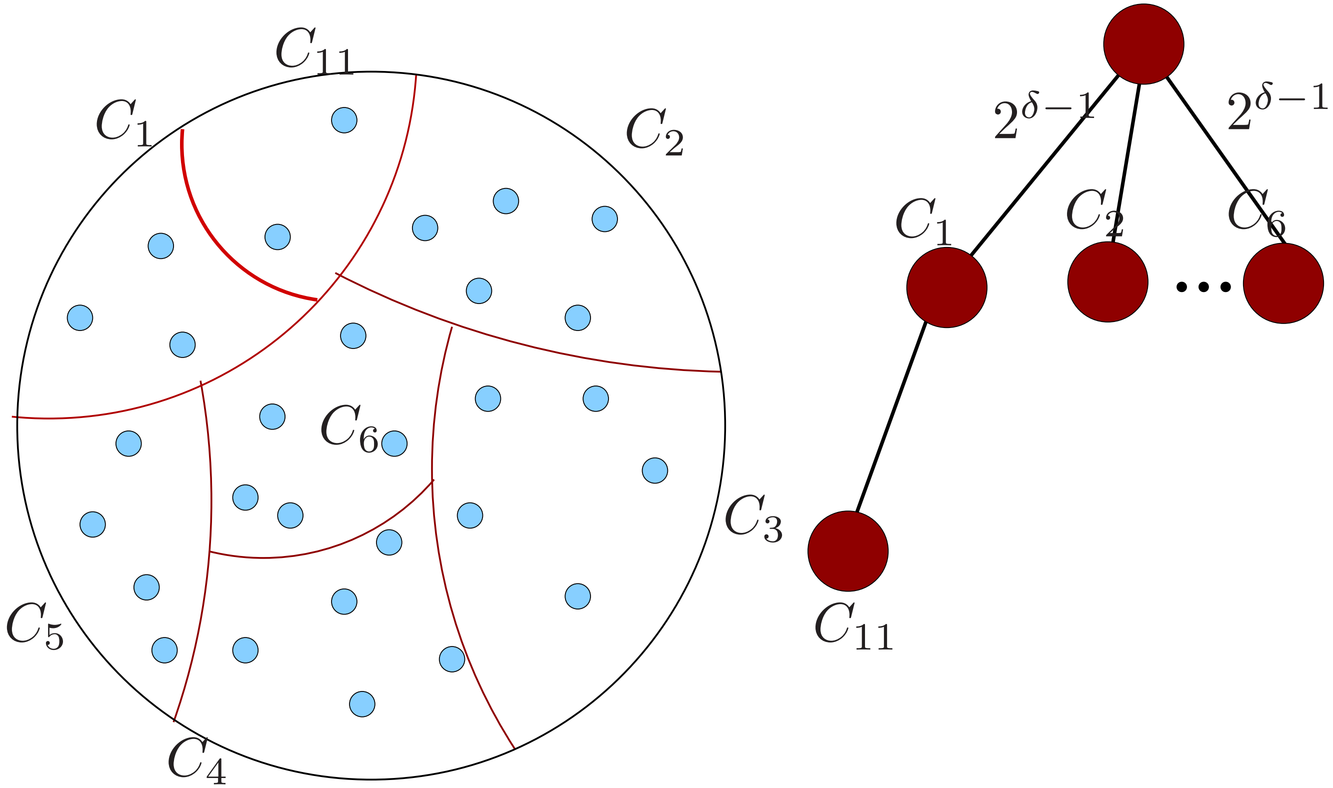
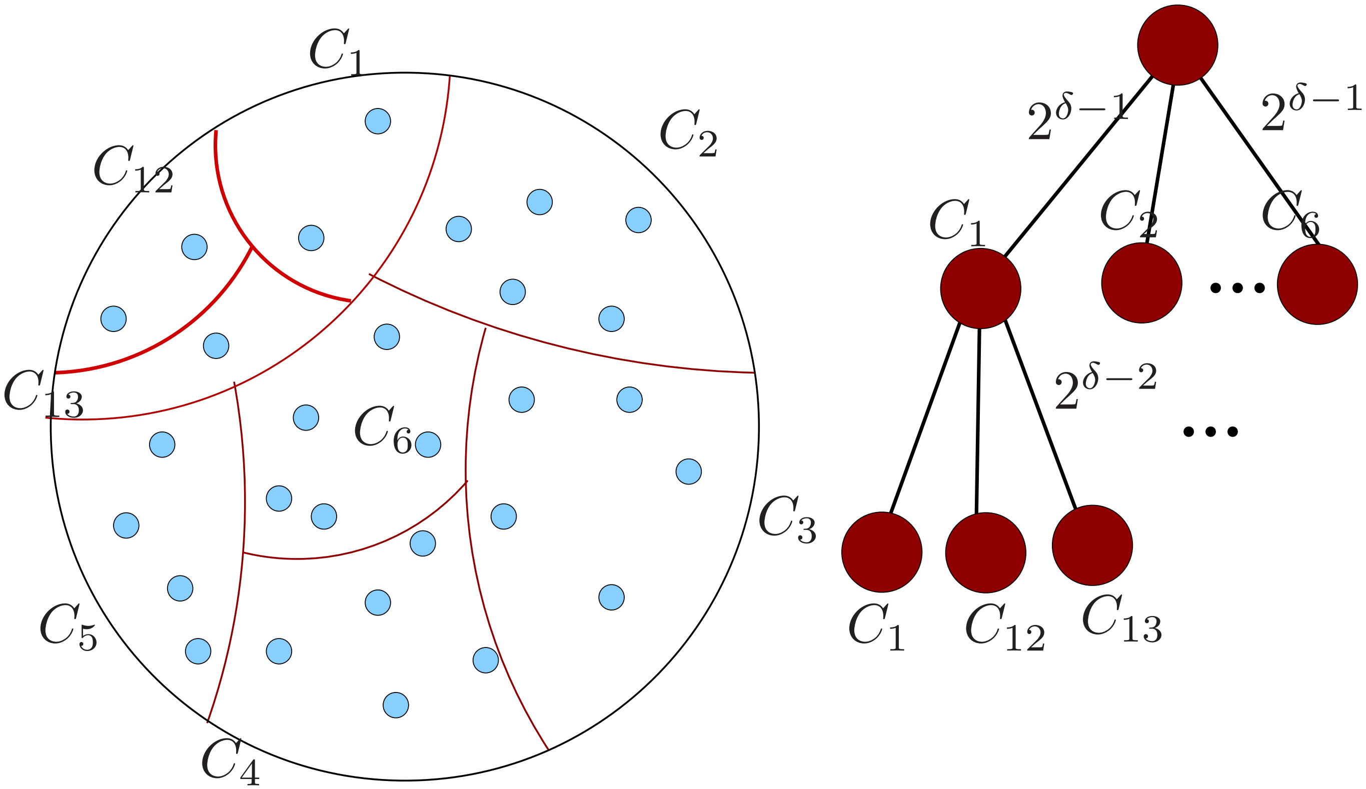


