Inventory and Price Control under Time-consistent Coherent and Markov Risk Measure–Unabridged Version

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Abstract

We use the recently proposed concept of time-consistent coherent and Markov risk measure on the study of a risk-averse firm's inventory and price control activities. In our shock-driven setting which is different from the state-driven setting where the measure is first introduced, we show the suitability of dynamic programming formulations. On this basis, we examine pure inventory and joint inventory-price control problems. The resulting model calls for worst-case analysis over a convex set of demand-distribution scenarios in every period. We achieve structural characterizations for optimal policies that are reminiscent of their risk-neutral counterparts. Monotone properties are derived for the pricing policy when the convex risk set constitutes a lattice under a suitably defined partial order. We also introduce the concept of optimism using the strong set order between risk sets. Two risk measures thus ranked produce inventory and pricing decisions that can be ranked themselves.

Keywords: Coherent Risk Measure; Dynamic programming; Inventory and Price Control; Lattice and Supermodularity; Strong Set Order

1 Introduction

Classical inventory and price control (IPC) problems focused on maximizing the expected total profit in the long run. For both pure inventory and joint inventory-price control models, tremendous advances have been made on the understanding of behaviors of profit-to-go functions and characters of optimal control policies; see, e.g., Karlin [14], Scarf [24], Federgruen and Heching [11], and Chen and Simchi-Levi [6]. However, as noted in Schweitzer and Cachon [25], a great number of firms are concerned with avoiding excess swings in their fortunes and intent on protecting themselves from grave downside risks. So it is important to incorporate risk aversion into IPC problems.

There have been some works on the optimal control of inventory-pricing activities with risk considerations. For instance, Bouakiz and Sobel [3] applied exponential utility to the total payoff of an inventory control problem and obtained an optimal time-varying base-stock policy. Meanwhile, Chen et al. [5] used the maximization of the total expected utility from consumptions in all periods as a risk criterion for IPC models. When the utility function is further additive with respect to period-wise contributions and the utility in each period is an exponential function of that period's consumption level, policy shapes from the risk-neutral case was shown to be preservable. Using the same utility function, Chen and Sun [8] dealt with a situation where there is ambiguity on demand distributions. They showed the optimality of simple policy forms when the time horizon is infinite.

These existing approaches have their limitations. First, the exponential utility function does not appeal to a wide swath of decision makers, as it violates conditions like convexity and positive homogeneity; also, being a single functional form, it lacks the flexibility needed to cater to disparate risk attitudes. Second, although widely accepted in the economics literature (Smith [27]), the utility-on-consumption criterion tacitly assumed that the concerned firm has access to a financial market for borrowing and lending with a risk-free saving and borrowing interest rate.

It is desirable to have an approach that is flexible, that does not add operational requirements on the concerned firm, and that lends itself to a general framework beyond the IPC application. We find one based on Ruszczyński's [21] recent work concerning the time-consistent coherent and Markov risk measure. In this approach, the riskiness of a sequence of random cost terms is assessed directly without resorting to an attendant consumption sequence or a financial market. The assessment is done in a backward-going fashion that is conducive to dynamic programming (DP) formulation.

The introduction of coherent risk measures can be credited to Artzner et al. [1], who also presented duality results that associated such measures with supremums over convex density sets. Using tools in convex analysis and optimization theory in vector spaces of measurable functions, researchers like Delbaen [10], Föllmer and Schied [12], Cheridito, Delbaen, and Kupper [9], Rockafellar, Uryasev, and Zabarankin [19], and Ruszczyński and Shapiro [22] made steady progresses on the dual representation theory. The flexibility and versatility of coherent risk measures have been widely accepted. These measures encompass conditional values at risk at various percentile points, as well as certain mean-deviation and mean-semideviation risk functions, the latter of which could for instance be used to reflect Markowitz's [15] classical mean-variance trade-offs; see, e.g., Ruszczyński and Shapiro [22].

The just mentioned paper also dealt with optimization with risk considerations. Multi-period optimization problems involving risk measures were studied too, for instance, by Artzner et al. [2] and Riedel [18]. That DP could be used in such problems was realized by Ruszczyński and Shapiro [23], who introduced conditional convex risk mappings, derived their representations in terms of conditional expectations, and developed DP relationships. By reducing the dependence of each period-wise risk measure on the entire history to that on the state-control pair only, Ruszczyński [21] made the resultant DP more tractable. Meanwhile, Çavuş and Ruszczyński [4] considered an infinite-horizon version of the problem.

We note that Ruszczyński's [21] general setup is not directly applicable to our situation. To explain this briefly, let the initial state $\mathbf{x}_1$ be known in both his setting and our general setting that countenances IPC as a special case. In the former, the σ-fields are produced from the endogenous sequence of states (such as inventory levels) $\mathbf{X}_2, \mathbf{X}_3, ...$ In the latter setting of ours, the σ-fields are produced from the exogenous sequence of independently generated shocks (such as demand levels) $\mathbf{\Delta}_1, \mathbf{\Delta}_2, ...$ For convenience, let us call Ruszczyński's setup state-driven and our setting shock-driven. Due partially to the nonlinearity of the risk operators, it will become clear that the two setups should be treated differently under risk considerations, even though they are all but the same in the risk neutral case.

In this paper, we first formulate the problem of using time-consistent coherent and Markov risk measure on a shock-driven system. We then show its compatibility with dynamic programming (Theorem 1). When bringing the DP to our specific risk-averse IPC situations, we derive policy characteristics for both pure inventory and joint inventory-price control problems. Like their risk-neutral counterparts, we find (s, S) and (s, S, A, P) are still basic policy forms (Propositions 1 and 2). When there is no fixed setup cost and the convex density set M_t involved in the definition of our risk measure possesses the lattice-like property, concrete links between higher inventory levels and lower prices can be established (Proposition 3). Moreover, we find the further imposition of demand additiveness will make the monotone trend mild (Proposition 4).

All our structural insights on optimal control policies can be extended to infinite-horizon settings (Propositions 5 and 6). We are interested in the comparative statics comparison between two systems under different risk measures as well. When $M_t^1 \leq M_t^2$ in the sense of the strong set order as proposed by Veinott [30], one can interpret that system 2 has a more optimistic expectation on future demand levels than system 1. If so, we can show that monotone trends exist in movements of policy parameters with regard to changes in the underlying risk measure (Propositions 7 and 8).

In a number of ways this paper can add to people's understanding of IPC problems. First, Proposition 2 as well as its without-setup corollary have verified the robustness of the ordering policy form known from risk-neutral IPC when now the decision maker's attitude may come from a much wider spectrum. Second, Proposition 3 and 4 have shown that monotone pricing trends could emerge in risk-averse situations. Indeed known for the more special risk-neutral case, these patterns are now tied to certain structures of the involved convex density sets. Third, Proposition 8 can be used to predict changes in ordering and pricing decisions when the decision maker's risk attitude changes. For instance, when it changes from risk-neutral to one involving more pessimistic demand outlooks, then the proposition will forecast not only less immediate ordering but also changes in pricing that will result with a lower average next-period inventory level.

The remainder of the paper is organized as follows. We concentrate on the general formulation for a shock-driven system in Section 2 and deal with our version of risk-averse IPC in Section 3. We spend Section 4 on the link between lattice-like density sets and monotone pricing properties. Section 5 is devoted to infinite-horizon analysis. The optimism concept is introduced in Section 6 to facilitate comparative statics comparisons between two systems under different risk measures. We show connections between the shock- and state-driven systems in Section 7. The paper is concluded in Section 8, where we also briefly discuss consequences of adopting slightly more general convex risk measures. Relatively more routine or tedious proofs have been relegated to appendices.

2 Formulation for a Shock-driven System

2.1 General Setup

Let $\Re = (-\infty, +\infty)$ be the real line endowed with the Euclidean topology, and let $\bar{\Re} = [-\infty, +\infty]$ be the extended real line endowed with the order topology wherein convergence is either in the ordinary sense or to one of the two infinite points. Note $\bar{\Re}$ is compact. We need a state space and a control space. Let the state space $\hat{X}$ be a closed subset of some $\bar{n}$-dimensional extended real space $\bar{\Re}^{\bar{n}}$. For instance, $\hat{X}$ is $\bar{\Re}$ for an inventory control problem involving backlogging. Let the control space $\hat{U}$ be a closed subset of some $\bar{\Re}^{\bar{m}}$. Let $T = 2, 3, ...$ be the length of the time horizon. In our problems, $1, 2, ..., T-1$ will be regular periods and T will be the terminal period. We introduce multi-functions (correspondences) $U_t : \hat{X} \rightrightarrows \hat{U}$ that are measurable with respect to the Borel σ-fields $\mathcal{B}(\hat{X})$ and $\mathcal{B}(\hat{U})$. For each period $t = 1, 2, ..., T-1$, we let each $U_t(\mathbf{x})$ be a closed subset of $\hat{U}$ that specifies the control range for the state $\mathbf{x} \in \hat{X}$.

For our shock-driven system, we further let a closed subset $\hat{D}$ of some $\Re^{\bar{l}}$ be the (demand) shock space. For period $t = 1, 2, ..., T-1$, the cost (or negative profit) function c_t is a measurable mapping from $\mathrm{graph}(U_t) \times \hat{D}$ to $\bar{\Re}$, where $\mathrm{graph}(U_t) = \{(\mathbf{x}, \mathbf{u}) \in \hat{X} \times \hat{U} \mid \mathbf{u} \in U_t(\mathbf{x})\}$. A measurable mapping c_T^0 from $\hat{X}$ to $\Re$ represents the terminal cost. For $t = 1, 2, ..., T-1$, there exists another measurable mapping $\mathbf{s}_{t+1}$ from $\mathrm{graph}(U_t) \times \hat{D}$ to $\hat{X}$, so that $\mathbf{s}_{t+1}(\mathbf{x}, \mathbf{u}, \delta)$ stands for the period-$(t+1)$ state. The period-t random shock $\mathbf{\Delta}_t$ is supposed to have a distribution R_t defined on the measurable space $(\hat{D}, \mathcal{B}(\hat{D}))$.

We define the probability space $(\Omega, \mathcal{F}, P)$ so that $\Omega = \hat{D}^{T-1}$, $\mathcal{F} = \mathcal{B}(\hat{D}^{T-1})$, and $P = \prod_{t=1}^{T-1} R_t$. Let $\mathbf{x}_1 \in \hat{X}$ with finite components and hence a finite Euclidean norm $\| \mathbf{x}_1 \|$ be given as an initial state. Define random variables $\mathbf{X}_1, \mathbf{\Delta}_1, ..., \mathbf{\Delta}_{T-1}$ so that $\mathbf{X}_1(\omega) = \mathbf{x}_1, \mathbf{\Delta}_1(\omega) = \delta_1, ..., \mathbf{\Delta}_{T-1}(\omega) = \delta_{T-1}$ for each $\omega = \delta_{[1,T-1]} = (\delta_t | t = 1, 2, ..., T-1) \in \Omega$. For each $t = 1, 2, ..., T$, we can let $\mathcal{F}_t$ be the σ-field generated by the generic random vector $(\mathbf{X}_1, \mathbf{\Delta}_1, ..., \mathbf{\Delta}_{t-1})$, which is just $\mathcal{B}(\hat{D}^{t-1})$. Note this implies $\mathcal{F}_1 = \{\emptyset, \Omega\}$. The sequence $(\mathcal{F}_t \mid t = 1, 2, ..., T)$ forms a filtration on the measurable space $(\Omega, \mathcal{F})$.

The system's state in any period t is a result of the past control sequence $\mathbf{U}_{[1,t-1]} = (\mathbf{U}_s | s = 1, 2, ..., t-1)$ and past shock history $\delta_{[1,t-1]} = (\delta_s | s = 1, 2, ..., t-1)$. Here, $\mathbf{U}_{[1,0]}$ or $\delta_{[1,0]}$ should be understood as "an empty string". Given $\mathbf{U}_{[1,t-1]}$, the state can be viewed as an $\mathcal{F}_t$-measurable random variable $\mathbf{X}_t^{\mathbf{U}_{[1,t-1]}}$. In deference to IPC conventions, we let the system experience control $\mathbf{U}_t$ ahead of shock $\mathbf{\Delta}_t$ in every period $t = 1, 2, ..., T-1$. Thus, the adaptive control $\mathbf{U}_t$ in period t has to not only be measurable in $\delta_{[1,t-1]}$, but also have its range in $U_t(\mathbf{X}_t^{\mathbf{U}_{[1,t-1]}}(\cdot))$. After subsequently experiencing the realized shock δ_t in the period, the system's state will evolve to $\mathbf{X}_{t+1}^{\mathbf{U}_{[1,t]}}(\delta_{[1,t]})$ following the transition rule $\mathbf{s}_{t+1}(\cdot, \cdot, \cdot)$.

For a formal treatment, we need a condition on the control ranges U_t.

Assumption 1 *For $t = 1, 2, ..., T-1$, each $U_t(\cdot)$ is upper hemi-continuous. Due to the closedness of each $U_t(\mathbf{x})$ and the compactness of $\hat{U}$, this is equivalent to the closedness of each* $\mathrm{graph}(U_t)$*: when sequence $(\mathbf{x}_k, \mathbf{u}_k)$ in $\hat{X} \times \hat{U}$ converges to $(\mathbf{x}, \mathbf{u}) \in \hat{X} \times \hat{U}$ and each $\mathbf{u}_k \in U_t(\mathbf{x}_k)$, we have $\mathbf{u} \in U_t(\mathbf{x})$.*

As $\mathrm{graph}(U_t)$ is a closed and hence measurable subset of $\hat{X} \times \hat{U}$, the Borel σ-field $\mathcal{B}(\mathrm{graph}(U_t))$ is well defined and it is the collection of elements in $\mathcal{B}(\hat{X} \times \hat{U})$ that happen to be subsets of graph(U_t). Now we can make the following iterative definitions over periods $t = 1, 2, ..., T-1$. First, if $t = 1$, let

$$\mathcal{U}_{[1,1]} = \mathcal{U}_1 = U_1(\mathbf{x}_1); \tag{1}$$

otherwise, for $\mathbf{U}_{[1,t-1]} \in \mathcal{U}_{[1,t-1]}$, let

$$\begin{aligned} &\mathcal{U}_t(\mathbf{U}_{[1,t-1]}) \text{ be the space of measurable mappings } \mathbf{U}_t(\cdot) \text{ from } \hat{D}^{t-1} \text{ to } \hat{U} \\ &\qquad \text{that make each } (\mathbf{X}_t^{\mathbf{U}_{[1,t-1]}}, \mathbf{U}_t)(\cdot) \text{ measurable from } \hat{D}^{t-1} \text{ to graph}(U_t). \end{aligned} \tag{2}$$

Next, let

$$\mathcal{U}_{[1,t]} = \{(\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_t) \mid \mathbf{U}_1 \in \mathcal{U}_1, \mathbf{U}_2 \in \mathcal{U}_2(\mathbf{U}_1), ..., \mathbf{U}_t \in \mathcal{U}_t(\mathbf{U}_{[1,t-1]})\}. \tag{3}$$

Finally, for every $\mathbf{U}_{[1,t]} \in \mathcal{U}_{[1,t]}$ and $\delta_{[1,t]} \in \hat{D}^t$, let

$$\mathbf{X}_{t+1}^{\mathbf{U}_{[1,t]}}(\delta_{[1,t]}) = \mathbf{s}_{t+1}\left(\mathbf{X}_t^{\mathbf{U}_{[1,t-1]}}(\delta_{[1,t-1]}), \mathbf{U}_t(\delta_{[1,t-1]}), \delta_t\right). \tag{4}$$

This way, every $\mathcal{U}_{[1,t]}$ is the space of control policies in periods 1 through t, and every $\mathbf{X}_{t+1}^{\mathbf{U}_{[1,t]}}$ represents the random state at the beginning of period $t+1$ when the policy adopted in periods 1 to t is $\mathbf{U}_{[1,t]} \in \mathcal{U}_{[1,t]}$.

The policy space for our control problem is $\mathcal{U} \equiv \mathcal{U}_{[1,T-1]}$. Suppose $\alpha \in [0,1]$ is a given discount factor. Then to each control $\mathbf{U} \in \mathcal{U}$, there corresponds an adapted cost sequence $Z^{\mathbf{U}} = (Z_t^{\mathbf{U}_{[1,t-1]}} \mid t = 1, 2, ..., T)$, with

$$\begin{cases} Z_1 & = 0, \\ Z_2^{\mathbf{U}_1} & = c_1(\mathbf{x}_1, \mathbf{U}_1, \mathbf{\Delta}_1), \\ Z_3^{\mathbf{U}_{[1,2]}} & = \alpha \cdot c_2(\mathbf{X}_2^{\mathbf{U}_1}, \mathbf{U}_2, \mathbf{\Delta}_2), \\ \cdots & \cdots\cdots, \\ Z_{T-1}^{\mathbf{U}_{[1,T-2]}} & = \alpha^{T-3} \cdot c_{T-2}(\mathbf{X}_{T-2}^{\mathbf{U}_{[1,T-3]}}, \mathbf{U}_{T-2}, \mathbf{\Delta}_{T-2}), \\ Z_T^{\mathbf{U}_{[1,T-1]}} & = \alpha^{T-2} \cdot c_{T-1}(\mathbf{X}_{T-1}^{\mathbf{U}_{[1,T-2]}}, \mathbf{U}_{T-1}, \mathbf{\Delta}_{T-1}) + \alpha^{T-1} \cdot c_T^0(\mathbf{X}_T^{\mathbf{U}_{[1,T-1]}}). \end{cases} \tag{5}$$

The total expected cost $e(Z^{\mathbf{U}})$ can be expressed as in

$$\begin{aligned} e(Z^{\mathbf{U}}) = \mathbb{E}[\textstyle\sum_{t=1}^T Z_t^{\mathbf{U}_{[1,t-1]}}] &= \mathbb{E}[c_1(\mathbf{x}_1, \mathbf{U}_1, \mathbf{\Delta}_1) + \alpha \cdot \mathbb{E}[c_2(\mathbf{X}_2^{\mathbf{U}_1}, \mathbf{U}_2, \mathbf{\Delta}_2) \\ &+ \cdots + \alpha \cdot \mathbb{E}[c_{T-1}(\mathbf{X}_{T-1}^{\mathbf{U}_{[1,T-2]}}, \mathbf{U}_{T-1}, \mathbf{\Delta}_{T-1}) + \alpha \cdot c_T^0(\mathbf{X}_T^{\mathbf{U}_{[1,T-1]}})] \cdots]]. \end{aligned} \tag{6}$$

The traditional risk-neutral control is concerned with solving $J^* = \inf_{\mathbf{U} \in \mathcal{U}} e(Z^{\mathbf{U}})$.

We now introduce a few boundedness conditions.

Assumption 2 *For $t = 1, 2, ..., T-1$,*

$$\mid c_t(\mathbf{x}, \mathbf{u}, \delta) \mid \leq A_t \cdot \parallel \mathbf{x} \parallel + B_t \cdot \parallel \mathbf{u} \parallel + C_t \cdot \parallel \delta \parallel + D_t,$$

for some positive constants A_t, B_t, C_t, and D_t.

Assumption 3 *For some positive constants A_T^0 and D_T^0,*

$$| c_T^0(\mathbf{x}) | \leq A_T^0 \cdot || \mathbf{x} || + D_T^0.$$

Assumption 4 *For $t = 1, 2, ..., T-1$,*

$$|| \mathbf{s}_{t+1}(\mathbf{x}, \mathbf{u}, \delta) || \leq E_t \cdot || \mathbf{x} || + F_t \cdot || \mathbf{u} || + G_t \cdot || \delta || + H_t,$$

for some positive constants E_t, F_t, G_t, and H_t.

