Questions on the Book "Gödel's Proof"

Jian Yang, MSIS, Rutgers Business School

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I have a few questions regarding Nagel and Newman's [1] rendition of Gödel's famous paper. Hope you would help me answer them. Thank you.

On p.84–85 of Nagel and Newman [1], $\mathrm{Dem}(x, z)$ is defined as the formula within PM expressing the meta-mathematical demonstration that the sequence of formulas with Gödel number x constitutes a proof in PM of the formula with Gödel number z. My understanding is that $\mathrm{Dem}(x, z)$ is specific to the **numbers** x and z; so even just x as a number changes while z stays intact, $\mathrm{Dem}(x, z)$ could be vastly different. Basically, to demonstrate that an alternative proof proves the same conclusion may require a different demonstration. Also, I could foresee that for one conclusion with Gödel number z one particular axiom may need to be invoked, while for another, that axiom may not be needed. Thus the following question.

Question 1 *I am totally fine with Dem(x, z) for a particular pair of numbers x and z. However, Is Dem$(\cdot, \cdot)$, which is required to adapt to different numbers x and z, well defined? For instance, is it "**universal**" enough just to anticipate the changes in string lengths of the formulas corresponding to x and z?*

On p.87–89, $\mathrm{sub}(x, 17, x)$ is defined as the Gödel **number** of the formula obtained by taking the formula with Gödel number x and, wherever there are occurrences of the symbol 'y' in that formula, replacing them by the numeral for x. On p.90, the formal **expression** inside PM that mirrors the Gödel sub function is designated as $\mathrm{Sub}(x, 16, x)$. Again, these seem to have been defined for particular **numbers** x rather than numerical **variables** x and y. Formula (1) of p.96 on the other hand states

$$\neg(\exists x)\mathrm{Dem}(x, \mathrm{Sub}(y, 17, y)).$$

Question 2 *This is related to Question 1. If Dem$(\cdot, z)$ has not been defined "universally", I cannot see how (1) can be given in PM and be associated with a Gödel number n. For a given number z, I understand Dem$(13, z)$ = Dem$(`x', z)$ has a Gödel number; I understand*

Dem$(8000000, z)$ *has a likely* **different** *Gödel number. So isn't Dem*(x, z)*'s Gödel number a function of* x *the number? If so, for* $\neg(\exists x)$*Dem*(x, z) *to have* **a** *Gödel number, shouldn't* x *there be understood as just the symbol '*x*'? But this does not seem satisfying either.*

Question 3 *Is Sub in (1) a typo? Should it be sub because what Dem*(x, z) *needs in* z *is a number rather than an expression. The meaning will certainly still be associated with the expression whose Gödel number is* z*.*

Question 4 *Is* y *in (1) a number/numeral or rather the symbol '*y*'? If the latter, Isn't Sub*$(y, 17, y)$ *merely '*$ss \cdots s0$*' with 17 s's?*

According to the text on p.96, it appears as if the y stands for an arbitrary number; however, a formula to appear on p.97 seems to require y to be treated as just the symbol 'y'. Now, for the (G) defined on p.97 as

$$\neg(\exists x)\mathrm{Dem}(x, \mathrm{Sub}(n, 17, n)),$$

I have doubts similar to those raised in Questions 2 to 4.

Also, pertaining to the conclusion that G's Gödel number $g = \mathrm{sub}(n, 17, n)$, I guess an implicit requirement is that y in (1) be treated as the **symbol** 'y'. This would have answered Question 4, but in an uncomfortable way.

Probably more importantly, the $g = \mathrm{sub}(n, 17, n)$ conclusion probably also requires that, to demonstrate that a proof can reach a conclusion where a symbol say 'y' has been replaced by a numeral '$ss \cdots s0$', all one has to do is replacing the 'y' in the original demonstration by the same numeral. This leads me to the following potentially **serious** question.

Question 5 *When* z *becomes sub*$(z, 17, z)$*, will Dem*(x, z) *(or* $\neg(\exists x)$*Dem*(x, z)*) with a Gödel number* n *simply become Sub*$(n, 17, n)$ *with a Gödel number sub*$(n, 17, n)$*? What if the demonstration Dem*(x, z) *(or* $\neg(\exists x)$*Dem*(x, z)*) already contains the symbol '*y*' even when the expression whose Gödel number is* z *contains no '*y*' in the first place? if so, will the taciturn agreement have led to over-substitutions which would destroy the demonstration? Of course, the changes required of the original demonstration could amount to much more than mere conclusion-related substitutions in the first place.*

References

[1] Nagel, E. and J.R. Newsman. 2001. *Gödel's Proof, Revised Version*, Edited with a New Foreword by D.R. Hofstadter. New York University press, New York and London.