C_{11} : vertices of C_1 within distance $2^{\delta-3}$ from x_1

Construction of level $\delta - 2$

$$\pi = x_1, x_2, \ldots x_n$$



Deterministic Construction of tree metric : Top Down approach

The highest level is δ, and consists of cluster $S = \{V\}$.

$i \leftarrow \delta - 1;$

$\pi = x_1, x_2, ..., x_n;$

While ($i >= 0$) do $\{$

$\beta_i \leftarrow 2^{i-1};$

For each cluster S at level $i + 1$ do

For $j = 1$ to n do

1. Create a new cluster consisting of all unassigned vertices of S which are with in distance β_i from vertex x_j.

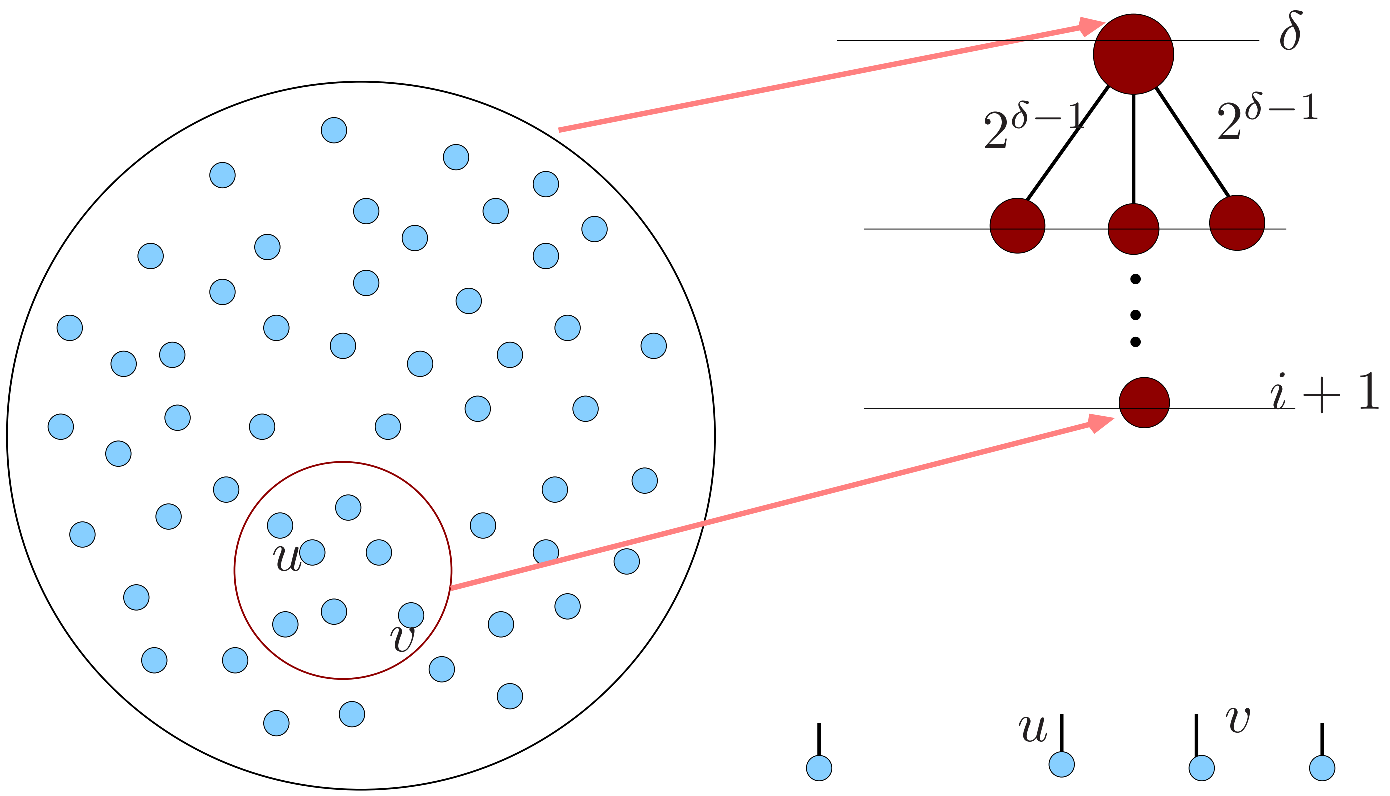
2. Assign length $2\beta_i$ to the edge between the node for S and the child node corresponding to the new cluster.

This defines the level i of the tree.

$i \leftarrow i - 1 \}$

Analysing the algorithm from perspective of an edge (u, v)

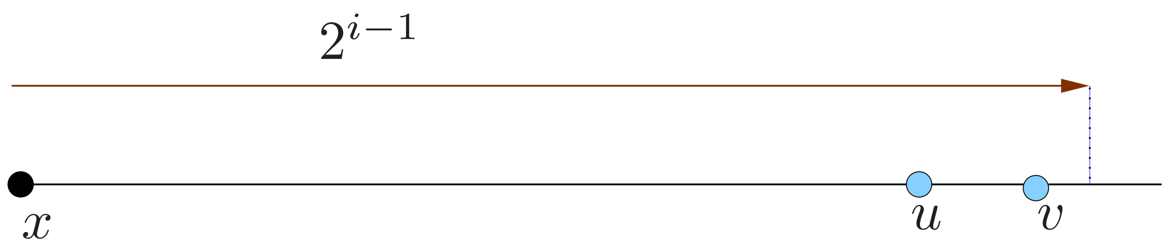
$\pi = x_1, x_2, ..., x_n$



only vertices within distance 2^{i-1} from (u, v) matter

Analysing the algorithm from perspective of an edge (u, v)

Case 1



Case 1 : ???