Assumption 5 *For $t = 1, 2, ..., T-1$, control ranges are bounded, so that*

$$|| \mathbf{u} || \leq I_t \cdot || \mathbf{x} || + J_t, \qquad \forall \mathbf{u} \in U_t(\mathbf{x}),$$

for some positive constants I_t and J_t.

Due to Assumptions 2 to 5, we can use (2) and (4) to iteratively derive the ensuing bounds. For any $t = 1, 2, ..., T-1$, there exist positive constants $A_t^0, A_t^1, ..., A_t^t, A_t^{t+1}$, so that

$$| [c_t(\mathbf{X}_t^{\mathbf{U}_{[1,t-1]}}, \mathbf{U}_t, \mathbf{\Delta}_t)](\delta_{[1,t]}) | \leq A_t^0 \cdot || \mathbf{x}_1 || + \sum_{s=1}^{t} A_t^s \cdot || \delta_s || + A_t^{t+1}, \tag{7}$$

for any $\delta_{[1,t]} \in \hat{D}^t$ and $\mathbf{U}_{[1,t]} \in \mathcal{U}_{[1,t]}$; also, there exist positive constants $A_T^0, A_T^1, ..., A_T^T$, so that

$$| [c_T^0(\mathbf{X}_T^{\mathbf{U}_{[1,T-1]}})](\delta_{[1,T-1]}) | \leq A_T^0 \cdot || \mathbf{x}_1 || + \sum_{s=1}^{T-1} A_T^s \cdot || \delta_s || + A_T^T, \tag{8}$$

for any $\delta_{[1,T-1]} \in \hat{D}^{T-1}$ and $\mathbf{U}_{[1,T-1]} \in \mathcal{U}_{[1,T-1]}$.

For a particular $p \in (1, +\infty)$, we also suppose the following.

Assumption 6 *For $t = 1, 2, ..., T-1$,*

$$\mathbb{E}[(|| \mathbf{\Delta}_t ||)^p] = \int_{\hat{D}} (|| \delta ||)^p \cdot R_t(d\delta) < +\infty.$$

If we understand $|| \mathbf{\Delta}_t ||$ as a real-valued function f measurable on $\hat{D}$, then Assumption 6 means that $|| f ||^p < +\infty$ and hence f belongs to the Banach space $\mathbb{L}_t^p \equiv \mathbb{L}^p(\hat{D}, \mathcal{B}(\hat{D}), R_t)$. This, the fact that $|| \mathbf{x}_1 || < +\infty$, as well as (7) and (8), will together lead to the $\mathbb{L}_t^p$- and hence $\mathbb{L}^1(\hat{D}, \mathcal{B}(\hat{D}), R_t)$-membership of the cost terms involved in (5). Thus, the nested expectations in (6) are well defined.

2.2 Risk Measure

For any probability space $(\Omega^0, \mathcal{F}^0, P^0)$ and filtration $(\mathcal{F}^0_t \mid t = 1, 2, ..., T)$, we can define conditional risk measures ρ_{tT} for $t = 1, 2, ..., T$, so that for each sequence $(Z_s \mid s = t, t+1, ..., T) \in \prod_{s=t}^T \mathbb{L}^p(\Omega^0, \mathcal{F}^0_s, P^0)$, the resulting $\rho_{tT}(Z_t, ..., Z_T) \in \mathbb{L}^p(\Omega^0, \mathcal{F}^0_t, P^0)$ gives the decision maker's time-t assessment of his risk exposure from period t to the terminal period T. Ruszczyński [21] studied conditional risk measures that satisfy the following three properties:

(I) $\rho_{tT}(Z_t, Z_{t+1}, ..., Z_T) = Z_t + \rho_{tT}(0, Z_{t+1}, ..., Z_T)$;

(II) $\rho_{tT}(0, ..., 0) = 0$;

(III) $\rho_{tT}(Z_t, ..., Z_s, ..., Z_T) = \rho_{tT}(Z_t, ..., Z_{s-1}, \rho_{sT}(Z_s, ..., Z_T), 0, ..., 0)$.

When letting $\rho_t(Z_{t+1}) = \rho_{tT}(0, Z_{t+1}, 0, ..., 0)$ for such a time-consistent measure, it has been shown that the following recursive relation would hold:

$$\rho_{tT}(Z_t, ..., Z_T) = Z_t + \rho_t(Z_{t+1} + \rho_{t+1}(Z_{t+2} + \cdots + \rho_{T-2}(Z_{T-1} + \rho_{T-1}(Z_T)) \cdots)). \tag{9}$$

Due to this, we call $(\rho_t \mid t = 1, 2, ..., T-1)$, where each ρ_t is of the nature $\mathbb{L}^p(\Omega^0, \mathcal{F}^0_{t+1}, P^0) \to \mathbb{L}^p(\Omega^0, \mathcal{F}^0_t, P^0)$ and satisfies $\rho_t(0) = 0$, a time-consistent risk measure for cost sequences adapted to the filtration $(\mathcal{F}^0_t \mid t = 1, 2, ..., T)$.

Like Ruszczyński [21], we let $(\rho_t \mid t = 1, 2, ..., T-1)$ be coherent as well. That is, for each t, we require ρ_t to satisfy four axioms characteristic of coherency (Artzner et al. [1]):

(i) Positive Homogeneity: $\rho_t(\beta Z) = \beta \cdot \rho_t(Z)$, for any $Z \in \mathbb{L}^p(\Omega^0, \mathcal{F}^0_{t+1}, P^0)$ and $\beta \geq 0$;

(ii) Monotonicity: If $Z \leq W$ then $\rho_t(Z) \leq \rho_t(W)$, for any $Z, W \in \mathbb{L}^p(\Omega^0, \mathcal{F}^0_{t+1}, P^0)$;

(iii) Translation Equivariance: $\rho_t(Z + W) = Z + \rho_t(W)$, for any $Z \in \mathbb{L}^p(\Omega^0, \mathcal{F}^0_t, P^0)$ and $W \in \mathbb{L}^p(\Omega^0, \mathcal{F}^0_{t+1}, P^0)$;

(iv) Convexity: $\rho_t((1-\lambda)Z + \lambda W) \leq (1-\lambda) \cdot \rho_t(Z) + \lambda \cdot \rho_t(W)$, for any $Z, W \in \mathbb{L}^p(\Omega, \mathcal{F}_{t+1}, P)$ and $\lambda \in [0, 1]$.

For separable metric spaces A and B, it is known that $\mathcal{B}(A \times B) = \mathcal{B}(A) \times \mathcal{B}(B)$, the smallest σ-field containing all $A' \times B'$ for $A' \in \mathcal{B}(A)$ and $B' \in \mathcal{B}(B)$; see Parthasarathy [16] (Theorem 1.10). So our particular filtration is of the product form in that every $\mathcal{F}_{t+1} = \mathcal{F}_t \times \mathcal{B}(\hat{D})$. Besides, our particular probability P is of the product form as well. Because of these, $\rho = (\rho_t \mid t = 1, 2, ..., T-1)$ enjoys a simple representation.

Let $q \in (1, +\infty)$ be such that $1/p + 1/q = 1$. For any probability space $(\Omega^0, \mathcal{F}^0, P^0)$, let $\mathbb{M}^q(\Omega^0, \mathcal{F}^0, P^0)$ be the subset of $\mathbb{L}^q(\Omega^0, \mathcal{F}^0, P^0)$ that correspond to densities:

$$\mathbb{M}^q(\Omega^0, \mathcal{F}^0, P^0) = \left\{ m \in \mathbb{L}^q(\Omega^0, \mathcal{F}^0, P^0) \mid m \geq 0, \text{ and } \int_{\Omega^0} m(\omega^0) \cdot P^0(d\omega^0) = 1 \right\}. \tag{10}$$

Now come back to our particular case involving $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t | t = 1, 2, ..., T)$. According to Theorem 2.2 of Ruszczyński and Shapiro [22], there would exist for any $\delta_{[1,t-1]} \in \hat{D}^{t-1}$ a set

$\mathcal{M}_t(\delta_{[1,t-1]}) \subseteq \mathbb{M}_t^q \equiv \mathbb{M}^q(\hat{D}, \mathcal{B}(\hat{D}), R_t)$ that is convex and closed under the weak topology, such that

$$\begin{aligned} [\rho_t(Z)](\delta_{[1,t-1]}) &= \sup_{m \in \mathcal{M}_t(\delta_{[1,t-1]})} \mathbb{E}^m[Z(\delta_{[1,t-1]}, \Delta_t)] \\ &= \sup_{m \in \mathcal{M}_t(\delta_{[1,t-1]})} \int_{\hat{D}} Z(\delta_{[1,t-1]}, \delta_t) \cdot m(\delta_t) \cdot R_t(d\delta_t), \end{aligned} \tag{11}$$

for any $Z \in \mathbb{L}_{[1,t]}^p \equiv \mathbb{L}^p(\hat{D}^t, \mathcal{B}(\hat{D}^t), \prod_{s=1}^t R_s)$ and $\delta_{[1,t-1]} \in \hat{D}^{t-1}$.

It is important to ensure $\rho_t(Z)$'s $\mathbb{L}_{[1,t]}^p$-in-$\mathbb{L}_{[1,t-1]}^p$-out status, or something to that effect. To this end, note that $\mathbb{L}_t^q \equiv \mathbb{L}^q(\hat{D}, \mathcal{B}(\hat{D}), R_t)$ is a separable metric space under the $|| \cdot ||^q$-induced topology. So $\mathbb{M}_t^q$ is separable as well, and hence $\mathcal{B}(\mathbb{M}_t^q)$ is meaningful. For $m \in \mathbb{M}_t^q$, define section $\mathcal{D}_{[1,t-1]}(m)$ of the multi-function $\mathcal{M}_t$'s graph as follows:

$$\begin{aligned} \mathcal{D}_{[1,t-1]}(m) &= \{\delta_{[1,t-1]} \in \hat{D}^{t-1} | m \in \mathcal{M}_t(\delta_{[1,t-1]})\} \\ &= \{\delta_{[1,t-1]} \in \hat{D}^{t-1} | (\delta_{[1,t-1]}, m) \in \text{graph}(\mathcal{M}_t)\}. \end{aligned} \tag{12}$$

For a measurable map f from $\hat{D}^t$ to $\Re$, let $|| \ f(\delta_{[1,t']}, \cdot) \ ||^p$ at any $t' = 0, 1, ..., t-1$ and $\delta_{[1,t']} \in \hat{D}^{t'}$ be defined through

$$|| \ f(\delta_{[1,t']}, \cdot) \ ||^p = \left[\int_{\hat{D}^{t-t'}} (| \ f(\delta_{[1,t']}, \delta_{[t'+1,t]}) \ |)^p \cdot \left(\prod_{s=t'+1}^{t} R_s \right) (d\delta_{[t'+1,t]}) \right]^{1/p}. \tag{13}$$

In the above, $\delta_{[1,0]}$ should again be understood as "an empty string". Now let $\mathbb{L}_{[1,t]}^{p+}$ be the space of such f's with $|| \ f(\delta_{[1,t']}, \cdot) \ ||^p < +\infty$ for every $t' = 0, 1, ..., t-1$ and $\delta_{[1,t']} \in \hat{D}^{t'}$. Clearly, $\mathbb{L}_{[1,t]}^{p+}$ is a linear subspace of $\mathbb{L}_{[1,t]}^p$ and $\mathbb{L}_{[1,1]}^{p+}$ is merely $\mathbb{L}_{[1,1]}^p = \mathbb{L}_1^p$. With reasonable hypotheses, the $\mathbb{L}_{[1,t]}^{p+}$-in-$\mathbb{L}_{[1,t-1]}^{p+}$-out nature of (11) can be confirmed.

Lemma 1 *Suppose for some $t = 2, 3, ..., T-1$, the following are true:*

(i) $\mathcal{D}_{[1,t-1]}(m)$ is a member of $\mathcal{B}(\hat{D}^{t-1})$ for every $m \in \mathbb{M}_t^q$,

(ii) $\sup\{|| \ m \ ||^q \ | m \in \bigcup_{\delta_{[1,t-1]} \in \hat{D}^{t-1}} \mathcal{M}_t(\delta_{[1,t-1]})\} < +\infty$.

Then, (11) will turn any $Z \in \mathbb{L}_{[1,t]}^{p+}$ into $\rho_t(Z) \in \mathbb{L}_{[1,t-1]}^{p+}$.

Hypothesis (i) in Lemma 1 will be true when $\mathcal{M}_t : \mathcal{D}^{t-1} \rightrightarrows \mathbb{M}_t^q$ is measurable, because if so,

$$\mathcal{D}_{[1,t-1]}(m) = \{\delta_{[1,t-1]} \in \hat{D}^{t-1} | \{m\} \cap \mathcal{M}_t(\delta_{[1,t-1]}) \neq \emptyset\} \in \mathcal{B}(\hat{D}^{t-1}), \tag{14}$$

for any $m \in \mathbb{M}_t^q$ which renders $\{m\} \in \mathcal{B}(\mathbb{M}_t^q)$. Currently, the premise that $Z \in \mathbb{L}_{[1,t]}^{p+}$, stronger than the mere membership of Z in $\mathbb{L}_{[1,t]}^p$, remains indispensable to our measurability proof. This is the main reason why the lemma is not exactly $\mathbb{L}_{[1,t]}^p$-in-$\mathbb{L}_{[1,t-1]}^p$-out.

Besides time-consistency and coherency, we emphasize Markov properties for our risk measure. The particular Markov case we consider has every period $t = 1, 2, ..., T-1$ and state-control pair $(\mathbf{x},\mathbf{u}) \in \text{graph}(U_t)$ associated with a convex subset $M_t(\mathbf{x},\mathbf{u})$ of $\mathbb{M}_t^q$ that is also closed under the weak topology. The assembly $M_{[1,T-1]}(\cdot,\cdot) = (M_t(\mathbf{x},\mathbf{u})|t=1,2,...,T-1,\ (\mathbf{x},\mathbf{u}) \in \text{graph}(U_t))$ can be regarded as the generator of our risk measure. Now under each control $\mathbf{U} \in \mathcal{U}$, the generator will produce a time-consistent coherent risk measure $\rho^{\mathbf{U}} = (\rho_t^{\mathbf{U}_{[1,t]}}|t=1,2,...,T-1)$, so that

$$\left[\rho_t^{\mathbf{U}_{[1,t]}}(Z)\right](\delta_{[1,t-1]}) = \sup_{m \in \mathcal{M}_t^{\mathbf{U}_{[1,t]}}(\delta_{[1,t-1]})} \mathbb{E}^m[Z(\delta_{[1,t-1]},\Delta_t)], \tag{15}$$

where

$$\mathcal{M}_t^{\mathbf{U}_{[1,t]}}(\delta_{[1,t-1]}) = M_t\left(\mathbf{X}_t^{\mathbf{U}_{[1,t-1]}}(\delta_{[1,t-1]}), \mathbf{U}_t(\delta_{[1,t-1]})\right). \tag{16}$$

Then, by (5) and (9), the total risk exposure $\rho^{\mathbf{U}}(Z^{\mathbf{U}})$ is expressible by

$$\begin{aligned}\rho^{\mathbf{U}}(Z^{\mathbf{U}}) &= Z_1 + \rho_1^{\mathbf{U}_1}(Z_2^{\mathbf{U}_1} + \rho_2^{\mathbf{U}_{[1,2]}}(Z_3^{\mathbf{U}_{[1,2]}} + \cdots + \rho_{T-1}^{\mathbf{U}_{[1,T-1]}}(Z_T^{\mathbf{U}_{[1,T-1]}})\cdots))\\ &= \rho_1^{\mathbf{U}_1}(c_1(\mathbf{x}_1,\mathbf{U}_1,\boldsymbol{\Delta}_1) + \rho_2^{\mathbf{U}_{[1,2]}}(\alpha \cdot c_2(\mathbf{X}_2^{\mathbf{U}_1},\mathbf{U}_2,\boldsymbol{\Delta}_2) + \cdots\\ &\quad + \rho_{T-1}^{\mathbf{U}_{[1,T-1]}}(\alpha^{T-2} \cdot c_{T-1}(\mathbf{X}_{T-1}^{\mathbf{U}_{[1,T-2]}},\mathbf{U}_{T-1},\boldsymbol{\Delta}_{T-1}) + \alpha^{T-1} \cdot c_T^0(\mathbf{X}_T^{\mathbf{U}_{[1,T-1]}}))\cdots)),\end{aligned} \tag{17}$$

with each $\rho_t^{\mathbf{U}_{[1,t]}}$ term defined through (15) and (16). Comparing this with (6), we can see that the risk-neutral $e(Z^{\mathbf{U}})$ is a special case of the current $\rho^{\mathbf{U}}(Z^{\mathbf{U}})$ with the $\rho^{\mathbf{U}}$ there being $\mathbf{U}$-independent and for $t=1,2,...,T-1$, $\delta_{[1,t-1]} \in \hat{D}^{t-1}$, and $Z \in \mathbb{L}_{[1,t]}^p$,

$$\left[\rho_t^{\mathbf{U}_{[1,t]}}(Z)\right](\delta_{[1,t-1]}) = \mathbb{E}[Z(\delta_{[1,t-1]},\boldsymbol{\Delta}_t)] = \int_{\hat{D}} Z(\delta_{[1,t-1]},\delta_t) \cdot R_t(d\delta_t). \tag{18}$$

The generator for this special $\rho^{\mathbf{U}}$ is simply the $M_{[1,T-1]}(\cdot,\cdot)$ with each $M_t(\mathbf{x},\mathbf{u})$ being $\{1\}$, where 1 stands for the all-one unit rate function in $\mathbb{M}_t^q$. In general, we are concerned with

$$J^* = \inf_{\mathbf{U} \in \mathcal{U}} \rho^{\mathbf{U}}(Z^{\mathbf{U}}). \tag{19}$$

Let us introduce two conditions on the Markov risk generator.

Assumption 7 *For $t=1,2,...,T-1$, each $M_t(\cdot,\cdot)$ is measurable. That is, for any $M' \in \mathcal{B}(\mathbb{M}_t^q)$,*

$$\{(\mathbf{x},\mathbf{u}) \in \text{graph}(U_t)|M' \cap M_t(\mathbf{x},\mathbf{u}) \neq \emptyset\} \in \mathcal{B}(\text{graph}(U_t)).$$

Assumption 8 *For $t=1,2,...,T-1$, more than the boundedness of each individual $M_t(\mathbf{x},\mathbf{u})$, we require the uniform boundedness of all $M_t(\mathbf{x},\mathbf{u})$'s in $\mathbb{M}_t^q$, so that*

$$\sup\left\{ \|\, m \,\|^q \ |m \in \bigcup_{(\mathbf{x},\mathbf{u}) \in \text{graph}(U_t)} M_t(\mathbf{x},\mathbf{u})\right\} < +\infty.$$

These conditions are related to hypotheses (i) and (ii) of Lemma 1. Let

$$\mathcal{D}_{[1,t-1]}^{\mathbf{U}_{[1,t]}}(m) = \left\{\delta_{[1,t-1]} \in \hat{D}^{t-1} | m \in \mathcal{M}_t^{\mathbf{U}_{[1,t]}}(\delta_{[1,t-1]})\right\}, \tag{20}$$

where $\mathcal{M}_t^{\mathbf{U}_{[1,t]}}(\delta_{[1,t-1]})$ is involved in the definition of $\rho_t^{\mathbf{U}_{[1,t]}}$ in (15). By (16), it follows that

$$\mathcal{D}_{[1,t-1]}^{\mathbf{U}_{[1,t]}}(m) = \left(\mathbf{X}_t^{\mathbf{U}_{[1,t-1]}}, \mathbf{U}_t\right)^{-1} (\{(\mathbf{x}, \mathbf{u}) \in \text{graph}(U_t) | m \in M_t(\mathbf{x}, \mathbf{u})\}). \tag{21}$$

By (2), we know $(\mathbf{X}_t^{\mathbf{U}_{[1,t-1]}}, \mathbf{U}_t)(\cdot)$ is a measurable map from $\hat{D}^{t-1}$ to graph(U_t). In view of (14) and (21), this and Assumption 7 will together lead to hypothesis (i), that $\mathcal{D}_{[1,t-1]}^{\mathbf{U}_{[1,t]}}(m)$ is a member of $\mathcal{B}(\hat{D}^{t-1})$ for every m. Also, by (2) and (16),

$$\bigcup_{\delta_{[1,t-1]} \in \hat{D}^{t-1}} \mathcal{M}_t^{\mathbf{U}_{[1,t]}}(\delta_{[1,t-1]}) \subseteq \bigcup_{(\mathbf{x},\mathbf{u}) \in \text{graph}(U_t)} M_t(\mathbf{x}, \mathbf{u}). \tag{22}$$

So Assumption 8 will yield hypothesis (ii), that $\{|| \, m \, ||^q \, | m \in \bigcup_{\delta_{[1,t-1]} \in \hat{D}^{t-1}} \mathcal{M}_t^{\mathbf{U}_{[1,t]}}(\delta_{[1,t-1]})\}$ is finite. Then, Lemma 1 will guarantee the $\mathbb{L}_{[1,t]}^{p+}$-in-$\mathbb{L}_{[1,t-1]}^{p+}$-out nature of each of the $\rho_t^{\mathbf{U}_{[1,t]}}$ operators used in (17). From Assumption 6, as well as (7) and (8), we can see that $c_t(\mathbf{X}_t^{\mathbf{U}_{[1,t-1]}}, \mathbf{U}_t, \mathbf{\Delta}_t) \in \mathbb{L}_{[1,t]}^{p+}$ for $t = 1, 2, ..., T-1$ and $c_T^0(\mathbf{X}_T^{\mathbf{U}_{[1,T-1]}})](\delta_{[1,T-1]}) \in \mathbb{L}_{[1,T-1]}^{p+}$. So through an induction from inside out on the right-hand side, we can ensure the validity of the entire (17).

2.3 Dynamic Programming

We now examine the suitability of Markov policies and DP formulation for the optimization problem (19). When the decision $\mathbf{U}_t(\delta_{[1,t-1]})$ depends on past information $\delta_{[1,t-1]}$ always through $\mathbf{X}_t^{\mathbf{U}_{[1,t-1]}}$, we say the policy $\mathbf{U}_t$ is Markov. More precisely, for $t = 1, 2, ..., T-1$, define $\mathcal{U}_t^M$ so that

$$\begin{gathered} \mathcal{U}_t^M \text{ is the space of measurable mappings } \mathbf{u}_t \text{ from } \hat{X} \text{ to } \hat{U} \\ \text{such that } \mathbf{u}_t(\mathbf{x}) \in U_t(\mathbf{x}) \text{ for each } \mathbf{x} \in \hat{X}. \end{gathered} \tag{23}$$

Given $\mathbf{U}_{[1,t-1]} \in \mathcal{U}_{[1,t-1]}$, we call period-$t$ policy $\mathbf{U}_t \in \mathcal{U}_t(\mathbf{U}_{[1,t-1]})$ Markov, when for every $\delta_{[1,t-1]} \in \hat{D}^{t-1}$,

$$\mathbf{U}_t(\delta_{[1,t-1]}) = \mathbf{u}_t\left(\mathbf{X}_t^{\mathbf{U}_{[1,t-1]}}(\delta_{[1,t-1]})\right), \quad \text{for some } \mathbf{u}_t \in \mathcal{U}_t^M. \tag{24}$$

Naturally, we call policy $\mathbf{U} = (\mathbf{U}_1, ..., \mathbf{U}_{T-1}) \in \mathcal{U}$ Markov when for $t = 1, 2, ..., T-1$, each $\mathbf{U}_t$ is Markov. Let $\mathcal{U}^M = \prod_{t=1}^{T-1} \mathcal{U}_t^M$ be the space of Markov policies, so that every $\mathbf{u} \in \mathcal{U}^M$ is representable by an element $\mathbf{u} = (\mathbf{u}_t(\mathbf{x}) \mid t = 1, 2, ..., T-1, \mathbf{x} \in \hat{X})$ of $\mathcal{U}^M$.

Indeed, every $\mathbf{u} \in \mathcal{U}^M$ will lead to a policy $\mathbf{U} \in \mathcal{U}$ that is Markov. First, we can let $\mathbf{U}_1 = \mathbf{u}_1(\mathbf{x}_1)$. Then, we can go through an iterative procedure, in which for every $t = 2, 3, ..., T-1$,

$$\mathbf{U}_t(\delta_{[1,t-1]}) = \mathbf{u}_t\left(\mathbf{X}_t^{\mathbf{U}_{[1,t-1]}}(\delta_{[1,t-1]})\right), \tag{25}$$

at every $\delta_{[1,t-1]} \in \hat{D}^{t-1}$.

We can use induction to show that the thus obtained $\mathbf{U}_t(\cdot)$'s satisfy (2) and hence $\mathbf{U} = (\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_{T-1}) \in \mathcal{U}$. First, by (4), we know that $\mathbf{X}_2^{\mathbf{U}_1}(\cdot) = \mathbf{s}_2(\mathbf{x}_1, \mathbf{u}_1(\mathbf{x}_1), \cdot)$ is measurable in $\delta_{[1,1]} \equiv \delta_1$. Then for $t = 2, 3, ..., T-1$, we can pose the induction hypothesis that $\mathbf{X}_t^{\mathbf{U}_{[1,t-1]}}(\cdot)$ is measurable in $\delta_{[1,t-1]}$. Let $\mathbf{i}(\cdot)$ be the identity map from $\hat{X}$ to $\hat{X}$. Due to (23), for any $X' \in \mathcal{B}(\hat{X})$ and $U' \in \mathcal{B}(\hat{U})$,

$$(\mathbf{i}, \mathbf{u}_t)^{-1}((X' \times U') \cap \mathrm{graph}(U_t)) = X' \cap (\mathbf{u}_t)^{-1}(U') \in \mathcal{B}(\hat{X}). \tag{26}$$

Since $\mathcal{B}(\mathrm{graph}(U_t)) = \{G \cap \mathrm{graph}(U_t) | G \in \mathcal{B}(\hat{X} \times \hat{U})\}$ and $\mathcal{B}(\hat{X} \times \hat{U}) = \mathcal{B}(\hat{X}) \times \mathcal{B}(\hat{U})$, this means that $\mathbf{x}_t \to (\mathbf{x}_t, \mathbf{u}_t(\mathbf{x}_t))$ is a measurable map from $\hat{X}$ to $\mathrm{graph}(U_t)$. Thus, by the induction hypothesis, we know

$$\delta_{[1,t-1]} \to \mathbf{X}_t^{\mathbf{U}_{[1,t-1]}}(\delta_{[1,t-1]}) \to \left(\mathbf{X}_t^{\mathbf{U}_{[1,t-1]}}(\delta_{[1,t-1]}), \mathbf{u}_t(\mathbf{X}_t^{\mathbf{U}_{[1,t-1]}}(\delta_{[1,t-1]}))\right) \tag{27}$$

is a measurable map from $\hat{D}^{t-1}$ to $\mathrm{graph}(U_t)$. But by (25), this means that $(\mathbf{X}_t^{\mathbf{U}_{[1,t-1]}}, \mathbf{U}_t)(\cdot)$ is measurable as a map from $\hat{D}^{t-1}$ to $\mathrm{graph}(U_t)$. Finally, by (4) again, we can obtain that $\mathbf{X}_{t+1}^{\mathbf{U}_{[1,t]}}(\cdot)$ is measurable in $\delta_{[1,t]}$.

To proceed further, we bring up more assumptions.

Assumption 9 *For $t = 1, 2, ..., T-1$ and $\delta \in \hat{D}$, each $c_t(\cdot, \cdot, \delta)$ is lower semi-continuous. That is, when $(\mathbf{x}_k, \mathbf{u}_k)$ in $\mathrm{graph}(U_t)$ converges to $(\mathbf{x}, \mathbf{u}) \in \mathrm{graph}(U_t)$, we have*

$$\liminf_{k \to +\infty} c_t(\mathbf{x}_k, \mathbf{u}_k, \delta) \geq c_t(\mathbf{x}, \mathbf{u}, \delta).$$

Assumption 10 *$c_T^0(\cdot)$ is lower semi-continuous.*

Assumption 11 *For $t = 1, 2, ..., T-1$ and $\delta \in \hat{D}$, each $\mathbf{s}_{t+1}(\cdot, \cdot, \delta)$ is continuous. That is, when $(\mathbf{x}_k, \mathbf{u}_k)$ in $\mathrm{graph}(U_t)$ converges to $(\mathbf{x}, \mathbf{u}) \in \mathrm{graph}(U_t)$, we have*

$$\lim_{k \to +\infty} \mathbf{s}_{t+1}(\mathbf{x}_k, \mathbf{u}_k, \delta) = \mathbf{s}_{t+1}(\mathbf{x}, \mathbf{u}, \delta).$$

Assumption 12 *For $t = 1, 2, ..., T-1$, each $M_t(\cdot, \cdot)$ is lower hemi-continuous. That is, suppose sequence $((\mathbf{x}_k, \mathbf{u}_k) \mid k = 1, 2, ...)$ in $\mathrm{graph}(U_t)$ converges to some $(\mathbf{x}, \mathbf{u}) \in \mathrm{graph}(U_t)$,* *with some $m \in M_t(\mathbf{x}, \mathbf{u})$. Then, for each $k = 1, 2, ...$ there exists $m_k \in M_t(\mathbf{x}_k, \mathbf{u}_k)$, so that the sequence $(m_k \mid k = 1, 2, ...)$ in $\mathbb{M}_t^q$ converges to m in the strong sense, in that*

$$\lim_{k \to +\infty} \|\, m_k - m \,\|^q = 0.$$

The following paves the way for a DP formulation of (19).

Theorem 1 *The problem (19) has an optimal solution with its optimal value achieved at $v_1(\mathbf{x}_1)$ under any given starting state $\mathbf{x}_1 \in \hat{X}$; in turn, $v_1(\cdot)$ results from the following DP:*

$$v_T(\mathbf{x}) = c_T^0(\mathbf{x}), \qquad \forall \mathbf{x} \in \hat{X}, \tag{28}$$

and for $t = T - 1, T - 2, ..., 1$,

$$v_t(\mathbf{x}) = \min_{\mathbf{u} \in U_t(\mathbf{x})} \sup_{m \in M_t(\mathbf{x},\mathbf{u})} \mathbb{E}^m[(c_t + \alpha \cdot v_{t+1} \circ \mathbf{s}_{t+1})(\mathbf{x}, \mathbf{u}, \mathbf{\Delta}_t)], \qquad \forall \mathbf{x} \in \hat{X}. \tag{29}$$

Moreover, an optimal Markov policy $\mathbf{u}^* \in \mathcal{U}^M$ exists and satisfies

$$\mathbf{u}_t^*(\mathbf{x}) \in \mathrm{argmin}_{\,\mathbf{u} \in U_t(\mathbf{x})} \sup_{m \in M_t(\mathbf{x},\mathbf{u})} \mathbb{E}^m[(c_t + \alpha \cdot v_{t+1} \circ \mathbf{s}_{t+1})(\mathbf{x}, \mathbf{u}, \mathbf{\Delta}_t)], \tag{30}$$

for every $t = T - 1, T - 2, ..., 1$ and $\mathbf{x} \in \hat{X}$.

In the proof of Theorem 1, we have borrowed from Ruszczyński [21] the idea of starting with the innermost layer and gradually coming out from it. The DP (28) and (29) could potentially find its use in many situations. Here, especially in Sections 3 to 6, we concentrate on its application to IPC.

3 Inventory and Price Control

3.1 $\mathcal{C}$-convexity and Preservation Properties

In IPC applications, the state space $\hat{X}$ contains inventory levels. Also, the control space $\hat{U}$ is in the form of $\hat{X} \times \hat{Z}$, where $\hat{Z}$ contains price control levers. We can let $\hat{Z}$ be a singleton when it comes to pure inventory control. For $t = 1, 2, ..., T - 1$ and $\mathbf{x} \in \hat{X}$, let the control range be some $U_t(\mathbf{x}) = Y_t(\mathbf{x}) \times Z_t$, so that each control $\mathbf{u} \in U_t(\mathbf{x})$ has two components, a post-ordering inventory component $\mathbf{y} \in Y_t(\mathbf{x}) \subseteq \hat{U}$ and a pricing component $\mathbf{z} \in Z_t \subseteq \hat{Z}$. Also, we let

$$c_t(\mathbf{x}, \mathbf{y}, \mathbf{z}, \delta) = -\bar{\mathbf{c}}_t \cdot \mathbf{x} + \bar{k}_t \cdot \mathbf{1}(\mathbf{y} \neq \mathbf{x}) + \tilde{c}_t(\mathbf{y}, \mathbf{z}, \delta), \tag{31}$$

and

$$\mathbf{s}_{t+1}(\mathbf{x},\mathbf{y},\mathbf{z},\delta)=\tilde{\mathbf{s}}_{t+1}(\mathbf{y},\mathbf{z},\delta). \tag{32}$$

The positive constants $\bar{k}_t$ stand for fixed order setup costs and the functions $\tilde{c}_t$ represent costs related to ordering and inventory holding activities, as well as the negative of revenues from sales. Meanwhile, the functions $\tilde{\mathbf{s}}_{t+1}$ still reflect state transitions.

We adopt a special time-consistent coherent and Markov risk measure with a state-control-independent generator $M_{[1,T-1]}=(M_t|t=1,2,...,T-1)$. The risk measure's state- or control- dependency would destroy most structural results for the current IPC setting. We can now replace Assumptions 8 and 12 with simply the boundedness of each M_t.

The DP (28) and (29) can be simplified, so that (28) is kept intact, while for $t=T-1,T-2,...,1$,

$$v_t(\mathbf{x})=-\bar{\mathbf{c}}_t\cdot\mathbf{x}+u_t(\mathbf{x})\wedge[\bar{k}_t+\min_{\mathbf{y}\in Y_t(\mathbf{x})}u_t(\mathbf{y})],\qquad \forall\mathbf{x}\in\hat{X}, \tag{33}$$

where

$$u_t(\mathbf{y})=\min_{\mathbf{z}\in Z_t}q_t(\mathbf{y},\mathbf{z}),\qquad \forall\mathbf{y}\in\hat{X}, \tag{34}$$

$$q_t(\mathbf{y},\mathbf{z})=\sup_{m\in M_t}w_t^m(\mathbf{y},\mathbf{z}),\qquad \forall\mathbf{y}\in\hat{X},\ \mathbf{z}\in\hat{Z}, \tag{35}$$

and

$$w_t^m(\mathbf{y},\mathbf{z})=\mathbb{E}^m[(\tilde{c}_t+\alpha\cdot v_{t+1}\circ\tilde{\mathbf{s}}_{t+1})(\mathbf{y},\mathbf{z},\mathbf{\Delta}_t)],\qquad \forall m\in M_t,\ \mathbf{y}\in\hat{X},\ \mathbf{z}\in\hat{Z}. \tag{36}$$

Here, $x\wedge y$ means $\min\{x,y\}$ and $x\vee y$ means $\max\{x,y\}$. In the above, (33), (34), and (36) are commonly seen in traditional risk-neutral IPC. Only (35) is new, as in the traditional case, M_t is a singleton containing only the all-one uniform-rate density function.

We now introduce a property relevant to IPC problems and then supply its preservation under (35). Suppose $\hat{W}$ is a convex subset of $\Re^j$ for some positive integer j. Let $\mathcal{C}$ be a mapping of the nature $\Re^2\times\hat{W}^2\times[0,1]\to\Re$ that is increasing in its first two arguments. For real-valued function f defined on $\hat{W}$, we say it is $\mathcal{C}$-convex when for any $\mathbf{w}^0,\mathbf{w}^1\in\hat{W}$ and $\lambda\in[0,1]$,

$$f((1-\lambda)\mathbf{w}^0+\lambda\mathbf{w}^1)\le\mathcal{C}(f(\mathbf{w}^0),f(\mathbf{w}^1),\mathbf{w}^0,\mathbf{w}^1,\lambda).$$

Note that $\mathcal{C}$-convexity would be symmetric $\bar{k}$-convexity for a given positive parameter $\bar{k}$, as proposed by Chen and Simchi-Levi [6], when

$$\mathcal{C}(f^0,f^1,\mathbf{w}^0,\mathbf{w}^1,\lambda)=(1-\lambda)\cdot f^0+\lambda\cdot f^1+\max\{\lambda,1-\lambda\}\cdot\bar{k},$$

and $\bar{k}$-convexity when

$$\mathcal{C}(f^0,f^1,w^0,w^1,\lambda)=(1-\lambda)\cdot[f^0+\bar{k}\cdot\mathbf{1}(w^0>w^1)]+\lambda\cdot[f^1+\bar{k}\cdot\mathbf{1}(w^0<w^1)].$$

The following lemma says that $\mathcal{C}$-convexity is preserved through a (35)-like operator.

Lemma 2 *Let I be an arbitrary index set. For each $i \in I$, suppose $f(\cdot, i)$ is a $\mathcal{C}$-convex function from $\hat{W}$ to $\Re$. Let $g(\mathbf{w}) = \sup_{i\in I} f(\mathbf{w}, i)$ be real-valued. Then, $g(\mathbf{w})$ is $\mathcal{C}$-convex from $\hat{W}$ to $\Re$ as well.*

Caution is also required of the preservation through (36), though it is relatively better known. A special case of $\mathcal{C}$-convexity is c-convexity when for a real-valued function c defined on $\hat{W}^2 \times [0, 1]$,

$$\mathcal{C}(f^0, f^1, \mathbf{w}^0, \mathbf{w}^1, \lambda) = (1-\lambda)\cdot f^0 + \lambda \cdot f^1 + c(\mathbf{w}^0, \mathbf{w}^1, \lambda).$$

By letting $c(\mathbf{w}^0, \mathbf{w}^1, \lambda) = \max\{\lambda, 1-\lambda\} \cdot \bar{k}$ and $c(w^0, w^1, \lambda) = (1-\lambda)\cdot \bar{k} \cdot \mathbf{1}(w^0 > w^1) + \lambda \cdot \bar{k} \cdot \mathbf{1}(w^0 < w^1)$, respectively, we can recover symmetric $\bar{k}$-convexity and $\bar{k}$-convexity.

Lemma 3 *Let $(\Omega^0, \mathcal{F}^0, P^0)$ be an arbitrary probability space. Suppose $f : \hat{W} \times \Omega^0 \to \Re$ is such that each $f(\mathbf{w}, \cdot)$ is integrable and each $f(\cdot, \omega)$ is c-convex. Let $g(\mathbf{w}) = \int_{\Omega^0} f(\mathbf{w}, \omega) \cdot P^0(d\omega)$. Then, $g(\mathbf{w})$ is c-convex on $\hat{W}$ as well.*

In the remainder of this section, we treat both pure inventory and joint inventory-price control problems that involve one single ordering stage, zero ordering lead time, and backlogging of unsatisfied demands. There are positive unit holding cost rates $\bar{h}_t$ and positive unit backlogging cost rates $\bar{b}_t$ for periods $t = 1, 2, ..., T-1$. For convenience, we let $\tilde{h}_t(x) = \bar{h}_t \cdot x^+ + \bar{b}_t \cdot x^-$. Also, default demand distributions R_t satisfy Assumption 6.