Analysing the algorithm from perspective of an edge (u, v)

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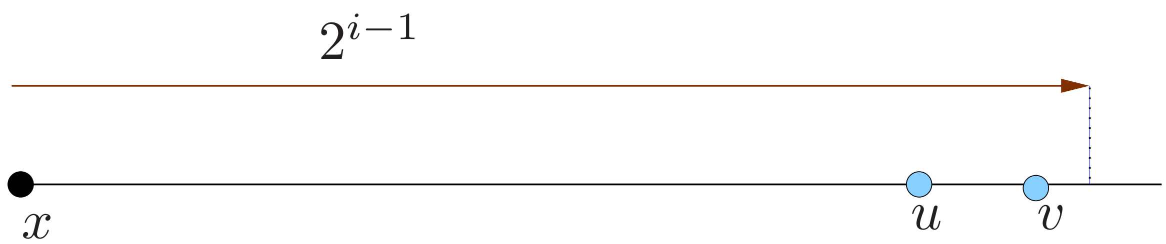
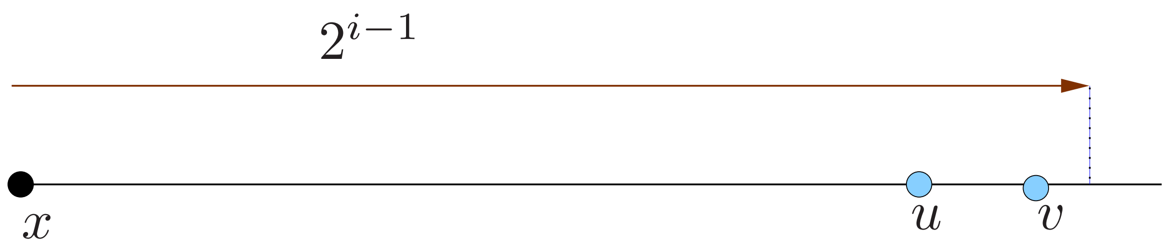


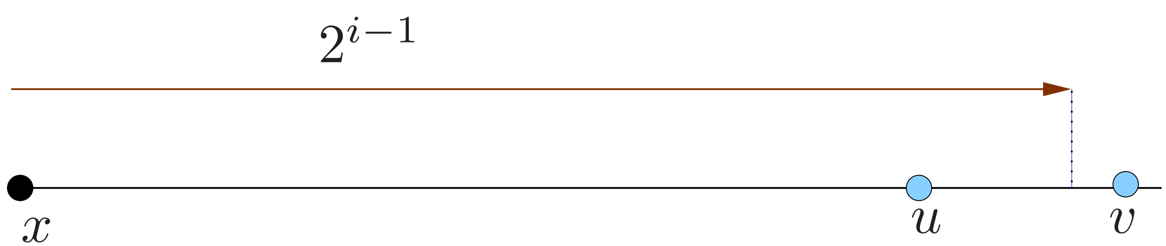
Case 1 : the edge (u, v) is **retained** at level i as well !

Analysing the algorithm from perspective of an edge (u, v)

Case 1



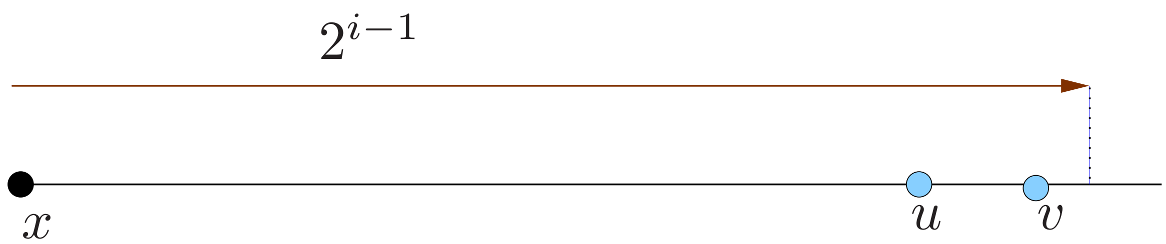
Case 2



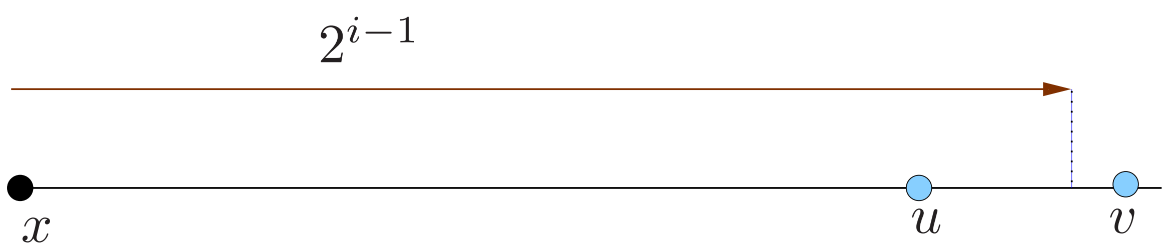
Case 2 : ???

Analysing the algorithm from perspective of an edge (u, v)

Case 1



Case 2

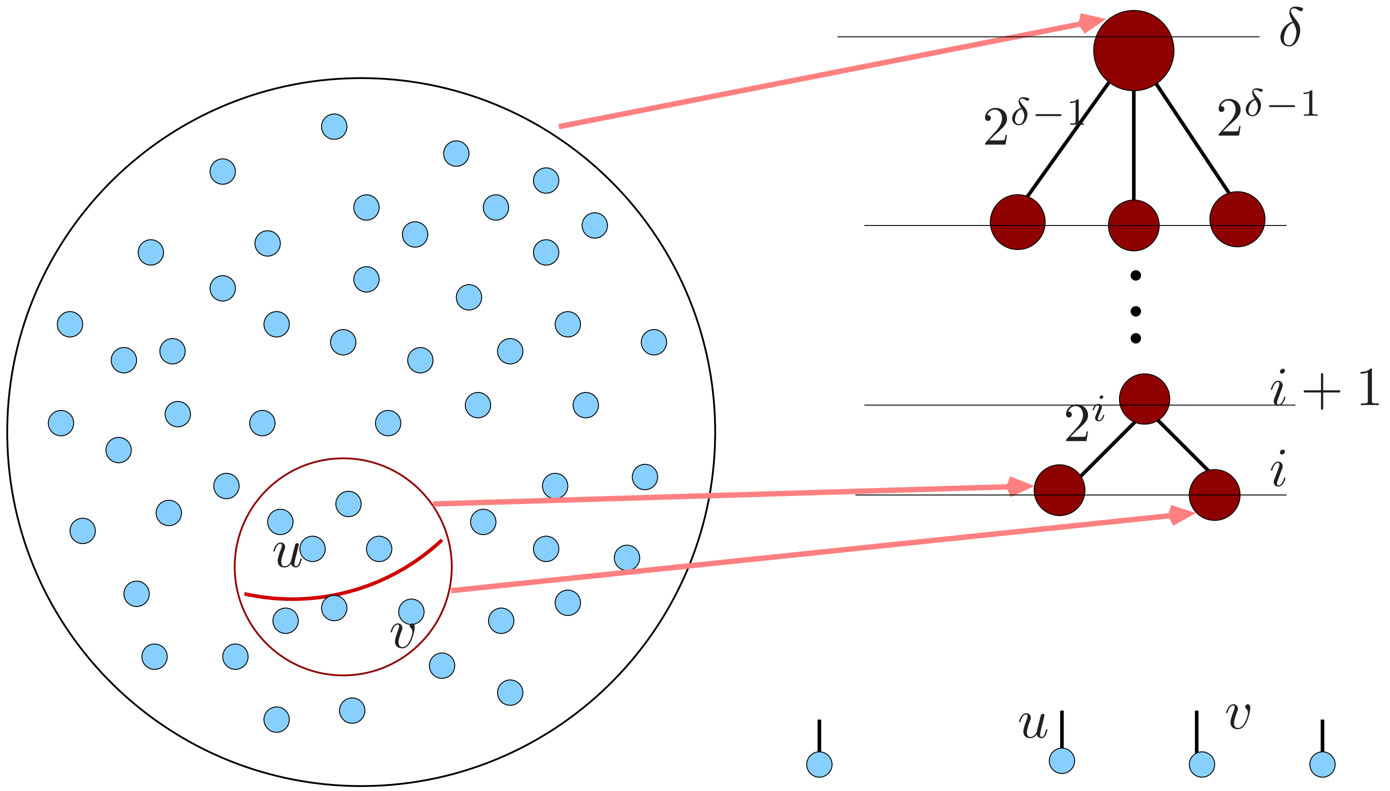


Case 2 : the edge (u, v) is **cut** at level i !!

Analysing the algorithm from perspective of an edge (u, v)

Case 2 : Careful look

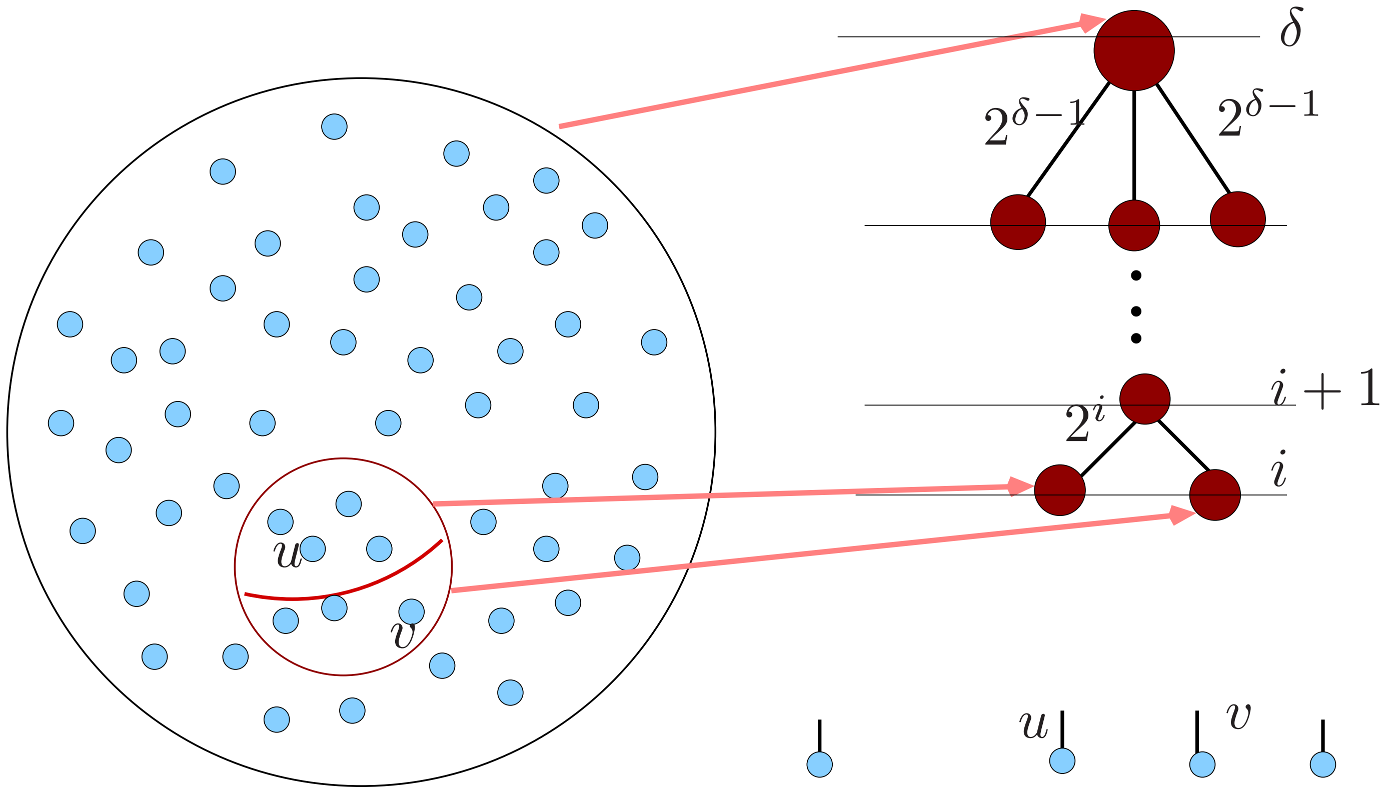
$$\pi = x_1, x_2, ..., x_n$$



Analysing the algorithm from perspective of an edge (u, v)