3.2 Pure Inventory Control

Here, we consider a firm that orders periodically to keep up with random demand. In every period t, it raises its inventory from the starting level x to an order-up-to level y at the cost $\bar{k}_t \cdot \mathbf{1}(y > x) + \bar{c}_t \cdot (y - x)$. Then, the demand level δ is realized, which causes the firm to either carry a $(y-\delta)^+$ quantity in holding to the next period at the cost $\bar{h}_t \cdot (y-\delta)^+$ or owe a backlog of $(y-\delta)^-$ to the next period at the cost $\bar{b}_t \cdot (y-\delta)^-$. The firm's starting inventory level for period $t+1$ is simply $y - \delta$.

Formally, $\hat{X} = \bar{\Re}$, $\hat{Z}$ is an ignorable singleton, and $\hat{D} = \Re^+$. For every $t = 1, 2, ..., T-1$, we have $\tilde{c}_t(y, \delta) = \bar{c}_t \cdot y + \tilde{h}_t(y - \delta)$, $Y_t(x) = [x, +\infty]$, $Z_t = \hat{Z}$, and $\tilde{s}_{t+1}(y, \delta) = y - \delta$. Finally, $c^0_T(x) = -\bar{c}_T \cdot x$ for some positive unit ordering cost $\bar{c}_T$.

For this setting, (33) to (36) can be written as

$$v_t(x) = -\bar{c}_t \cdot x + u_t(x) \wedge [\bar{k}_t + \min_{y \in [x, +\infty]} u_t(y)], \tag{37}$$

where

$$u_t(y) = \bar{c}_t \cdot y + \sup_{m \in M_t} w_t^m(y), \tag{38}$$

and

$$w_t^m(y) = \mathbb{E}^m[(\tilde{h}_t + \alpha \cdot v_{t+1})(y - \Delta_t)]. \tag{39}$$

Based on the above formulation, we can understand M_t as containing all potential demand densities perceivable by the firm. The latter in turn makes decisions in anticipation of the worst outcome.

Note $[x, +\infty]$ serving as $Y_t(x)$ is in violation of Assumption 5. To ensure that Theorem 1's conclusion is still valid, we make the following assumptions.

IPC Assumption 1 $\bar{c}_t + \bar{h}_t - \alpha \bar{c}_{t+1} > 0$ *for* $t = 1, 2, ..., T-1$.

IPC Assumption 2 *Distributions* m *in* M_t *have uniformly small tails. That is, for any* $t = 1, 2, ..., T-1$ *and* $\epsilon > 0$, *there exists* $\delta \in \Re^+$, *such that for any* $m \in M_t$,

$$\mathbb{E}^m[\mathbf{1}(\Delta_t \geq \delta)] = \int_\delta^{+\infty} m(\delta') \cdot R_t(d\delta') < \epsilon.$$

IPC Assumption 1 appears often in the inventory control literature; see, e.g., Veinott [29] and Gavirneni [13]. It says that there is no speculative motive for the firm to order at an earlier time than needed. Technically, it promulgates that unit ordering costs do not rise too fast over time. Meanwhile, IPC Assumption 2 just specifies that predictions on demand are not outlandish. The following shows that these conditions would usher in upper bounds for order-up-to levels, which would in turn validate the DP from (37) to (39).

Lemma 4 *Consider the DP made up of (28) for* $t = T$, *as well as (37) to (39) for* $t = T-1, T-2, ..., 1$. *Then, Assumptions 1 to 12, except Assumption 5, are all true. Moreover, for* $t = T, T-1, ..., 1$, $v_t(\cdot)$ *has an increasing rate above* $-(\sum_{\tau=t}^{T-1} \alpha^{\tau-t} \cdot \bar{b}_\tau + \alpha^{T-t} \cdot \bar{c}_T)$, *while* $(\bar{c}_t + v_t)(x)$ *has a positive increasing rate when* x *is large enough. Effectively, Assumption 5 can be supplanted, and the conclusion of Theorem 1 would remain valid for the current DP.*

In Lemma 4, the lower bound on $u_t(\cdot)$'s increasing rate means that any extra item in period t is worth at most $\sum_{\tau=t}^{T-1} \alpha^{\tau-t} \cdot \bar{b}_\tau + \alpha^{T-t} \cdot \bar{c}_T$ to the firm. With demand being always finite, the firm with a post-ordering inventory level above $-\infty$ in a period would end the period with an inventory level above $-\infty$. Also, with Lemma 4's upper bound on the optimal ordering range, the firm with a starting inventory level below $+\infty$ would have its post-ordering inventory level below $+\infty$ as well. Hence, we can treat $x, y \in \Re$ as opposed to $x, y \in \bar{\Re}$ in (28) and (37) to (39).

Let us make an additional assumption.

IPC Assumption 3 $\bar{k}_1 \geq \alpha\bar{k}_2 \geq \cdots \geq \alpha^{T-2}\bar{k}_{T-1} \geq \alpha^{T-1}\bar{k}_T = 0$.

This assumption states that setup costs do not rise too fast over time. It is standard for $\bar{k}$-convexity and (s,S) policies under non-stationary settings. We can now characterize optimal ordering policies by capitalizing on results in Section 3.1.

Proposition 1 *Consider the DP treated in Lemma 4. Then, for $t = T, T-1, ..., 1$, $v_t(\cdot)$ is $\bar{k}_t$-convex. Consequently, for $t = T-1, T-2, ..., 1$, one optimal solution $y_t^*(x)$ to (37) is of the (s,S)-form $x + (S_t - x)\cdot \mathbf{1}(x < s_t)$ for some constants s_t and S_t satisfying $s_t \leq S_t$.*

Proof: By (28), $v_T(x) = c_T^0(x) = -\bar{c}_T \cdot x$ is obviously $\bar{k}_T$-convex in x. Suppose for some $t = T-1, T-2, ..., 1$, we know that $v_{t+1}(\cdot)$ is $\bar{k}_{t+1}$-convex. Then, from this, (39), and Lemma 3, we can obtain the $\alpha\bar{k}_{t+1}$-convexity of $w_t^m(\cdot)$. Due to IPC Assumption 3, this means that $w_t^m(\cdot)$ is $\bar{k}_t$-convex. This would then lead to the $\bar{k}_t$-convexity of $u_t(\cdot)$ due to (38) and Lemma 2.

Finally, due to (37) and known inventory literature such as Porteus [17], we can let $S_t = \inf\ \text{argmin}\{u_t(y) \mid y \in \Re\}$, $s_t = \inf\{y \in (-\infty, S_t] \mid u_t(y) \leq \bar{k}_t + u_t(S_t)\}$, $y_t^*(x) = x + (S_t - x)\cdot \mathbf{1}(x < s_t)$, and make sure that $v_t(x) = -\bar{c}_t \cdot x + \bar{k}_t \cdot \mathbf{1}(y_t^*(x) > x) + u_t(y_t^*(x))$ is $\bar{k}_t$-convex in x. The induction process is thus completed. ∎

Proposition 1 makes the same conclusion for the current risk-averse case as Scarf [24] had done for the risk-neutral case. Of course, the actual s_t and S_t levels depend on the particular density sets $M_{t'}$ adopted by the firm. When there is no fixed setup cost, all concerned functions are 0-convex and hence convex. An optimal policy will be base-stock with $y_t^*(x) = S_t \vee x$, where S_t is just as defined in the above proof. This is in agreement with Karlin [14] for the risk-neutral case.

3.3 Joint Inventory-price Control

In this case involving pricing, we inherit much of the same developments from Section 3.2. The only notable difference is we now allow demand to be controlled by the firm's own pricing. Suppose it can choose sales price p from a given interval $[\underline{p}_t, \overline{p}_t]$; also, there exist continuous and strictly decreasing function $\tilde{z}_t(\cdot)$ and random variable Θ_t, so that the random demand in period t follows the form

$$\Delta_t(p) = \tilde{z}_t(p) + \Theta_t. \tag{40}$$

It will be more convenient to treat the demand lever $z = \tilde{z}_t(p)$ as being under control. For that matter, we let $\tilde{p}_t(\cdot)$ be the inverse function of $\tilde{z}_t(\cdot)$, $\underline{z}_t = \tilde{z}_t(\overline{p}_t)$, and $\overline{z}_t = \tilde{z}_t(\underline{p}_t)$.

Model primitives for the current case include $\hat{X} = \bar{\Re}$, $\hat{Z} = [\min_{t=1}^{T-1} \underline{z}_t, \max_{t=1}^{T-1} \overline{z}_t]$, and $\hat{D} = \Re$. We let each $Y_t(x) = [x, +\infty]$ and each $Z_t = [\underline{z}_t, \overline{z}_t]$. As mentioned before, we shall write $\mathbf{u} = (y, z)$, so that y continues to be the order-up-to level but z stands for the demand lever. For every $t = 1, 2, ..., T-1$, suppose

$$\tilde{c}_t(y, z, \theta) = -\tilde{p}_t(z) \cdot (z + \theta) + \bar{c}_t \cdot y + \tilde{h}_t(y - z - \theta), \tag{41}$$

and

$$\tilde{s}_{t+1}(y, z, \theta) = y - z - \theta. \tag{42}$$

We still have $c_T^0(x) = -\bar{c}_T \cdot x$. In (41), the first term on the right-hand side is the negative of the revenue from sales, the second term reflects ordering cost, and the third term stems from inventory holding-backlogging. Both this last term and (42) follow from (40), indicating that the realized demand is $z + \theta$. Define revenue function $\tilde{r}_t^m(z)$, so that

$$\tilde{r}_t^m(z) = \tilde{p}_t(z) \cdot \mathbb{E}^m[z + \Theta_t] = \tilde{p}_t(z) \cdot z + \mathbb{E}^m[\Theta_t] \cdot \tilde{p}_t(z). \tag{43}$$

We assume that $\tilde{r}_t^m(\cdot)$ is concave at every $t = 1, 2, ..., T-1$ and $m \in M_t$. This concavity assumption is already common for risk-neutral joint control; see Federgruen and Heching [11] and Chen and Simchi-Levi [6].

For this setting, (33) to (36) can be written as (37), but with

$$u_t(y) = \bar{c}_t \cdot y + \min_{z \in [\underline{z}_t, \overline{z}_t]} q_t(y, z), \tag{44}$$

where

$$q_t(y, z) = \sup_{m \in M_t} w_t^m(y, z), \tag{45}$$

and

$$w_t^m(y, z) = -\tilde{r}_t^m(z) + \mathbb{E}^m[(\tilde{h}_t + \alpha \cdot v_{t+1})(y - z - \Theta_t)]. \tag{46}$$

We inherit IPC Assumptions 1 and 3, but update IPC Assumption 2 to the following.

IPC Assumption 4 *Distributions m in M_t have uniformly small tails. That is, for any $t = 1, 2, ..., T-1$ and $\epsilon > 0$, there exists $\delta \in \Re^+$, such that for any $m \in M_t$,*

$$\mathbb{E}^m[\mathbf{1}(\Theta_t \geq \delta)] = \int_\delta^{+\infty} m(\theta) \cdot R_t(d\theta) < \epsilon.$$

We can again derive a finite upper bound for the order-up-to level.

Lemma 5 *Consider the DP made up of (28) for $t = T$, as well as (37) and (44) to (46) for $t = T-1, T-2, ..., 1$. Then, Assumptions 1 to 12, except Assumption 5, are all true.*

Moreover, for $t = T, T-1, ..., 1$, $v_t(\cdot)$ has an increasing rate above $-(\sum_{\tau=t}^{T-1} \alpha^{\tau-t} \cdot \bar{b}_\tau + \alpha^{T-t} \cdot \bar{c}_T)$, while $(\bar{c}_t + v_t)(x)$ has a positive increasing rate when x is large enough. Effectively, Assumption 5 can be supplanted, and the conclusion of Theorem 1 would remain valid for the current DP.

With Lemma 5, we can as before treat $x, y \in \Re$ as opposed to $x, y \in \bar{\Re}$ in the DP. Policy characterization then follows.

Proposition 2 *Consider the DP treated in Lemma 5. Then, for $t = T, T-1, ..., 1$, $v_t(\cdot)$ is symmetric $\bar{k}_t$-convex. Consequently, for $t = T-1, T-2, ..., 1$, there exist constants s_t and S_t with $s_t \leq S_t$ and a set $A_t \subseteq [s_t, (s_t + S_t)/2]$, so that one optimal solution $y_t^*(x)$ to (37) is equal to $x + (S_t - x) \cdot \mathbf{1}(x < s_t \text{ or } x \in A_t)$.*

Proof: By (28), $v_T(x) = c_T^0(x) = -\bar{c}_T \cdot x$ is obviously symmetric $\bar{k}_T$-convex in x. Suppose for some $t = T-1, T-2, ..., 1$, we know that $v_{t+1}(\cdot)$ is symmetric $\bar{k}_{t+1}$-convex. Then, from this, (46), the concavity of $\tilde{r}_t^m(\cdot)$, and Lemma 3, we can obtain the symmetric $\alpha\bar{k}_{t+1}$-convexity of $w_t^m(y, z)$ in (y, z). By IPC Assumption 3, we have the symmetric $\bar{k}_t$-convexity of $w_t^m(\cdot, \cdot)$. Using (45) and Lemma 2, we can obtain the symmetric $\bar{k}_t$-convexity of $q_t(y, z)$ in (y, z).

Given y^0, y^1, suppose $z^0, z^1 \in [\underline{z}_t, \overline{z}_t]$ minimizes $q_t(y^0, \cdot)$ and $q_t(y^1, \cdot)$, respectively. For any $\lambda \in [0, 1]$, we have $(1-\lambda)z^0 + \lambda z^1 \in [\underline{z}_t, \overline{z}_t]$. Hence due to (44) and the symmetric $\bar{k}_t$-convexity of $q_t(\cdot, \cdot)$,

$$\begin{aligned} u_t((1-\lambda)y^0 + \lambda y^1) &\leq \bar{c}_t \cdot ((1-\lambda)y^0 + \lambda y^1) + q_t((1-\lambda)y^0 + \lambda y^1, (1-\lambda)z^0 + \lambda z^1) \\ &\leq (1-\lambda) \cdot [\bar{c}_t \cdot y^0 + q_t(y^0, z^0)] + \lambda \cdot [\bar{c}_t \cdot y^1 + q_t(y^1, z^1)] + \max\{1-\lambda, \lambda\} \cdot \bar{k}_t \\ &= (1-\lambda) \cdot u_t(y^0) + \lambda \cdot u_t(y^1) + \max\{1-\lambda, \lambda\} \cdot \bar{k}_t. \end{aligned} \tag{47}$$

Therefore, $u_t(\cdot)$ is symmetric $\bar{k}_t$-convex.

Finally, due to (37) and Chen and Simchi-Levi [6], we can let $S_t = \inf\ \text{argmin}\{u_t(y) \mid y \in \Re\}$ and $s_t = \inf\{y \in (-\infty, S_t] \mid u_t(y) \leq \bar{k}_t + u_t(S_t)\}$, and also identify some $A_t \subseteq [s_t, (s_t + S_t)/2]$. We can then let $y_t^*(x) = x + (S_t - x) \cdot \mathbf{1}(x < s_t \text{ or } x \in A_t)$, and make sure that $v_t(x) = -\bar{c}_t \cdot x + \bar{k}_t \cdot \mathbf{1}(y_t^*(x) > x) + u_t(y_t^*(x))$ is symmetric $\bar{k}_t$-convex in x. We can thus complete the induction process. ∎

Regarding inventory policy, Proposition 2 draws the same conclusion for the current risk-averse case as Chen and Simchi-Levi [6] did for the risk-neutral case. Again, actual policy parameters would depend on the particular density sets $M_{t'}$ taken by the firm. It is worth noting that the proposition's proof uses the notion of symmetric $\bar{k}$-convexity for two-dimensional functions as well as one-dimensional ones. Like convexity but unlike $\bar{k}$-convexity, the extension of $\bar{k}$-convexity, originally proposed by Chen and Simchi-Levi [6] for single-dimensional functions, is quite natural.

When there is no fixed setup cost, all concerned functions are symmetric 0-convex and hence jointly convex or simply convex. An optimal policy will be base-stock with $y_t^*(x) = S_t \vee x$, where S_t is just as defined in the above proof. This is not different from that provided by Federgruen and Heching [11] for the risk-neutral case.

4 Lattice Risk Sets and Monotone Pricing Trends

When there is no fixed setup cost and the risk set possesses a lattice-like property, we show that monotone pricing trends would emerge.

4.1 The Setup

When there is no fixed setup cost, we are guaranteed by Proposition 2 to have a convex cost-to-go function $v_{t+1}(\cdot)$. Let us carry out the transformation $\tilde{w}_t^m(\upsilon, z) = w_t^m(-\upsilon, z)$. By (46),

$$\tilde{w}_t^m(\upsilon, z) = -\tilde{r}_t^m(z) + \mathbb{E}^m[(\tilde{h}_t + \alpha \cdot v_{t+1})(-\upsilon - z - \Theta_t)]. \tag{48}$$

The supermodularity of $\tilde{w}_t^m(\cdot, \cdot)$ is then easy to obtain.

Lemma 6 *$\tilde{w}_t^m(\upsilon, z)$ is supermodular in $(\upsilon, z) \in \Re \times [\underline{z}_t, \overline{z}_t]$.*

Proof: By (48), we have $\tilde{w}_t^m(\upsilon, z) = -\tilde{r}_t^m(z) + \mathbb{E}^m[f(-\upsilon - z - \Theta_t)]$, where $f(x) = (\tilde{h}_t + \alpha \cdot v_{t+1})(x)$. Note that $f(\cdot)$ is convex. So, for $\Delta\upsilon, \Delta z \geq 0$,

$$\begin{aligned} &\tilde{w}_t^m(\upsilon + \Delta\upsilon, z + \Delta z) + \tilde{w}_t^m(\upsilon, z) - \tilde{w}_t^m(\upsilon + \Delta\upsilon, z) - \tilde{w}_t^m(\upsilon, z + \Delta z) \\ &\quad = \mathbb{E}^m[f(-\upsilon - z - \Theta_t - \Delta\upsilon - \Delta z)] + \mathbb{E}^m[f(-\upsilon - z - \Theta_t)] \\ &\qquad -\mathbb{E}^m[f(-\upsilon - z - \Theta_t - \Delta\upsilon)] - \mathbb{E}^m[f(-\upsilon - z - \Theta_t - \Delta z)] \geq 0. \end{aligned} \tag{49}$$

Thus, $\tilde{w}_t^m(\upsilon, z)$ is supermodular in (υ, z). ∎

Define $\tilde{q}_t(\upsilon, z) = q_t(-\upsilon, z)$, so that following (45),

$$\tilde{q}_t(\upsilon, z) = q_t(-\upsilon, z) = \sup_{m \in M_t} w_t^m(-\upsilon, z) = \sup_{m \in M_t} \tilde{w}_t^m(\upsilon, z). \tag{50}$$

From (44) and (50), it is quite clear that monotone trends of $z_t^*(\cdot)$ would come out of modularity properties of $\tilde{q}_t(\cdot, \cdot)$. However, even with the supermodularity of $\tilde{w}_t^m(\cdot, \cdot)$ as given in Lemma 6, that of $\tilde{q}_t(\cdot,\cdot)$ is not guaranteed if either $\tilde{w}_t^m(v,z)$ possesses no further properties in terms of (m,v) and (m,z) or M_t is not well structured.

To see the former, let $f:\{1,2\}^3\rightarrow\Re$ be such that

$$f(u,v,1)=\left(\begin{array}{cc} 3 & 1.9 \\ 2 & 1 \end{array}\right), \qquad f(u,v,2)=\left(\begin{array}{cc} 2.9 & 2.1 \\ 1.6 & 0.9 \end{array}\right). \tag{51}$$

Also, let $g(u,v)=\sup_{i=1,2} f(u,v,i)$. Then, we have

$$g(u,v)=\left(\begin{array}{cc} 3 & 2.1 \\ 2 & 1 \end{array}\right). \tag{52}$$

It is clear that both $f(u,v,1)$ and $f(u,v,2)$ are supermodular in (u,v); yet, $g(u,v)$ is submodular in (u,v).