Case 2 : what is $d_T(u,v)$ **= ???**

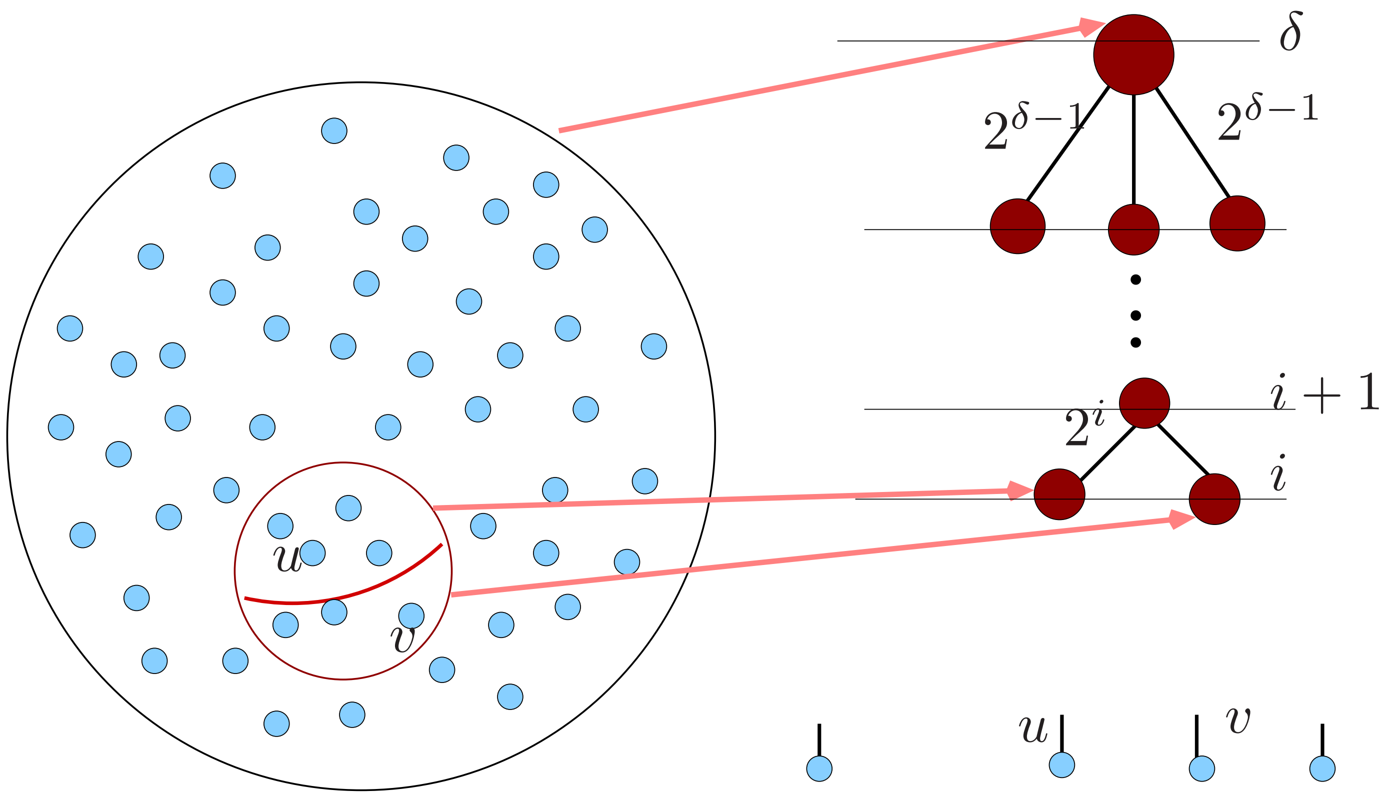
$\pi = x_1, x_2, ..., x_n$



Analysing the algorithm from perspective of an edge (u, v)

Case 2 : $d_T(u, v) = 2\sum_{j \le i} 2^j = 2^{i+2}$

$\pi = x_1, x_2, ..., x_n$



Some more Observations

1. In the tree T, exactly one vertex cuts the edge (u, v).
2. If a vertex cuts the edge (u, v) at level i, then $d_T(u, v) \leq 2^{i+2}$.
3. A vertex w has *potential* to cut (u, v) if and only if $\exists j$, $d(w, u) \leq 2^j < d(w, v)$.

What can be $\frac{d_T(u,v)}{d(u,v)}$?

What can be $\frac{d_T(u,v)}{d(u,v)}$?

huge if $d(u,v) <<< 2^{i+2}$

Part II : adding some randomization to the construction

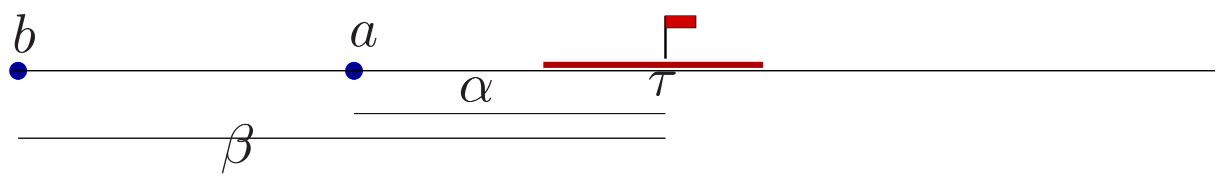
Main obstacle :

How to ensure that the vertices which are nearer to (u, v) have higher chances to cut the edge.