4.2 Lattice-like Risk Sets

For a density m^0 defined for any arbitrary probability space $(\Omega^0,\mathcal{F}^0,P^0)$, let us define distribution $\int m^0\cdot dP^0$ on the measurable space $(\Omega^0,\mathcal{F}^0)$ through

$$\left(\int m^0\cdot dP^0\right)(\Omega')=\int_{\Omega'} m^0(\omega^0)\cdot P^0(d\omega^0), \qquad \forall\Omega'\in\mathcal{F}^0. \tag{53}$$

According to Shaked and Shanthikumar [26] (Section 1.A), two probability measures Q^1 and Q^2 on $(\Re,\mathcal{B}(\Re))$ are ranked as $Q^1\leq Q^2$ in the usual stochastic sense when

$$Q^1([\theta,+\infty))\leq Q^2([\theta,+\infty)), \qquad \forall\theta\in\Re. \tag{54}$$

We can give a partial order to $\mathbb{M}_t^q$, so that $m^1\leq m^2$ for $m^1,m^2\in\mathbb{M}_t^q$ when

$$\int m^1\cdot dR_t\leq\int m^2\cdot dR_t \qquad \text{in the usual stochastic sense.} \tag{55}$$

For $m^1,m^2\in\mathbb{M}_t^q$, we can define m so that

$$\begin{aligned} m(\theta)=m^1(\theta)\cdot\mathbf{1}(\textstyle\int_\theta^{+\infty} m^1(\theta')\cdot R_t(d\theta')>\int_\theta^{+\infty} m^2(\theta')\cdot R_t(d\theta')) \\ +m^2(\theta)\cdot\mathbf{1}(\textstyle\int_\theta^{+\infty} m^1(\theta')\cdot R_t(d\theta')\leq\int_\theta^{+\infty} m^2(\theta')\cdot R_t(d\theta')). \end{aligned} \tag{56}$$

The thus defined m is a member of $\mathbb{M}_t^q$, and $m=m^1\vee m^2$, the least upper bound for both m^1 and m^2: for any $\theta\in\Re$,

$$\int_\theta^{+\infty} m(\theta')\cdot R_t(d\theta')=\left[\int_\theta^{+\infty} m^1(\theta')\cdot R_t(d\theta')\right]\bigvee\left[\int_\theta^{+\infty} m^2(\theta')\cdot R_t(d\theta')\right]. \tag{57}$$

This means that $m = m^1 \vee m^2 \in \mathbb{M}_t^q$. Symmetrically, we can verify that $m^1 \wedge m^2$, the greatest lower bound for m^1 and m^2, belongs to $\mathbb{M}_t^q$ as well. Therefore, $\mathbb{M}_t^q$ under the partial order specified in (55) is a lattice.

For any $m \in \mathbb{M}_t^q$, let us define $\tilde{\Theta}_t^m$ as a Borel-measurable mapping from $[0,1]$ to $\Re$, so that for any $\lambda \in [0,1]$,

$$\tilde{\Theta}_t^m(\lambda) = \inf\left\{\theta \in \Re \mid \left(\int m \cdot dR_t\right)((-\infty,\theta]) = \int_{-\infty}^{\theta} m(\theta') \cdot R_t(d\theta') \geq \lambda\right\}. \tag{58}$$

The defined entity is basically the λ-quantile of $\int m \cdot dR_t$. For $f \in \mathbb{L}^p(\Re, \mathcal{B}(\Re), R_t)$,

$$\mathbb{E}^m[f(\Theta_t)] = \int_0^1 f(\tilde{\Theta}_t^m(\lambda)) \cdot d\lambda. \tag{59}$$

In addition, $m^1 \leq m^2$ is translatable to

$$\tilde{\Theta}_t^{m^1}(\lambda) \leq \tilde{\Theta}_t^{m^2}(\lambda), \qquad \forall \lambda \in [0,1]. \tag{60}$$

Hence, for every $\lambda \in [0,1]$,

$$\tilde{\Theta}_t^{m^1 \wedge m^2}(\lambda) = \tilde{\Theta}_t^{m^1}(\lambda) \wedge \tilde{\Theta}_t^{m^2}(\lambda), \qquad \tilde{\Theta}_t^{m^1 \vee m^2}(\lambda) = \tilde{\Theta}_t^{m^1}(\lambda) \vee \tilde{\Theta}_t^{m^2}(\lambda). \tag{61}$$

We have the following useful result.

Lemma 7 *$\tilde{r}_t^m(z)$ is submodular in $(m,z) \in \mathbb{M}_t^q \times [\underline{z}_t, \overline{z}_t]$.*

Proof: For any $m^1, m^2 \in \tilde{M}_t^q$ and $z^1, z^2 \in [\underline{z}_t, \overline{z}_t]$ which, without loss of generality, is assumed to satisfy $z^1 \leq z^2$, we have from (43), (59), and (61) that

$$\begin{array}{l} \tilde{r}_t^{m^1 \vee m^2}(z^1 \vee z^2) + \tilde{r}_t^{m^1 \wedge m^2}(z^1 \wedge z^2) - \tilde{r}_t^{m^1}(z^1) - \tilde{r}_t^{m^2}(z^2) \\ \quad = \tilde{r}_t^{m^1 \vee m^2}(z^2) + \tilde{r}_t^{m^1 \wedge m^2}(z^1) - \tilde{r}_t^{m^1}(z^1) - \tilde{r}_t^{m^2}(z^2) \\ \quad = \mathbb{E}^{m^1 \vee m^2}[\Theta_t] \cdot \tilde{p}_t(z^2) + \mathbb{E}^{m^1 \wedge m^2}[\Theta_t] \cdot \tilde{p}_t(z^1) - \mathbb{E}^{m^1}[\Theta_t] \cdot \tilde{p}_t(z^1) - \mathbb{E}^{m^2}[\Theta_t] \cdot \tilde{p}_t(z^2), \end{array} \tag{62}$$

which is equal to $\int_0^1 a(\lambda) \cdot d\lambda$ with

$$\begin{array}{c} a(\lambda) = [\tilde{\Theta}_t^{m^1}(\lambda) \vee \tilde{\Theta}_t^{m^2}(\lambda)] \cdot \tilde{p}_t(z^2) + [\tilde{\Theta}_t^{m^1}(\lambda) \wedge \tilde{\Theta}_t^{m^2}(\lambda)] \cdot \tilde{p}_t(z^1) \\ -\tilde{\Theta}_t^{m^1}(\lambda) \cdot \tilde{p}_t(z^1) - \tilde{\Theta}_t^{m^2}(\lambda) \cdot \tilde{p}_t(z^2). \end{array} \tag{63}$$

At a $\lambda \in [0,1]$ with $\tilde{\Theta}_t^{m^1}(\lambda) \leq \tilde{\Theta}_t^{m^2}(\lambda)$, we have $a(\lambda) = 0$. Otherwise,

$$a(\lambda) = [\tilde{\Theta}_t^{m^1}(\lambda) - \tilde{\Theta}_t^{m^2}(\lambda)] \cdot [\tilde{p}_t(z^2) - \tilde{p}_t(z^1)]. \tag{64}$$

Since $\tilde{p}_t(\cdot)$ is decreasing, $\tilde{p}_t(z^2) - \tilde{p}_t(z^1)$ must be negative. Hence, $a(\lambda) \leq 0$ here. After integration, we will have (62)'s negativity and hence the submodularity of $\tilde{r}_t^m(z)$ in (m,z). ∎

Here is a result that can help the derivation of the later Lemmas 9 and 11.

Lemma 8 *Suppose $i(\cdot)$ is a convex function defined on some real interval $[\underline{x}, \overline{x}]$ and $j_t^m(x) = \mathbb{E}^m[i(x+\Theta_t)]$ for any t, $m \in \mathbb{M}_t^q$, and $x \in [\underline{x}, \overline{x}]$. Then, $j_t^m(x)$ is supermodular in (m, x).*

Proof: For any t, any $m^1, m^2 \in \mathbb{M}_t^q$, $x \in [\underline{x}, \overline{x}]$, and positive Δx satisfying $x + \Delta x \leq \overline{x}$,

$$\begin{aligned} & j_t^{m^1 \vee m^2}(x+\Delta x) + j_t^{m^1 \wedge m^2}(x) - j_t^{m^1}(x) - j_t^{m^2}(x+\Delta x) \\ & = \mathbb{E}^{m^1 \vee m^2}[i(x+\Delta x+\Theta_t)] + \mathbb{E}^{m^1 \wedge m^2}[i(x+\Theta_t)] \\ & \quad -\mathbb{E}^{m^1}[i(x+\Theta_t)] - \mathbb{E}^{m^2}[i(x+\Delta x+\Theta_t)], \end{aligned} \tag{65}$$

which, according to (59) and (61), is equal to $\int_0^1 a(\lambda) \cdot d\lambda$ with

$$\begin{aligned} a(\lambda) = & \, i(x+\Delta x+\tilde{\Theta}_t^{m^1}(\lambda) \vee \tilde{\Theta}_t^{m^2}(\lambda)) + i(x+\tilde{\Theta}_t^{m^1}(\lambda) \wedge \tilde{\Theta}_t^{m^2}(\lambda)) \\ & -i(x+\tilde{\Theta}_t^{m^1}(\lambda)) - i(x+\Delta x+\tilde{\Theta}_t^{m^2}(\lambda)). \end{aligned} \tag{66}$$

At a $\lambda \in [0,1]$ with $\tilde{\Theta}_t^{m^1}(\lambda) \leq \tilde{\Theta}_t^{m^2}(\lambda)$, we have $a(\lambda) = 0$. Otherwise,

$$a(\lambda) = i(x+\Delta x+\tilde{\Theta}_t^{m^1}(\lambda)) + i(x+\tilde{\Theta}_t^{m^2}(\lambda)) - i(x+\tilde{\Theta}_t^{m^1}(\lambda)) - i(x+\Delta x+\tilde{\Theta}_t^{m2}(\lambda)), \tag{67}$$

which is positive due to the convexity of $i(\cdot)$. After integration, we will have (65)'s positivity and hence the supermodularity of $j_t^m(x)$ in (m, x). ∎

Now we can claim that $\tilde{w}_t^m(\upsilon, z)$ is also supermodular in both (m, υ) and (m, z).

Lemma 9 *$\tilde{w}_t^m(\upsilon, z)$ is supermodular in $(m, \upsilon, z) \in \mathbb{M}_t^q \times \Re \times [\underline{z}_t, \overline{z}_t]$.*

Proof: Note the function $f(x) = (\tilde{h}_t + \alpha \cdot v_{t+1})(-x)$ is convex. Meanwhile, (48) could be understood as

$$\tilde{w}_t^m(\upsilon, z) = -\tilde{r}_t^m(z) + \tilde{u}_t^m(\upsilon, z), \quad \text{with} \quad \tilde{u}_t^m(\upsilon, z) = \mathbb{E}^m[f(\upsilon + z + \Theta_t)]. \tag{68}$$

At a fixed $z \in [\underline{z}_t, \overline{z}_t]$, we see that $\tilde{u}_t^m(\upsilon, z) = \mathbb{E}^m[i(\upsilon + \Theta_t)]$ with $i(x) = f(x+z)$. The latter is convex since $f(\cdot)$ is. So by Lemma 8, we know that $\tilde{u}_t^m(\upsilon, z)$ is supermodular in (m, υ). At a fixed $\upsilon \in \Re$, we see that $\tilde{u}_t^m(\upsilon, z) = \mathbb{E}^m[i(z+\Theta_t)]$ with $i(x) = f(\upsilon + x)$. The latter is convex since $f(\cdot)$ is. So by Lemma 8, we know that $\tilde{u}_t^m(\upsilon, z)$ is supermodular in (m, z).

From Lemma 6, we know that $\tilde{w}_t^m(\upsilon, z)$ is supermodular in (υ, z). Also, in view of Lemma 7 and the properties just proved, we can see that $\tilde{w}_t^m(\upsilon, z)$ is supermodular in both (m, υ) and (m, z). Therefore, $\tilde{w}_t^m(\upsilon, z)$ is supermodular in each of its three arguments, and also has increasing differences in (m, υ, z). So following Theorem 2.6.2 of Topkis [28], we can obtain that $\tilde{w}_t^m(v,z)$ is supermodular in (m,v,z). For details, note that

$$
\begin{aligned}
&\tilde{w}_t^{m_1}(v_1,z_1)-\tilde{w}_t^{m_1\wedge m_2}(v_1\wedge v_2,z_1\wedge z_2)\\
&\quad=\tilde{w}_t^{m_1}(v_1,z_1)-\tilde{w}_t^{m_1}(v_1,z_1\wedge z_2)\\
&\qquad\qquad+\tilde{w}_t^{m_1}(v_1,z_1\wedge z_2)-\tilde{w}_t^{m_1}(v_1\wedge v_2,z_1\wedge z_2)\\
&\qquad\qquad+\tilde{w}_t^{m_1}(v_1\wedge v_2,z_1\wedge z_2)-\tilde{w}_t^{m_1\wedge m_2}(v_1\wedge v_2,z_1\wedge z_2)\\
&\quad\leq\tilde{w}_t^{m_1\vee m_2}(v_1\vee v_2,z_1)-\tilde{w}_t^{m_1\vee m_2}(v_1\vee v_2,z_1\wedge z_2)\\
&\qquad\qquad+\tilde{w}_t^{m_1\vee m_2}(v_1,z_2)-\tilde{w}_t^{m_1\vee m_2}(v_1\wedge v_2,z_2)\\
&\qquad\qquad+\tilde{w}_t^{m_1}(v_2,z_2)-\tilde{w}_t^{m_1\wedge m_2}(v_2,z_2)\\
&\quad\leq\tilde{w}_t^{m_1\vee m_2}(v_1\vee v_2,z_1\vee z_2)-\tilde{w}_t^{m_1\vee m_2}(v_1\vee v_2,z_2)\\
&\qquad\qquad+\tilde{w}_t^{m_1\vee m_2}(v_1\vee v_2,z_2)-\tilde{w}_t^{m_1\vee m_2}(v_2,z_2)\\
&\qquad\qquad+\tilde{w}_t^{m_1\vee m_2}(v_2,z_2)-\tilde{w}_t^{m_2}(v_2,z_2)\\
&\quad=\tilde{w}_t^{m_1\vee m_2}(v_1\vee v_2,z_1\vee z_2)-\tilde{w}_t^{m_2}(v_2,z_2),
\end{aligned}
\tag{69}
$$

where the first inequality is due to the various increasing differences, and the second inequality is due to the various individual supermodularities. ∎

To move forward, we make the following assumption on our risk measure.

IPC Assumption 5 *For $t=1,2,...,T-1$, each M_t is a sublattice within the lattice $\mathbb{M}_t^q$, and hence a lattice in its own right.*

This assumption means that $m^1,m^2\in M_t$ will lead to both $m^1\vee m^2\in M_t$ and $m^1\wedge m^2\in M_t$. With it, we can obtain the desired supermodularity of $\tilde{q}_t(\cdot,\cdot)$.

Lemma 10 *Under IPC Assumption 5, $\tilde{q}_t(v,z)$ is supermodular in $(v,z)\in\Re\times[\underline{z}_t,\overline{z}_t]$.*

Proof: Since IPC Assumption 5 has asserted that M_t is a lattice, Lemma 9 can be understood as declaring $\tilde{w}_t^m(v,z)$ a supermodular function on the lattice $M_t\times\Re\times[\underline{z}_t,\overline{z}_t]$. Consider the optimization problem on the right-hand side of (50). According to Theorem 2.7.6 of Topkis [28], especially by identifying M_t with his X, $\Re\times[\underline{z}_t,\overline{z}_t]$ with his T, $M_t\times\Re\times[\underline{z}_t,\overline{z}_t]$ with his S, and $\tilde{w}_t^m(v,z)$ with his $f(x,t)$, we can establish that $\tilde{q}_t(v,z)$ as defined in (50) is supermodular in (v,z). ∎

Now we can obtain the monotone-pricing property.

Proposition 3 *Consider the DP made up of (28) for $t=T$, as well as (37) with $\bar{k}_t=0$, and (44) to (46) for $t=T-1,T-2,...,1$. Then, under IPC Assumption 5, one optimal solution $z_t^*(y)$ to (44) is increasing in y for $t=T-1,T-2,...,1$.*

Proof: Let $z_t^0(\upsilon)$ be the largest optimal solution to the problem $\min_{z\in[\underline{z}_t,\overline{z}_t]} \tilde{q}_t(\upsilon, z)$. Then

$$\tilde{q}_t(\upsilon, z) > \tilde{q}_t(\upsilon, z_t^0(\upsilon)), \qquad \forall z \in (z_t^0(\upsilon), \overline{z}_t]. \tag{70}$$

Suppose $\upsilon' \geq \upsilon$. Then, Lemma 10 will lead to

$$\tilde{q}_t(\upsilon', z) > \tilde{q}_t(\upsilon', z_t^0(\upsilon)), \qquad \forall z \in (z_t^0(\upsilon), \overline{z}_t]. \tag{71}$$

This means that $z_t^0(\upsilon') \leq z_t^0(\upsilon)$. In view of (50), we can let one optimal solution $z_t^*(y)$ to (44) be $z_t^0(-y)$. The thus constructed $z_t^*(y)$ is increasing in y. ∎

When $z_t^*(y)$ is increasing in y, the optimal price level $p_t^*(y) = \tilde{p}_t(z_t^*(y))$ will be decreasing in y. Hence, Proposition 3 delivers the message that sales price should be lowered when there is more inventory left. This is consistent with Theorem 2(a) of Federgruen and Heching [11] for the risk-neutral case. We can see that IPC Assumption 5 is a mild requirement. For instance, given $\beta \in [0, 1]$ and $\gamma \in (0, 1]$, we might have

$$M_t = \left\{ m \in \mathbb{M}_t^q \mid 1 - \beta \leq m(\theta) \leq 1 - \beta + \frac{\beta}{\gamma} \qquad \forall \theta \in \Re \right\}. \tag{72}$$

For $m^1, m^2 \in M_t$, we can check that $m^1 \vee m^2$ as defined through (56) is a member of M_t. Symmetrically, we can verify that $m^1 \wedge m^2 \in M_t$ as well. Therefore, M_t is a sublattice of the lattice $\mathbb{M}_t^q$, and hence induces a risk measure that satisfies IPC Assumption 5. The corresponding time-t risk measure ρ_t amounts to a mixture of the ordinary expectation and the conditional value at risk. For any random cost $Z \in \mathbb{L}^p(\Re^t, \mathcal{B}(\Re^t), \prod_{s=1}^t R_s)$,

$$\rho_t(Z(\theta_{[1,t-1]}, \Theta_t)) = (1 - \beta) \cdot \mathbb{E}[Z(\theta_{[1,t-1]}, \Theta_t)] + \beta \cdot \mathrm{CVaR}_{1-\gamma}[Z(\theta_{[1,t-1]}, \Theta_t)]. \tag{73}$$

At each fixed $\theta_{[1.t-1]} = (\theta_1, ..., \theta_{t-1}) \in \Re^{t-1}$, the conditional value at risk $\mathrm{CVaR}_{1-\gamma}[Z(\theta_{[1,t-1]}, \Theta_t)]$ is the average of the worst γ portion of the random cost $Z(\theta_{[1,t-1]}, \Theta_t)$. Note that $\gamma = 1$ corresponds to the risk-neutral case: $\mathrm{CVaR}_0[Z(\theta_{[1,t-1]}, \Theta_t)] = \mathbb{E}[Z(\theta_{[1,t-1]}, \Theta_t)]$.