A small probability exercise

There are two persons a and b aiming to shoot an arrow at a target which is a red strip of length τ. An arrow is said to shoot the target if it hits anywehre within the red strip. They aim at the center of the strip. Both of them are sharp shooters so that if everything goes fine, they will hit the center of the strip. However, due to randomness of wind speed, the arrow may miss its target. Assume that if the actual target is at distance x from the point from where it is shot, then it may land any where with in the interval $[x, 2x]$ from the point where it was shot. If a and b are at distance α and β respectively from the center of thetarget strip, then who is more likely to shoot the target. (see the following slides for more details)

A small probability exercise



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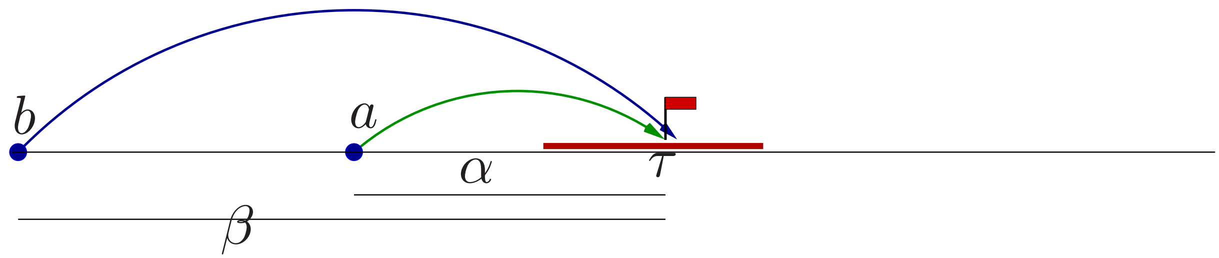

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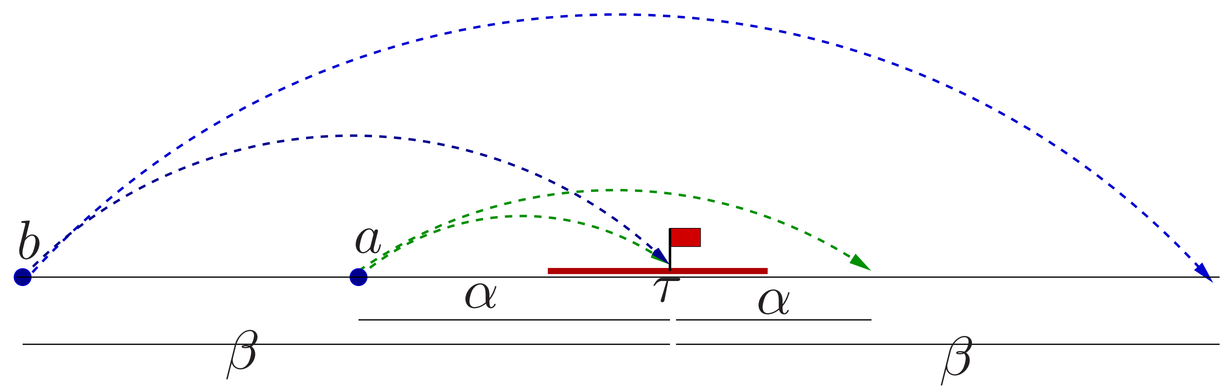
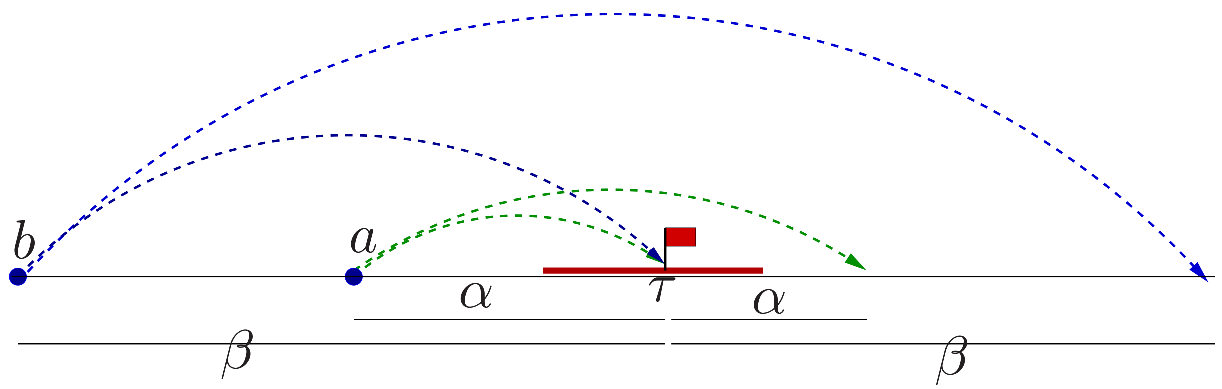


A small probability exercise

who has more chances to hit the target ?



A small probability exercise

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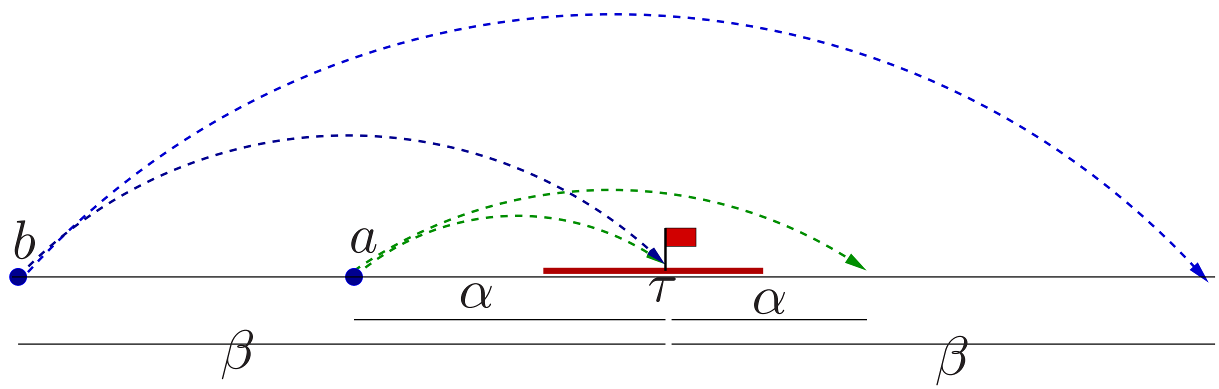

person closer to the target is more likely to hit the target

The randomized construction

1. Generate the permutation π uniformly randomly
2. Select a random number y uniformly randomly in the interval $[1, 2]$ and replace β_i by $2^{i-1}y$.