4.3 Moderation in Pricing

Let us extend the definition of the revenue function $\tilde{r}_t^m(z)$ through linear extrapolation, so that it is concave on not only $[\underline{z}_t, \overline{z}_t]$ but also the entire $\Re$. Now define $\hat{w}_t^m(y, \zeta) = w_t^m(y, y+\zeta)$ for $(y, \zeta) \in \Re^2$, so that following (46),

$$\hat{w}_t^m(y, \zeta) = -\tilde{r}_t^m(y + \zeta) + \mathbb{E}^m[(\tilde{h}_t + \alpha \cdot v_{t+1})(-\zeta - \Theta_t)]. \tag{74}$$

Note that $-\zeta$ is an indicator of the period-$(t + 1)$ starting inventory level.

Lemma 11 *$\hat{w}_t^m(y,\zeta)$ is supermodular in $(m,y,\zeta)\in\mathbb{M}_t^q\times\Re^2$.*

Proof: Due to the concavity of $\tilde{r}_t^m(\cdot)$, for any $\Delta y,\Delta\zeta\ge 0$,

$$\begin{aligned}&\hat{w}_t^m(y+\Delta y,\zeta+\Delta\zeta)+\hat{w}_t^m(y,\zeta)-\hat{w}_t^m(y+\Delta y,\zeta)-\hat{w}_t^m(y,z+\Delta\zeta)\\&\quad=\tilde{r}_t^m(y+\zeta+\Delta y)+\tilde{r}_t^m(y+\zeta+\Delta\zeta)-\tilde{r}_t^m(y+\zeta+\Delta y+\Delta\zeta)-\tilde{r}_t^m(y+\zeta)\ge 0.\end{aligned}\tag{75}$$

Therefore, $\hat{w}_t^m(y,\zeta)$ is supermodular in (y,ζ).

Since $v_{t+1}(\cdot)$ is convex, $f(x)=(\tilde{h}_t+\alpha\cdot v_{t+1})(-x)$ is convex too. Now consider

$$g_t^m(\zeta)=\mathbb{E}^m[f(\zeta+\Theta_t)]=\mathbb{E}^m[(\tilde{h}_t+\alpha\cdot v_{t+1})(-\zeta-\Theta_t)].\tag{76}$$

According to Lemma 8, $g_t^m(\zeta)$ is supermodular in (m,ζ). But by (74) and (76), we have $\hat{w}_t^m(y,\zeta)=-\tilde{r}_t^m(y+\zeta)+g^m(\zeta)$. So Lemma 7 and the above would together lead to the supermodularity of $\hat{w}_t^m(y,\zeta)$ in (m,y) and (m,ζ).

Taken together, we know that $\hat{w}_t^m(y,\zeta)$ is supermodular in (m,y,ζ). ∎

Define $\hat{q}_t(y,\zeta)=q_t(y,y+\zeta)$ for $(y,\zeta)\in\Re^2$, so that following (45),

$$\hat{q}_t(y,\zeta)=q_t(y,y+\zeta)=\sup_{m\in M_t}w_t^m(y,y+\zeta)=\sup_{m\in M_t}\hat{w}_t^m(y,\zeta).\tag{77}$$

Lemma 12 *$\hat{q}_t(y,\zeta)$ is supermodular in $(y,\zeta)\in\Re^2$.*

Proof: With IPC Assumption 5, we can use the same reasoning as that used in the proof of Lemma 10. ∎

Just because Lemma 10 leads to Proposition 3, Lemma 12 will lead to a monotone pricing result as well. However, as its feasible region is more complex, the proof of the following result is more involved.

Proposition 4 *Consider the DP made up of (28) for $t=T$, as well as (37) with $\bar{k}_t=0$, (44), (45), and (46) for $t=T-1,T-2,...,1$. Then, one optimal solution $z_t^*(y)$ to (44) will render $y-z_t^*(y)$ increasing in y.*

Proof: Consider any $t=T-1,T-2,...,1$. In Lemma 12, we have established that $\hat{q}_t(y,\zeta)$ as defined through (77) is supermodular on $\Re^2$. Define function $f(y,w)$ and set $W(y)$, so that

$$f(y,w)=-\hat{q}_t(y,-w),\qquad\text{and}\qquad W(y)=[y-\overline{z}_t,y-\underline{z}_t].\tag{78}$$

Note that $W(y)$, a subset of the lattice $\Re$, is increasing in y in the strong set order sense defined by Veinott [30], so that $w^1\in W(y^1)$ and $w^2\in W(y^2)$ for $y^1\le y^2$ would result with

$w^1 \wedge w^2 \in W(y^1)$ and $w^1 \vee w^2 \in W(y^2)$. Suppose we are given y^1, y^2 satisfying $y^1 \leq y^2$, $w^1 \in W(y^1)$, and $w^2 \in W(y^2)$. Then, when $w^1 \geq w^2$,

$$\begin{aligned}\hat{q}_t(y^1, -w^1) + \hat{q}_t(y^2, -w^2) &= \hat{q}_t(y^1, (-w^1) \wedge (-w^2)) + \hat{q}_t(y^2, (-w^1) \vee (-w^2)) \\ &\geq \hat{q}_t(y^1, (-w^1) \vee (-w^2)) + \hat{q}_t(y^2, (-w^1) \wedge (-w^2)),\end{aligned} \tag{79}$$

where the last inequality is due to the supermodularity of $\hat{q}_t(y, \zeta)$ in (y, ζ). On the other hand, when $w^1 < w^2$,

$$\hat{q}_t(y^1, -w^1) + \hat{q}_t(y^2, -w^2) = \hat{q}_t(y^1, (-w^1) \vee (-w^2)) + \hat{q}_t(y^2, (-w^1) \wedge (-w^2)). \tag{80}$$

In view of (78), (79) and (80) would together entail

$$f(y^1, w^1) + f(y^2, w^2) \leq f(y^1, w^1 \wedge w^2) + f(y^2, w^1 \vee w^2). \tag{81}$$

Due to Lemma 2.8.1 of Topkis [28], we know from the above properties regarding $W(\cdot)$ and $f(\cdot, \cdot)$ that one solution $w^*(y)$ to the optimization problem $\max_{w \in W(y)} f(y, w)$ has $w^*(\cdot)$ increasing in y. From (77) and (78), it follows that

$$\max_{w \in W(y)} f(y, w) = - \min_{\zeta \in [\underline{z}_t - y, \bar{z}_t - y]} \hat{q}_t(y, \zeta) = - \min_{z \in [\underline{z}_t, \bar{z}_t]} q_t(y, z), \tag{82}$$

with the last optimization problem being identical to (44). Hence, seeing that $z_t^*(y) = y - w^*(y)$ is one optimal solution to (44), we can let $y - z_t^*(y)$ be increasing in y. ∎

Proposition 4 advocates moderation in pricing. When the post-ordering inventory level y rises, the firm should follow the suggestion of Proposition 3 to lower its price and hence boost the demand lever $z_t^*(y)$. But the increase in $z_t^*(y)$ should not be so much as to erode the gain in y. We note this message was conveyed by Lemma 2 of Chen and Simchi-Levi [6] in a risk-neutral setting.

5 Infinite-horizon Problems

5.1 A General Treatment

For our infinite-horizon study, we first apply time-invariant parameters to the general shock-driven system studied in Section 2. For this purpose, let $U_t = U$, $c_t = c$, and $\mathbf{s}_{t+1} = \mathbf{s}$ for $t = 1, 2, ...$, for some measurable multi-function $U : \hat{X} \rightrightarrows \hat{U}$, measurable mapping c from $\text{graph}(U) \times \hat{D}$ to $\bar{\Re}$, and measurable mapping $\mathbf{s}$ from $\text{graph}(U) \times \hat{D}$ to $\hat{X}$. We still need a measurable mapping c^0 from $\hat{X}$ to $\bar{\Re}$ to represent the terminal cost, though. Also, let $\mathbf{x}_1 \in \hat{X}$

with $\| \mathbf{x}_1 \| < +\infty$ be the initial state, and let probability R defined on $(\hat{D}, \mathcal{B}(\hat{D}))$ be the distribution of all random shocks $\boldsymbol{\Delta}_t$. Moreover, let $\alpha \in [0,1)$ still be the discount factor.

We re-define measurable space $(\Omega, \mathcal{F})$ so that $\Omega = \hat{D}^\infty$ and $\mathcal{F} = \mathcal{B}(\hat{D}^\infty)$. Since $\hat{D}$, being a closed subset of $\Re^{\bar{l}}$, is a separable metric space, the above σ-field is well defined; see, e.g., Parthasarathy [16] (Theorem 1.10). On the space, we can define probability measure $P = R^\infty$. Define random variables $\mathbf{X}_1, \boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2, ...$ so that $\mathbf{X}_1(\omega) = \mathbf{x}_1, \boldsymbol{\Delta}_1(\omega) = \delta_1, \boldsymbol{\Delta}_2(\omega) = \delta_2, ...$ for each $\omega = (\delta_1, \delta_2, ...) \in \Omega$.

For each $t = 1, 2, ...$, we can let $\mathcal{F}_t$ be the σ-field generated by the generic random vector $(\mathbf{X}_1, \boldsymbol{\Delta}_1, ..., \boldsymbol{\Delta}_{t-1})$, which is just $\mathcal{B}(\hat{D}^{t-1})$. Note this makes $\mathcal{F}_1 = \{\emptyset, \Omega\}$. The sequence $(\mathcal{F}_t \mid t = 1, 2, ...)$ forms a filtration on the probability space $(\Omega, \mathcal{F}, P)$. For $t = 1, 2, ...$, we iteratively define period-t random states $\mathbf{X}_t^{\mathbf{U}_{[1,t-1]}}$ and period-1-to-t policy spaces $\mathcal{U}_{[1,t]}$ by following steps (1) to (4). We inherit t-independent versions of Assumptions 1 to 12.

For each $(\mathbf{x}, \mathbf{u}) \in \text{graph}(U)$, let $M(\mathbf{x}, \mathbf{u})$ be a convex subset of $\mathbb{M}^q(\hat{D}, \mathcal{B}(\hat{D}), R)$ that is also closed under the weak topology. For any $T = 1, 2, ...$, we adhere to the time-consistent coherent and Markov risk measure for a $(T-1)$-period problem that is generated by $M(\cdot, \cdot)$ in every period $t = 1, 2, ..., T-1$. Given policy $\mathbf{U}_{[1,T-1]} \in \mathcal{U}_{[1,T-1]}$ for the $(T-1)$-period problem, let $\rho_T^{\mathbf{U}_{[1,T-1]}}(Z_T^{\mathbf{U}_{[1,T-1]}})$, exactly amounting to the right-hand side of (17), stand for the total risk exposure.

Re-define $\mathcal{U}^M$ as the space of measurable mappings $\mathbf{u}$ from $\hat{X}$ to $\hat{U}$ such that $\mathbf{u}(\mathbf{x}) \in Y(\mathbf{x})$ for each $\mathbf{x} \in \hat{X}$. The space $(\mathcal{U}^M)^{T-1}$ now hosts Markov policies for a $(T-1)$-period problem. The following is a direct time-invariant consequence of Theorem 1.

Corollary 1 *Consider the DP*

$$v_0(\mathbf{x}) = c^0(\mathbf{x}), \qquad \forall \mathbf{x} \in \hat{X}, \tag{83}$$

and for $t = 1, 2, ...,$

$$v_t(\mathbf{x}) = \min_{\mathbf{u} \in U(\mathbf{x})} \sup_{m \in M(\mathbf{x}, \mathbf{u})} \mathbb{E}^m[(c + \alpha \cdot v_{t-1} \circ \mathbf{s})(\mathbf{x}, \mathbf{u}, \boldsymbol{\Delta})], \qquad \forall \mathbf{x} \in \hat{X}. \tag{84}$$

For each $T = 1, 2, ...,$ *the problem* $J_T^* = \inf_{\mathbf{U}_{[1,T-1]} \in \mathcal{U}_{[1,T-1]}} \rho_T^{\mathbf{U}_{[1,T-1]}}(Z_T^{\mathbf{U}_{[1,T-1]}})$ *has an optimal value* $v_{T-1}(\mathbf{x}_1)$*, where* $v_{T-1}(\cdot)$ *is defined by the DP; moreover, an optimal Markov policy* $\mathbf{u}^*_{[1,T-1]} = (\mathbf{u}^*_t(\mathbf{x}) \mid t = 1, 2, ..., T-1,\ \mathbf{x} \in \hat{X}) \in (\mathcal{U}^M)^{T-1}$ *exists and satisfies*

$$\mathbf{u}^*_t(\mathbf{x}) \in \text{argmin}_{\,\mathbf{u} \in U(\mathbf{x})} \sup_{m \in M(\mathbf{x}, \mathbf{u})} \mathbb{E}^m[(c + \alpha \cdot v_{t-1} \circ \mathbf{s})(\mathbf{x}, \mathbf{u}, \boldsymbol{\Delta})], \tag{85}$$

for every $t = 1, 2, ..., T-1$ *and* $\mathbf{x} \in \hat{X}$.

Each $v_t(\cdot)$ in the above corresponds to the $v_{T-t}(\cdot)$ in Theorem 1 for a $(T-1)$-period problem where $T = t+1, t+2, ...$ That $v_t(\cdot)$ can be defined without knowing which $(T-1)$-period problem it belongs to stems from the stationarity of parameters. For convenience, we now label periods in the reverse fashion. That is, we suppose a $(T-1)$-period problem starts with period $T-1$ and ends with period 0. The states on each sample path are denoted by $\mathbf{x}_{T-1}, ..., \mathbf{x}_0$ in the chronological order. For IPC applications, we use a state-control-independent density set M as the generator for risk measures.

5.2 Pure Inventory Control

We now treat pure inventory control. If we use the setting in Section 3.2 and let all parameters be time-invariant, the DP in Corollary 1 would become

$$v_0(x) = -\bar{c} \cdot x, \tag{86}$$

and for $t = 1, 2, ..., T-1$,

$$v_t(x) = -\bar{c} \cdot x + u_t(x) \wedge [\bar{k} + \min_{y \in [x,+\infty]} u_t(y)] \tag{87}$$

where

$$u_t(y) = \bar{c} \cdot y + \sup_{m \in M} w_t^m(y), \tag{88}$$

where

$$w_t^m(y) = \mathbb{E}^m[(\tilde{h} + \alpha \cdot v_{t-1})(y - \Delta)]. \tag{89}$$

Let us inherit the time-invariant version of IPC Assumption 1, which simply translates to $\bar{c} > 0$ or $\bar{h} > 0$ for the current time-invariant case. On the other hand, we need to upgrade the time-invariant version of IPC Assumption 2 to the following.

IPC Assumption 6 *Distributions m in M produce uniformly bounded averages. That is,*

$$\bar{\delta} = \sup_{m \in M} \mathbb{E}^m[\Delta] = \sup_{m \in M} \int_{\Re^+} \delta \cdot m(\delta) \cdot R(d\delta) < +\infty.$$

This again just means the absence of outlandish demand forecasts. With setup costs being time-invariant and $\alpha \in [0, 1)$, IPC Assumption 3 is automatically true.

We show that $v_t(\cdot)$ converges to some $v(\cdot)$. In the first step, we show that $v_t(x) - v_{t-1}(x)$ is bounded from below.

Lemma 13 *For $t = 1, 2, ...,$ it is true that*

$$v_t(x) - v_{t-1}(x) \geq \alpha^{t-1} \cdot (1-\alpha) \cdot \bar{c} \cdot (x - (t-1) \cdot \bar{\delta}). \tag{90}$$

Next, we set out to show that $v_t(x) - v_{t-1}(x)$ is also bounded from above. Before arriving to that conclusion, we need an intermediate result saying that order-up-to levels over different periods are uniformly bounded from above.

Lemma 14 *There exists positive constant $\overline{y}$, so that any $t = 1, 2, ...,$ any optimal solution to (87) in the pure inventory control problem satisfies $y_t^*(x) \leq x \vee \overline{y}$.*

We can then have the upper bound for $v_t(x) - v_{t-1}(x)$.

Lemma 15 *Let $A = (1-\alpha) \cdot \bar{c} + \bar{h} \vee \bar{b}$ and $B = \alpha \cdot \bar{c} + \bar{h} \vee \bar{b}$. For $t = 1, 2, ...,$*

$$v_t(x) - v_{t-1}(x) \leq \alpha^{t-1} \cdot (A \cdot \mid x \mid + (A \cdot (t-1) + B) \cdot \overline{\delta} + A \cdot (t-1) \cdot \overline{y}). \tag{91}$$

Lemmas 13 and 15 together lead to an upper bound for $\mid v_t(x) - v_{t-1}(x) \mid$, through which we can show that $\{v_t(x) \mid t = 0, 1, ...\}$ forms a Cauchy and hence convergent sequence.

Lemma 16 *As t tends to $+\infty$, each $v_t(x)$ for the pure inventory control problem converges to some $v(x)$, with*

$$\mid v(x) \mid \leq C \cdot (\mid x \mid + 1), \tag{92}$$

for some positive C. The convergence is uniform in any bounded x-region. Especially,

$$\mid v_t(x) - v(x) \mid \leq \frac{\alpha^t}{1-\alpha} \cdot [A \cdot \mid x \mid + ((t-1) + \frac{1}{1-\alpha}) \cdot B]. \tag{93}$$

Following (88) and (89), we define

$$w^m(y) = \mathbb{E}^m[(\tilde{h} + \alpha \cdot v)(y - \Delta)], \tag{94}$$

and then

$$u(y) = \bar{c} \cdot y + \sup_{m \in M} w^m(y). \tag{95}$$

More convergence thus follows.

Lemma 17 *As t tends to $+\infty$, each $w_t^m(y)$ converges to $w^m(y)$ and each $u_t(y)$ converges to $u(y)$. Also, the infinite-t version of (87) is true. That is,*

$$v(x) = -\bar{c} \cdot x + u(x) \wedge [\bar{k} + \min_{y \in [x, +\infty]} u(y)]. \tag{96}$$

Finally, we can show that policy characteristics identified in Proposition 1 for finite-t cases carry over to corresponding infinite-t cases.

Proposition 5 *The cost-to-go function $v(\cdot)$ defined in (96) is $\bar{k}$-convex. Consequently, one optimal solution $y^*(x)$ to (96) is of the (s,S)-form $x + (S-x)\cdot \mathbf{1}(x < s)$ for some constants s and S satisfying $s \leq S$. Especially, when $\bar{k} = 0$, the convexity of $v(\cdot)$ would follow; as a consequence, $y^*(x)$ would be of the base-stock form $S \vee x$ for some point S.*

Proof: Since $\bar{k}$-convexity is preserved under pointwise convergence, the $\bar{k}$-convexity of $v(\cdot)$ follows from Proposition 1 and Lemma 16 under the automatically satisfied IPC Assumption 3. The (s,S) shape can be deduced in the same fashion as the corresponding portion of Proposition 1. Convexity of $v(\cdot)$ for the special case with $\bar{k} = 0$ follows from the equivalence between convexity and 0-convexity. The base-stock policy shape can then be derived. ∎

5.3 Joint Inventory-price Control

We now treat joint inventory-price control. If we use the setting in Section 3.3 and let all parameters be time-invariant, the DP in Corollary 1 would constitute (86) for $t = 0$ and (87) for $t = 1, 2, ..., T-1$, where

$$u_t(y) = \bar{c} \cdot y + \min_{z \in [\underline{z}, \overline{z}]} q_t(y, z), \tag{97}$$

$$q_t(y, z) = \sup_{m \in M} w_t^m(y, z), \tag{98}$$

and

$$w_t^m(y, z) = -\tilde{r}^m(z) + \mathbb{E}^m[(\tilde{h} + \alpha \cdot v_{t-1})(y - z - \Theta)]. \tag{99}$$

Let us inherit the time-invariant version of IPC Assumption 1, which simply translates to $\bar{c} > 0$ or $\bar{h} > 0$ for the current time-invariant case. On the other hand, we need to upgrade the time-invariant version of IPC Assumption 4 to the following.