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Distribution of β_i ???

The randomized construction

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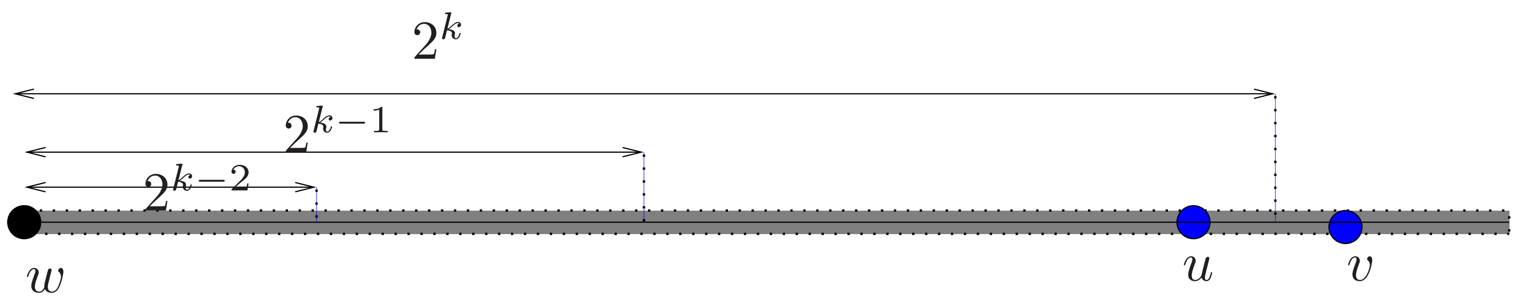
2. Select a random number y uniformly randomly in the interval $[1, 2]$ and replace β_i by $2^{i-1}y$.

Distribution of β_i : uniform in $[2^{i-1}, 2^i]$

Observations after randomization

1. In a tree T computed, exactly one vertex cuts the edge (u, v).
2. If a vertex cuts the edge (u, v) at level i, then $d_T(u, v) \leq 2^{i+3}$.
3. For all trees in $(\mathcal{S}, \mathcal{D})$, there exists only two possible levels at which a vertex can potentially cut the edge.

Analysis



Every vertex has potential to cut an edge (u, v)!!

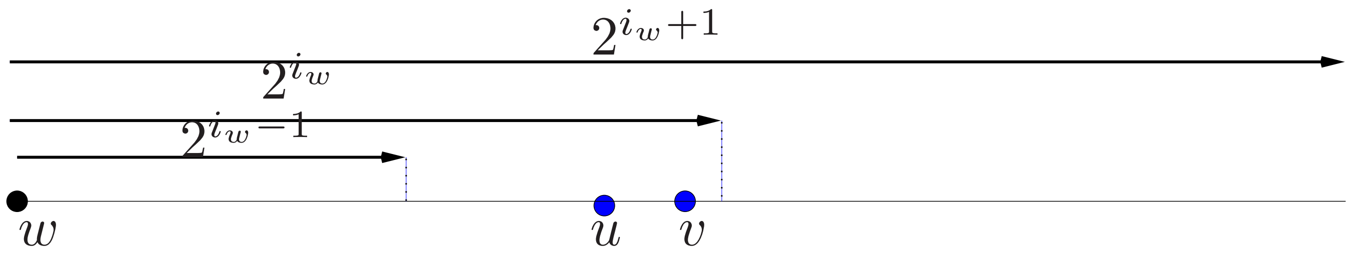
Analysis for an edge (u, v) and a vertex w

Let X_w be the indicator random variable which takes value 1 if vertex w cuts the edge (u, v) in T for $T \in (\mathcal{S}, \mathcal{D})$.

i_w : one of the two levels at which w can cut (u, v).

What is $\mathbf{P}[X_w = 1]$?

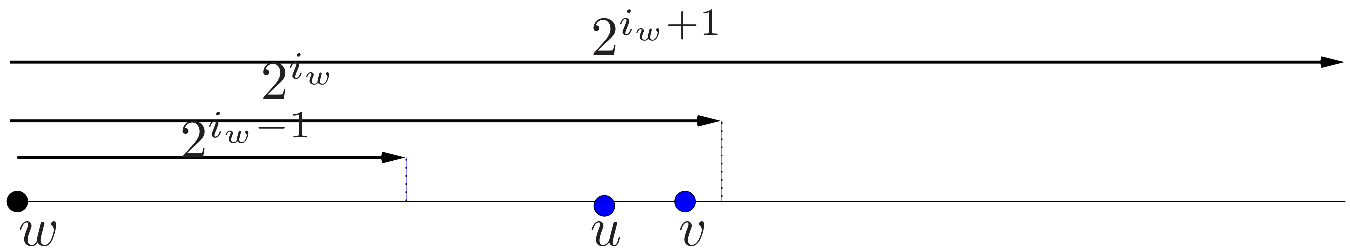
The necessary condition for w to cut (u, v) at level i_w ?



1. ???

2.

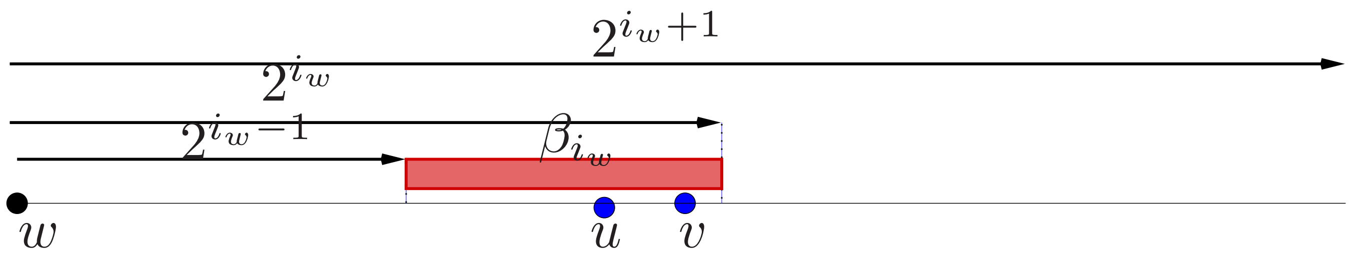
The necessary conditions for w to cut (u, v) at level i_w ?