IPC Assumption 7 *Distributions m in M yield uniformly bounded averages. That is,*

$$\overline{\delta} = \sup_{m \in M} \mathbb{E}^m[\overline{z} + \Theta] < +\infty.$$

With setup costs being time-invariant and $\alpha \in [0, 1)$, IPC Assumption 3 is automatically true. In addition, we need one new condition.

IPC Assumption 8 *In the worst case, the firm can manage not to lose too much money; on the other hand, it cannot manage to make too much either. That is, for some $\overline{r} \in \Re^+$,*

$$-\overline{r} \leq \min_{z \in [\underline{z}, \overline{z}]} \sup_{m \in M} \{\alpha \bar{c} \cdot \mathbb{E}^m[z + \Theta] - \tilde{r}^m(z)\} \leq \overline{r}.$$

Note that $\alpha\bar{c}\cdot\mathbb{E}^m[z+\Theta]-\tilde{r}^m(z)=(\alpha\bar{c}-\tilde{p}(z))\cdot\mathbb{E}^m[z+\Theta]$. So the above means marginal revenue is neither too bad nor too good when pitted against marginal cost $\alpha\bar{c}$ in the worst scenario. Again, we are to show the convergence of $v_t(\cdot)$ to some $v(\cdot)$. For this, we first show that $v_t(x)-v_{t-1}(x)$ is bounded from below.

Lemma 18 *For $t=1,2,...$, it is true that*

$$v_t(x)-v_{t-1}(x)\geq\alpha^{t-1}\cdot(1-\alpha)\cdot\bar{c}\cdot(x-(t-1)\cdot\overline{\delta})-\alpha^{t-1}\overline{r}. \tag{100}$$

To show that $v_t(x)-v_{t-1}(x)$ is bounded from above, we rely on the following intermediate result, which gives a uniform bound for order-up-to levels over different periods.

Lemma 19 *There exists positive constant $\overline{y}$, so that any $t=1,2,...$, any optimal solution to (87) in the joint inventory-price control problem satisfies $y_t^*(x)\leq x\vee\overline{y}$.*

We now arrive to an upper bound for $v_t(x)-v_{t-1}(x)$.

Lemma 20 *Let $A=(1-\alpha)\cdot\bar{c}+\bar{h}\vee\bar{b}$ and $B=\overline{r}/\overline{\delta}+\bar{h}\vee\bar{b}$. For $t=1,2,...$,*

$$v_t(x)-v_{t-1}(x)\leq\alpha^{t-1}\cdot(A\cdot\mid x\mid+(A\cdot(t-1)+B)\cdot\overline{\delta}+A\cdot(t-1)\cdot\overline{y}). \tag{101}$$

With Lemmas 18 and 20 providing an upper bound for $\mid v_t(x)-v_{t-1}(x)\mid$, we can verify that $\{v_t(x)\mid t=0,1,...\}$ forms a Cauchy and hence convergent sequence.

Lemma 21 *As t tends to $+\infty$, each $v_t(x)$ for the joint inventory-price control problem converges to some $v(x)$, with both (92) for some positive constant C and (93) for A and B just defined in the above being true.*

Following (97) to (99), we define

$$u(y)=\bar{c}\cdot y+\min_{z\in[\underline{z},\overline{z}]}q(y,z), \tag{102}$$

where

$$q(y,z)=\sup_{m\in M}w^m(y,z), \tag{103}$$

and

$$w^m(y,z)=-\tilde{r}^m(z)+\mathbb{E}^m[(\tilde{h}+\alpha\cdot v)(y-z-\Theta)]. \tag{104}$$

More convergence then follows.

Lemma 22 *As t tends to $+\infty$, each $w_t^m(y,z)$ converges to $w^m(y,z)$, each $q_t(y,z)$ converges to $q(y,z)$, and each $u_t(y)$ converges to $u(y)$. Also, (96) is true.*

Finally, we can show that policy characteristics identified in Propositions 2 to 3 for finite-t cases carry over to corresponding infinite-t cases.

Proposition 6 *The cost-to-go function $v(\cdot)$ in the joint inventory-price control problem is symmetric $\bar{k}$-convex. Consequently, one optimal solution $y^*(x)$ to (96) is of the (s,S,A)-form $x+(S-x)\cdot\mathbf{1}(x<s \text{ or } \in A)$ for some constants s and S satisfying $s\leq S$ and a set $A\subseteq[s,(s+S)/2]$. Especially, when $\bar{k}=0$, convexity of $v(\cdot)$ would follow; as a consequence, $y^*(x)$ would be of the base-stock form $S\vee x$ for some point S.*

Moreover, suppose $\bar{k}=0$ and M comes from $\mathbb{M}^q$, the time-invariant version of $\mathbb{M}^q_t$. Then, when M is lattice-like as defined in Section 4.2, one optimal $z^*(y)$ to (102) will increase in y. Finally, when demand is further additive as in Section 4.3, one optimal solution $z^*(y)$ satisfies that $y-z^*(y)$ is increasing in y.

Proof: Since symmetric $\bar{k}$-convexity is preserved under pointwise convergence, the symmetric $\bar{k}$-convexity of $v(\cdot)$ follows from Proposition 2 and Lemma 21 under the automatically satisfied IPC Assumption 3. The (s,S,A) shape can be deduced in the same fashion as the corresponding portion of Proposition 2. Convexity of $v(\cdot)$ for the special case with $\bar{k}=0$ follows from the equivalence between convexity and 0-convexity. The base-stock policy shape can then be derived. The special cases are provable using the preservation of supermodularity under pointwise convergence, as well as Propositions 3 and 4. ∎

When the firm is risk-neutral, Chen and Simchi-Levi [7] found that an (s,S) inventory policy could be optimal for the infinite-horizon case. Even when the firm is risk-averse and ambiguities exist for demand distributions, the same policy can be optimal for an infinite-horizon situation; see Chen and Sun [8]. In the latter, however, a linear-exponential utility function was assumed and the demand form was more special with $\bar{z}(p)=-p$.

6 Optimism and Higher Inventory Levels

6.1 Two Rankings of Risk Measures

In Sections 2 to 5, we dealt with optimal control under one risk measure. We now set out to conduct comparative statics studies concerning varying risk measures. For this purpose, we revert back to finite-horizon problems; also, we compare two systems, labeled by superscripts 1 and 2, that are otherwise identical but with two different risk measures.

We define two manners in which two risk measures can be ranked for the general setting considered in Section 2. The first manner has to do with the inclusion of convex density sets. For risk measures used in the two systems, we say one is less hesitant than the other, when regarding the two generators $M^1_{[1,T-1]}(\cdot,\cdot)$ and $M^2_{[1,T-1]}(\cdot,\cdot)$,

$$M^1_t(\mathbf{x},\mathbf{u}) \subseteq M^2_t(\mathbf{x},\mathbf{u}), \qquad \forall t = 1,2,...,T-1,\ (\mathbf{x},\mathbf{u}) \in \mathrm{graph}(U_t).$$

The conclusion reachable on the ranking of cost-to-go function values is straightforward.

Theorem 2 *Consider the DP with (28) at $t = T$ and (29) at $t = T-1, T-2, ...1$ for two shock-driven systems 1 and 2. Suppose system 1's risk measure is less hesitant than system 2's. Then, we have $v^1_t(\mathbf{x}) \le v^2_t(\mathbf{x})$ for every $t = T, T-1, ..., 1$ and $\mathbf{x} \in \hat{X}$.*

The above crude ranking does not imply much about the ranking between control policies. Thus, we next turn to a more refined ranking. Suppose a partial order has been adopted for $\mathbb{M}^q_t$, under which it forms a lattice. Then, between risk measures 1 and 2, we say one is less optimistic than another when for each $t = 1,2,...,T-1$ and $(\mathbf{x},\mathbf{u}) \in \mathrm{graph}(U_t)$, we have $M^1_t(\mathbf{x},\mathbf{u}) \le M^2_t(\mathbf{x},\mathbf{u})$ in the strong set order sense defined by Veinott [30], so that for any $m^1 \in M^1_t(\mathbf{x},\mathbf{u})$ and $m^2 \in M^2_t(\mathbf{x},\mathbf{u})$,

$$m^1 \wedge m^2 \in M^1_t(\mathbf{x},\mathbf{u}), \quad \text{and} \quad m^1 \vee m^2 \in M^2_t(\mathbf{x},\mathbf{u}). \tag{105}$$

We can obtain implications of the optimism ranking for certain IPC applications without setup costs. For these, the shock space $\hat{D}$, hosting demand levels, is either $\Re^+$ or $\Re$. We can rank demand distributions by the stochastic order. Thus, as in Section 4.2, we adopt the partial order for $\mathbb{M}^q_t$ so that $m^1 \le m^2$ is understood as $\int m^1 \cdot dR_t \le \int m^2 \cdot dR_t$ in the usual stochastic sense. Note that $\mathbb{M}^q_t$ is a lattice under this order.

6.2 Pure Inventory Control without Setup

We first deal with the no-setup special case of the pure inventory control problem treated in Section 3.2. For this setting with $\hat{D} = \Re^+$, we can verify that more optimistic demand outlooks do lead to higher ordering quantities.

Proposition 7 *Consider DPs with $v^1_T(x) = v^2_T(x) = -\bar{c}_T \cdot x$ and (37) with $\bar{k}_t = 0$, (38), and (39) at $t = T-1, T-2, ..., 1$ for two systems with different risk measures. Suppose risk measure 1 is less optimistic than risk measure 2. Then, $v^i_t(x)$ is supermodular in $(i, -x)$ for $t = T, T-1, ..., 1$; consequently, the base-stock points S^i_t satisfy $S^1_t \le S^2_t$.*

Proof: We prove by induction. First, as $v^1_T(x) = v^2_T(x) = -\bar{c}_T \cdot x$, we certainly have $v^i_T(x)$'s supermodularity in $(i, -x)$. Next, for some $t = T-1, T-2, ..., 1$, suppose $v^i_{t+1}(x)$ is supermodular in $(i, -x)$.

We first show that, for $m^1, m^2 \in \mathbb{M}^q(\Re^+, \mathcal{B}(\Re^+), R_t)$ and $y^1, y^2 \in \Re$ with $y^1 \geq y^2$,

$$w_t^{1,m^1 \wedge m^2}(y^1) + w_t^{2,m^1 \vee m^2}(y^2) \geq w_t^{1m^1}(y^2) + w_t^{2m^2}(y^1). \tag{106}$$

To this end, for any $m \in \mathbb{M}^q(\Re^+, \mathcal{B}(\Re^+), R_t)$, we define $\tilde{\Delta}_t^m$ as a Borel-measurable mapping from $[0,1]$ to $\Re^+$, so that for any $\lambda \in [0,1]$,

$$\tilde{\Delta}_t^m(\lambda) = \sup \left\{ \delta \in \Re^+ \mid (\int m \cdot dR_t)([0,\delta]) = \int_0^\delta m(\delta') \cdot R_t(d\delta') \leq \lambda \right\}. \tag{107}$$

The defined entity is basically the λ-quantile of $\int m \cdot dR_t$. For $f \in \mathbb{L}^p(\Re^+, \mathcal{B}(\Re^+), R_t)$,

$$\mathbb{E}^m[f(\Delta_t)] = \int_0^1 f(\tilde{\Delta}_t^m(\lambda)) \cdot d\lambda. \tag{108}$$

In addition, $m^1 \leq m^2$ is translatable to

$$\tilde{\Delta}_t^{m^1}(\lambda) \leq \tilde{\Delta}_t^{m^2}(\lambda), \qquad\qquad \forall \lambda \in [0,1]. \tag{109}$$

Hence, for every $\lambda \in [0,1]$,

$$\tilde{\Delta}_t^{m^1 \wedge m^2}(\lambda) = \tilde{\Delta}_t^{m^1}(\lambda) \wedge \tilde{\Delta}_t^{m^2}(\lambda), \qquad\qquad \tilde{\Delta}_t^{m^1 \vee m^2}(\lambda) = \tilde{\Delta}_t^{m^1}(\lambda) \vee \tilde{\Delta}_t^{m^2}(\lambda). \tag{110}$$

Now, let us keep the key observation (110) in mind. Since $y^1 \geq y^2$, we have

$$y^2 - \tilde{\Delta}_t^{m^1 \vee m^2}(\lambda) \leq y^2 - \tilde{\Delta}_t^{m^1}(\lambda),\ y^1 - \tilde{\Delta}_t^{m^2}(\lambda) \leq y^1 - \tilde{\Delta}_t^{m^1 \wedge m^2}(\lambda). \tag{111}$$

Meanwhile, the sum of the left- and right-hand sides of (111) equals that of the two middle terms. Hence, due to the convexity of $\tilde{h}_t(\cdot)$,

$$\tilde{h}_t(y^1 - \tilde{\Delta}_t^{m^1 \wedge m^2}(\lambda)) + \tilde{h}_t(y^2 - \tilde{\Delta}_t^{m^1 \vee m^2}(\lambda)) \geq \tilde{h}_t(y^2 - \tilde{\Delta}_t^{m^1}(\lambda)) + \tilde{h}_t(y^1 - \tilde{\Delta}_t^{m^2}(\lambda)), \tag{112}$$

where we really have equality for those λ's with $\tilde{\Delta}_t^{m^1}(\lambda) \geq \tilde{\Delta}_t^{m^2}(\lambda)$. From Proposition 1, we know that $v_{t+1}^1(\cdot)$ is convex. Thus, similarly to the above,

$$v_{t+1}^1(y^1 - \tilde{\Delta}_t^{m^1 \wedge m^2}(\lambda)) + v_{t+1}^1(y^2 - \tilde{\Delta}_t^{m^1 \vee m^2}(\lambda)) \geq v_{t+1}^1(y^2 - \tilde{\Delta}_t^{m^1}(\lambda)) + v_{t+1}^1(y^1 - \tilde{\Delta}_t^{m^2}(\lambda)). \tag{113}$$

On the other hand, note that

$$y^2 - \tilde{\Delta}_t^{m^1 \vee m^2}(\lambda) \leq y^1 - \tilde{\Delta}_t^{m^2}(\lambda). \tag{114}$$

So from the induction hypothesis on $v_{t+1}^i(x)$'s supermodularity in $(i, -x)$, we have

$$v_{t+1}^2(y^2 - \tilde{\Delta}_t^{m^1 \vee m^2}(\lambda)) - v_{t+1}^1(y^2 - \tilde{\Delta}_t^{m^1 \vee m^2}(\lambda)) \geq v_{t+1}^2(y^1 - \tilde{\Delta}_t^{m^2}(\lambda)) - v_{t+1}^1(y^1 - \tilde{\Delta}_t^{m^2}(\lambda)). \tag{115}$$

Combine (112), (113), and (115), and we obtain

$$
\begin{aligned}
&(\tilde{h}_t + \alpha \cdot v^1_{t+1})(y^1 - \tilde{\Delta}^{m^1 \wedge m^2}_t(\lambda)) + (\tilde{h}_t + \alpha \cdot v^2_{t+1})(y^2 - \tilde{\Delta}^{m^1 \vee m^2}_t(\lambda)) \\
&\qquad \geq (\tilde{h}_t + \alpha \cdot v^1_{t+1})(y^2 - \tilde{\Delta}^{m^1}_t(\lambda)) + (\tilde{h}_t + \alpha \cdot v^2_{t+1})(y^1 - \tilde{\Delta}^{m^2}_t(\lambda)).
\end{aligned} \tag{116}
$$

In the presence of (39) and (59), we can get (106) by integration over $\lambda \in [0,1]$.

We now show that $u^i_t(y)$ is supermodular in $(i, -y)$. Let y^1, y^2 with $y^1 \geq y^2$ be given. For any $\epsilon > 0$, let $m^1 \in M^1_t$ and $m^2 \in M^2_t$ be such that

$$
w^{1m^1}_t(y^2) \geq \sup_{m \in M^1_t} w^{1m}_t(y^2) - \epsilon, \quad \text{and} \quad w^{2m^2}_t(y^1) \geq \sup_{m \in M^2_t} w^{2m}_t(y^1) - \epsilon. \tag{117}
$$

Because $M^1_t \leq M^2_t$ in the strong set order sense, we know (105) is true. Hence, from (38),

$$
\begin{aligned}
u^1_t(y^1) + u^2_t(y^2) &= \bar{c}_t \cdot (y^1 + y^2) + \sup_{m \in M^1_t} w^{1m}_t(y^1) + \sup_{m \in M^2_t} w^{2m}_t(y^2) \\
&\geq \bar{c}_t \cdot (y^1 + y^2) + w^{1, m^1 \wedge m^2}_t(y^1) + w^{2, m^1 \vee m^2}_t(y^2) \\
&\geq \bar{c}_t \cdot (y^1 + y^2) + w^{1m^1}_t(y^2) + w^{2m^2}_t(y^1) \\
&\geq \bar{c}_t \cdot (y^1 + y^2) + \sup_{m \in M^1_t} w^{1m}_t(y^2) + \sup_{m \in M^2_t} w^{2m}_t(y^1) - 2\epsilon \\
&= u^1_t(y^2) + u^2_t(y^1) - 2\epsilon.
\end{aligned} \tag{118}
$$

In the above, the second inequality is due to (106), the third inequality is due to (117), and the last equality is again due to (38). As ϵ can be made arbitrarily small, this entails

$$
u^1_t(y^1) + u^2_t(y^2) \geq u^1_t(y^2) + u^2_t(y^1), \tag{119}
$$

that $u^i_t(y)$ is supermodular in $(i, -y)$.

Consider the base-stock points $S^i_t = \inf\ \text{argmin}\{u^i_t(y) \mid y \in \Re\}$ for $i = 1, 2$. By S^1_t's definition, we have $u^1_t(y) > u^1_t(S^1_t)$ for any $y < S^1_t$. But $u^i_t(y)$'s supermodularity in $(i, -y)$ means that

$$
u^2_t(y) - u^2(S^1_t) \geq u^1_t(y) - u^1_t(S^1_t) > 0. \tag{120}
$$

In view of S^2_t's definition, we must have $S^1_t \leq S^2_t$. From Proposition 1, we know that $v^i_t(x) = -\bar{c}_t \cdot x + u^i_t(S^i_t \vee x)$. So $v^i_t(x) + \bar{c}_t \cdot x$ is flat when $x < S^i_t$ and increasing when $x > S^i_t$. We can use a case-by-case comparison to show that $v^i_t(x)$ is supermodular in $(i, -x)$. We have thus completed the induction process. ∎

Note the $v^i_t(\cdot)$'s are cost-to-go functions. Hence in Proposition 7, the supermodularity of $v^i_t(x)$ in $(i, -x)$ means that better demand outlooks make higher inventory levels more appreciated, which in turn renders higher base-stock levels S_t more attractive.

A special case is when M^1_t is the singleton $\{1\}$, where 1 represents the all-one unit-rate function, and M^2_t contains densities m such that $\int m \cdot dR_t \geq \int 1 \cdot dR_t = R_t$ in the usual stochastic sense. Another case is when M_t^1 contains densities m with $\int m \cdot dR_t \leq R_t$ in the usual stochastic sense and M_t^2 is the singleton $\{1\}$. For both cases, we have $M_t^1 \leq M_t^2$ and hence $y_t^{1*}(x) \leq y_t^{2*}(x)$.