1. $d(w, u) \leq \beta_{i_w} < d(w, v)$

2.

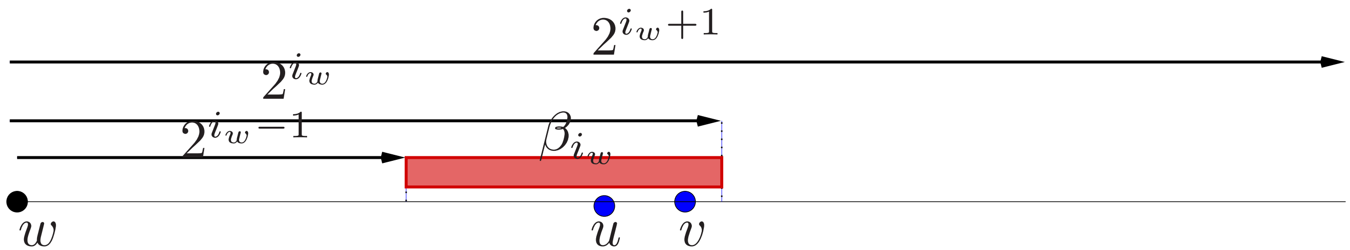
The necessary conditions for w to cut (u, v) at level i_w ?



1. $d(w, u) \leq \beta_{i_w} < d(w, v)$: **Probability =** ???

2.

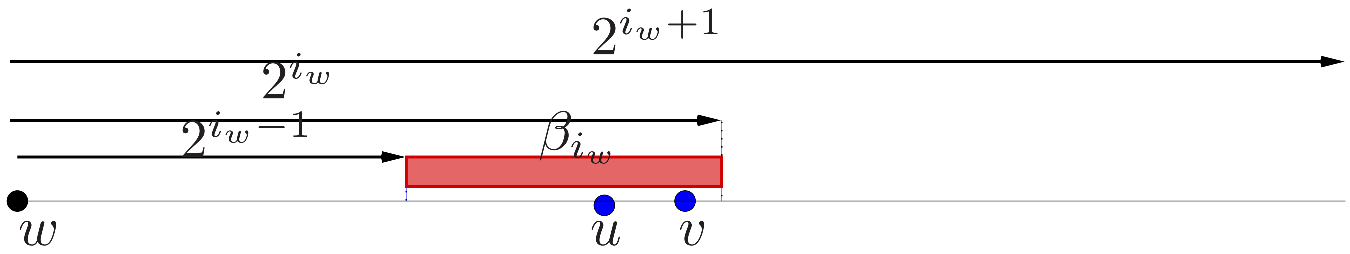
The necessary conditions for w to cut (u, v) at level i_w ?



1. $d(w, u) \leq \beta_{i_w} < d(w, v)$: **Probability =** $\frac{d(w,v)-d(w,u)}{2^{i_w-1}}$

2.

The necessary conditions for w to cut (u, v) at level i_w ?



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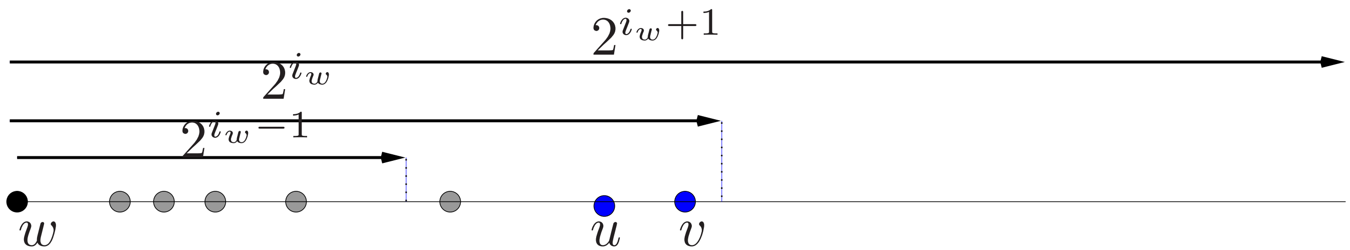
2. ???

The necessary conditions for w to cut (u, v) at level i_w ?



1. $d(w, u) \leq \beta_{i_w} < d(w, v)$: **Probability =** $\frac{d(w,v)-d(w,u)}{2^{i_w-1}}$

2. w must precede all vertices closer to (u, v) in π

If w is sth vertex closest to edge (u, v)

$$\mathbf{P}[X_w = 1] \quad \leq \quad \frac{1}{s} \frac{d(w,v) - d(w,u)}{2^{i_w - 1}}$$

If w is sth vertex closest to edge (u, v)

$$\begin{aligned}\mathbf{P}[X_w = 1] &\leq \frac{1}{s}\frac{d(w,v) - d(w,u)}{2^{i_w - 1}} \\ &\leq \frac{1}{s}\frac{d(u,v)}{2^{i_w - 1}}\end{aligned}$$

What is expected value of $d_T(u,v), T \in (\mathcal{S},\mathcal{D})$?

$$\mathbf{E}[d_T(u,v)] \quad = \quad \sum_w \mathbf{P}[X_w = 1]\, 2^{i_w+3}$$

What is expected value of $d_T(u,v), T \in (\mathcal{S},\mathcal{D})$?

$$
\begin{aligned}
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Derandomization

$$\text{average stretch} = \frac{2}{n(n-1)} \sum_{(u,v)} \frac{d_T(u,v)}{d(u,v)}$$

To compute a tree metric T which achieves average stretch $O(\log n)$.

What if we want a $T \subseteq G$ to achieve it ?

Open problem : A simpler and/or tight (randomized) construction ?