In the first case, M_t^1 stands for the risk neutral case with demand projection R_t, and M_t^2 stands for the risk averse case with optimistic/aggressive demand projections. In the second case, M_t^1 stands for the risk averse case with pessimistic/conservative demand projections, and M_t^2 stands for the risk neutral case. For these special cases, we have delivered the message that optimistic demand projections call for high inventory levels.

6.3 Joint Inventory-price Control without Setup

We next work on the special joint inventory-price control problem treated in Section 4.3. For this setting, we can verify that more optimistic demand outlooks entail not only higher ordering quantities, but also lower demand levers.

Proposition 8 *Consider DPs with $v_T^1(x) = v_T^2(x) = -\bar{c}_T \cdot x$, as well as (37) with $\bar{k}_t = 0$, (44), (45), and (46) at $t = T-1, T-2, ..., 1$ for two systems with different risk measures. Suppose risk measure 1 is less optimistic than risk measure 2. Then, $v_t^i(x)$ is supermodular in $(i, -x)$ for $t = T, T-1, ..., 1$; consequently, the order-up-to points $y_t^{i*}(x)$ satisfy $y_t^{1*}(x) \leq y_t^{2*}(x)$, and the demand levers $z_t^{i*}(y)$ satisfy $z_t^{1*}(y) \geq z_t^{2*}(y)$.*

Proof: We prove by induction. First, as $v_T^1(x) = v_T^2(x) = -\bar{c}_T \cdot x$, we certainly have $v_T^i(x)$'s supermodularity in $(i, -x)$. Next, for some $t = T-1, T-2, ..., 1$, suppose $v_{t+1}^i(x)$ is supermodular in $(i, -x)$.

Due to demand additivity, $\tilde{w}_t^m(\upsilon, z)$ defined in (48) really satisfies

$$\tilde{w}_t^m(\upsilon, z) = -\tilde{r}_t^m(z) + \mathbb{E}^m[(\tilde{h}_t + \alpha \cdot v_{t+1})(-\upsilon - z - \Delta_t)]. \tag{121}$$

We first show that, for $m^1, m^2 \in \mathbb{M}^q(\Re, \mathcal{B}(\Re), R_t)$ and $y, z^1, z^2 \in \Re$ with $z^1 \leq z^2$,

$$\tilde{w}_t^{1,m^1\wedge m^2}(y, z^1) + \tilde{w}_t^{2,m^1\vee m^2}(y, z^2) \geq \tilde{w}_t^{1m^1}(y, z^2) + \tilde{w}_t^{2m^2}(y, z^1). \tag{122}$$

Now, let us keep the key observation (61) in mind. Since $z^1 \leq z^2$, we have

$$-\upsilon - z^2 - \tilde{\Theta}_t^{m^1\vee m^2}(\lambda) \leq -\upsilon - z^2 - \tilde{\Theta}_t^{m^1}(\lambda), \; -\upsilon - z^1 - \tilde{\Theta}_t^{m^2}(\lambda) \leq -\upsilon - z^1 - \tilde{\Theta}_t^{m^1\wedge m^2}(\lambda). \tag{123}$$

Meanwhile, the sum of the left- and right-hand sides of (123) equals that of the two middle terms. Hence, due to the convexity of $\tilde{h}_t(\cdot)$,

$$\begin{aligned} &\tilde{h}_t(-\upsilon - z^1 - \tilde{\Theta}_t^{m^1\wedge m^2}(\lambda)) + \tilde{h}_t(-\upsilon - z^2 - \tilde{\Theta}_t^{m^1\vee m^2}(\lambda)) \\ &\qquad \geq \tilde{h}_t(-\upsilon - z^2 - \tilde{\Theta}_t^{m^1}(\lambda)) + \tilde{h}_t(-\upsilon - z^1 - \tilde{\Theta}_t^{m^2}(\lambda)), \end{aligned} \tag{124}$$

where we really have equality for those λ's with $\tilde{\Theta}_t^{m^1}(\lambda) \geq \tilde{\Theta}_t^{m^2}(\lambda)$. From Proposition 2, we know that $v_{t+1}^1(\cdot)$ is convex. Thus, similarly to the above,

$$\begin{aligned} & v_{t+1}^1(-\upsilon - z^1 - \tilde{\Theta}_t^{m^1 \wedge m^2}(\lambda)) + v_{t+1}^1(-\upsilon - z^2 - \tilde{\Theta}_t^{m^1 \vee m^2}(\lambda)) \\ & \qquad \geq v_{t+1}^1(-\upsilon - z^2 - \tilde{\Theta}_t^{m^1}(\lambda)) + v_{t+1}^1(-\upsilon - z^1 - \tilde{\Theta}_t^{m^2}(\lambda)). \end{aligned} \tag{125}$$

On the other hand, note that

$$-\upsilon - z^2 - \tilde{\Theta}_t^{m^1 \vee m^2}(\lambda) \leq -\upsilon - z^1 - \tilde{\Theta}_t^{m^2}(\lambda). \tag{126}$$

So from the induction hypothesis on $v_{t+1}^i(x)$'s supermodularity in $(i, -x)$, we have

$$\begin{aligned} & v_{t+1}^2(-\upsilon - z^2 - \tilde{\Theta}_t^{m^1 \vee m^2}(\lambda)) - v_{t+1}^1(-\upsilon - z^2 - \tilde{\Theta}_t^{m^1 \vee m^2}(\lambda)) \\ & \qquad \geq v_{t+1}^2(-\upsilon - z^1 - \tilde{\Theta}_t^{m^2}(\lambda)) - v_{t+1}^1(-\upsilon - z^1 - \tilde{\Theta}_t^{m^2}(\lambda)). \end{aligned} \tag{127}$$

Combine (124), (125), and (127), and we obtain

$$\begin{aligned} & (\tilde{h}_t + \alpha \cdot v_{t+1}^1)(-\upsilon - z^1 - \tilde{\Theta}_t^{m^1 \wedge m^2}(\lambda)) + (\tilde{h}_t + \alpha \cdot v_{t+1}^2)(-\upsilon - z^2 - \tilde{\Theta}_t^{m^1 \vee m^2}(\lambda)) \\ & \qquad \geq (\tilde{h}_t + \alpha \cdot v_{t+1}^1)(-\upsilon - z^2 - \tilde{\Theta}_t^{m^1}(\lambda)) + (\tilde{h}_t + \alpha \cdot v_{t+1}^2)(-\upsilon - z^1 - \tilde{\Theta}_t^{m^2}(\lambda)). \end{aligned} \tag{128}$$

Since $\tilde{p}_t(\cdot)$ is decreasing and $z^1 \leq z^2$, we know

$$\tilde{\Theta}_t^{m^1 \wedge m^2}(\lambda) \cdot \tilde{p}_t(z^1) + \tilde{\Theta}_t^{m^1 \vee m^2}(\lambda) \cdot \tilde{p}_t(z^2) \leq \tilde{\Theta}_t^{m^1}(\lambda) \cdot \tilde{p}_t(z^2) + \tilde{\Theta}_t^{m^2}(\lambda) \cdot \tilde{p}_t(z^1), \tag{129}$$

which is really equality when $\tilde{\Theta}_t^{m^1}(\lambda) \geq \tilde{\Theta}_t^{m^2}(\lambda)$. In the presence of (43), (59), and (121), we can get (122) by integrating the sum of (128) and (129) over $\lambda \in [0, 1]$.

We now show that $\tilde{q}_t^i(\upsilon, z)$ defined in (50) is supermodular in (i, z). Let υ, z^1, z^2 with $z^1 \leq z^2$ be given. For any $\epsilon > 0$, let $m^1 \in M_t^1$ and $m^2 \in M_t^2$ be such that

$$\tilde{w}_t^{1m^1}(\upsilon, z^2) \geq \sup_{m \in M_t^1} \tilde{w}_t^{1m}(\upsilon, z^2) - \epsilon, \quad \text{and} \quad \tilde{w}_t^{2m^2}(\upsilon, z^1) \geq \sup_{m \in M_t^2} \tilde{w}_t^{2m}(\upsilon, z^1) - \epsilon. \tag{130}$$

Because $M_t^1 \leq M_t^2$ in the strong set order sense, we know (105) is true. Hence, from (50),

$$\begin{aligned} \tilde{q}_t^1(\upsilon, z^1) + \tilde{q}_t^2(\upsilon, z^2) & = \sup_{m \in M_t^1} \tilde{w}_t^{1m}(\upsilon, z^1) + \sup_{m \in M_t^2} \tilde{w}_t^{2m}(\upsilon, z^2) \\ & \geq \tilde{w}_t^{1, m^1 \wedge m^2}(\upsilon, z^1) + \tilde{w}_t^{2, m^1 \vee m^2}(\upsilon, z^2) \\ & \geq \tilde{w}_t^{1m^1}(\upsilon, z^2) + \tilde{w}_t^{2m^2}(\upsilon, z^1) \\ & \geq \sup_{m \in M_t^1} \tilde{w}_t^{1m}(\upsilon, z^2) + \sup_{m \in M_t^2} \tilde{w}_t^{2m}(\upsilon, z^1) - 2\epsilon \\ & = \tilde{q}_t^1(\upsilon, z^2) + \tilde{q}_t^2(\upsilon, z^1) - 2\epsilon. \end{aligned} \tag{131}$$

In the above, the second inequality is due to (122), the third inequality is due to (130), and the last equality is again due to (50). As ϵ can be made arbitrarily small, this entails

$$\tilde{q}_t^1(\upsilon, z^1) + \tilde{q}_t^2(\upsilon, z^2) \geq \tilde{q}_t^1(\upsilon, z^2) + \tilde{q}_t^2(\upsilon, z^1), \tag{132}$$

that $\tilde{q}_t^i(\upsilon, z)$ is supermodular in (i, z). In view of (44) and (50), we can let $z_t^{1*}(y) \geq z_t^{2*}(y)$.

Similarly, we can have an inequality almost identical to (128), but with $\upsilon + z^i$ replaced by $\upsilon^i + z$. Then by similar logic, for $m^1, m^2 \in \mathbb{M}^q(\Re, \mathcal{B}(\Re), R_t)$ and $\upsilon^1, \upsilon^2, z \in \Re$ with $\upsilon^1 \leq \upsilon^2$,

$$\tilde{w}_t^{1,m^1 \wedge m^2}(\upsilon^1, z) + \tilde{w}_t^{2,m^1 \vee m^2}(\upsilon^2, z) \geq \tilde{w}_t^{1m^1}(\upsilon^2, z) + \tilde{w}_t^{2m^2}(\upsilon^1, z), \tag{133}$$

and hence $\tilde{q}_t^i(\upsilon, z)$ is supermodular in (i, υ). Combine the two supermodular properties with Lemma 10, and we can conclude that $\tilde{q}_t^i(\upsilon, z)$ is supermodular in (i, υ, z). Hence, $-\tilde{q}_t^i(-y, z)$ is supermodular in (i, y, z).

Now consider optimization problem

$$\tilde{u}_t^i(y) = \max_{z \in [\underline{z}_t, \bar{z}_t]} \{-\tilde{q}_t^i(-y, z)\}. \tag{134}$$

Just like in the proof of Lemma 10, we can use Theorem 2.7.6 of Topkis [28] to show that $\tilde{u}_t^i(y)$ is supermodular in (i, y). From (44) and (50),

$$\begin{aligned} u_t^i(y) &= \bar{c}_t \cdot y + \min_{z \in [\underline{z}_t, \bar{z}_t]} q_t^i(y, z) = \bar{c}_t \cdot y + \min_{z \in [\underline{z}_t, \bar{z}_t]} \tilde{q}_t^i(-y, z) \\ &= \bar{c}_t \cdot y - \max_{z \in [\underline{z}_t, \bar{z}_t]} \{-\tilde{q}_t^i(-y, z)\}, \end{aligned} \tag{135}$$

which is equal to $\bar{c}_t \cdot y - \tilde{u}_t^i(y)$ according to (134). Therefore, $u_t^i(y)$ is supermodular in $(i, -y)$.

Consider the base-stock points $S_t^i = \inf\ \operatorname{argmin}\{u_t^i(y) \mid y \in \Re\}$ for $i = 1, 2$. By S_t^1's definition, we have $u_t^1(y) > u_t^1(S_t^1)$ for any $y < S_t^1$. But $u_t^i(y)$'s supermodularity in $(i, -y)$ means that

$$u_t^2(y) - u_t^2(S_t^1) \geq u_t^1(y) - u_t^1(S_t^1) > 0. \tag{136}$$

In view of S_t^2's definition, we must have $S_t^1 \leq S_t^2$. This means that $y_t^{1*}(x) = S_t^1 \vee x \leq S_t^2 \vee x = y_t^{2*}(x)$. From Proposition 2, we know that $v_t^i(x) = -\bar{c}_t \cdot x + u_t^i(S_t^i \vee x)$. So $v_t^i(x) + \bar{c}_t \cdot x$ is flat when $x < S_t^i$ and increasing when $x > S_t^i$. We can use a case-by-case comparison to show that $v_t^i(x)$ is supermodular in $(i, -x)$. We have thus completed the induction process. ∎

In Proposition 8, the supermodularity of $v_t^i(x)$ in $(i, -x)$ again indicates that higher inventory levels are appreciated more under more aggressive demand outlooks. Note $z_t^{1*}(y) \geq z_t^{2*}(y)$ would lead to $y - z_t^{1*}(y) \leq y - z_t^{2*}(y)$. Proposition 2 says that each $y_t^{i*}(x)$ is increasing in x. When M_t^1 and M_t^2 are both lattices, we know from Proposition 3 that each $z_t^{i*}(y)$ is increasing in y and from Proposition 4 that each $y - z_t^{i*}(y)$ is increasing in y as well.

Now with $y_t^{1*}(x) \leq y_t^{2*}(x)$, we still cannot be certain about the relationship between $z_t^{1*}(y_t^{1*}(x))$ and $z_t^{2*}(y_t^{2*}(x))$. However, it follows that

$$y_t^{1*}(x) - z_t^{1*}(y_t^{1*}(x)) \leq y_t^{2*}(x) - z_t^{1*}(y_t^{2*}(x)) \leq y_t^{2*}(x) - z_t^{2*}(y_t^{2*}(x)). \tag{137}$$

Therefore, under more optimistic demand forecast, we should not only bring up the order-up-to level, but also promote the average next-period inventory level.

7 Connections between Two Systems

7.1 Parallel Description of a State-driven System

Let us describe the state-driven system treated in Ruszczyński's [21] in a language close to our own. The just-mentioned system can share with our shock-driven system the same dual p-q pair, horizon length T, state space $\hat{X}$, initial state $\mathbf{x}_1 \in \hat{X}$, control space $\hat{U}$, control ranges $(U_t(\mathbf{x})|t = 1, 2, ..., T-1,\ \mathbf{x} \in \hat{X})$ and hence graphs $(\text{graph}(U_t)|t = 1, 2, ..., T-1)$ defined in $\hat{X} \times \hat{U}$, and terminal cost c_T^0 defined on $\hat{X}$.

Recall that our shock-driven system still has shock space $\hat{D}$, state transition functions $(\mathbf{s}_{t+1}|t = 1, 2, ..., T-1)$ from $\hat{X} \times \hat{U} \times \hat{D}$ to $\hat{X}$, non-terminal costs $(c_t|t = 1, 2, ..., T-1)$ defined on $\hat{X} \times \hat{U} \times \hat{D}$, and base shock probabilities $(R_t|t = 1, 2, ..., T-1)$ in $\mathbb{P}(\hat{D})$, the space of probabilities defined on the measurable space $(\hat{D}, \mathcal{B}(\hat{D}))$. On these model primitives, we have built product-form probability space $(\Omega, \mathcal{F}, P)$ with $\Omega = \hat{D}^{T-1}$, $\mathcal{F} = \mathcal{B}(\hat{D}^{T-1}) = \prod_{t=1}^{T-1} \mathcal{B}(\hat{D})$, and $P = \prod_{t=1}^{T-1} R_t$, as well as product-form filtration $(\mathcal{F}_t|t = 1, 2, ..., T)$ with each $\mathcal{F}_t = \mathcal{B}(\hat{D}^{t-1})$.

The state-driven system has no shock space or state transition function. It has non-terminal costs $(\tilde{c}_t|t = 1, 2, ..., T-1)$ defined on $\hat{X} \times \hat{U}$ and base state probabilities $(\tilde{P}_{t+1}|t = 1, 2, ..., T-1)$ in $\mathbb{P}(\hat{X})$. Its state transitions are facilitated by density-valued transition kernels $(\tilde{Q}_t|t = 1, 2, ..., T-1)$ from $\hat{X} \times \hat{U}$ to $\mathbb{M}_t^q \equiv \mathbb{M}^q(\hat{X}, \mathcal{B}(\hat{X}), \tilde{P}_{t+1})$. On these model primitives, one can build product-form measurable space $(\tilde{\Omega}, \tilde{\mathcal{F}})$ with $\tilde{\Omega} = \hat{X}^{T-1}$ and $\tilde{\mathcal{F}} = \mathcal{B}(\hat{X}^{T-1})$, as well as product-form filtration $(\tilde{\mathcal{F}}_t|t = 1, 2, ..., T)$ with each $\tilde{\mathcal{F}}_t = \mathcal{B}(\hat{X}^{t-1})$. Any $\tilde{\omega} \in \tilde{\Omega}$ is representable by $\mathbf{x}_{[2,T]} = (\mathbf{x}_t|t = 2, 3, ..., T)$. For this system, however, no natural probability seems definable for the measurable space $(\tilde{\Omega}, \tilde{\mathcal{F}})$ until an adaptive control has been specified.

Note that (1) to (4) help to define the control space $\mathcal{U}$ for the shock-driven system. For the state-driven system, we can let $\tilde{\mathcal{U}}_1 = U_1(\mathbf{x}_1)$ and for $t = 2, 3, ..., T-1$, let $\tilde{\mathcal{U}}_t$ be the space of measurable maps $\tilde{\mathbf{U}}_t$ from $\mathbf{x}_{[2,t]} = (\mathbf{x}_s|s = 2, 3, ..., t)$ to $\hat{U}$ with $\tilde{\mathbf{U}}_t(\mathbf{x}_{[2,t]}) \in U_t(\mathbf{x}_t)$. The policy space for the current system is then $\tilde{\mathcal{U}} = \prod_{t=1}^{T-1} \tilde{\mathcal{U}}_t$. Much like (5) for the shock-driven system, a given policy $\tilde{\mathbf{U}} = (\tilde{\mathbf{U}}_t|t = 1, 2, ..., T-1) \in \tilde{\mathcal{U}}$ will lead to an adapted cost sequence $\tilde{Z}^{\tilde{\mathbf{U}}} = (\tilde{Z}_1^{\tilde{\mathbf{U}}_1}, ..., \tilde{Z}_{T-1}^{\tilde{\mathbf{U}}_{T-1}}, \tilde{Z}_T)$, this time with

$$\begin{cases} \tilde{Z}_1^{\tilde{\mathbf{U}}_1} & = \tilde{c}_1(\mathbf{x}_1, \tilde{\mathbf{U}}_1), \\ \tilde{Z}_2^{\tilde{\mathbf{U}}_2}(\mathbf{x}_2) & = \alpha \cdot \tilde{c}_2(\mathbf{x}_2, \tilde{\mathbf{U}}_2(\mathbf{x}_2)), \\ \tilde{Z}_3^{\tilde{\mathbf{U}}_3}(\mathbf{x}_{[2,3]}) & = \alpha^2 \cdot c_3(\mathbf{X}_3, \tilde{\mathbf{U}}_3(\mathbf{x}_{[2,3]})), \\ \cdots & \cdots\cdots, \\ \tilde{Z}_{T-1}^{\tilde{\mathbf{U}}_{T-1}}(\mathbf{x}_{[2,T-1]}) & = \alpha^{T-2} \cdot c_{T-1}(\mathbf{x}_{T-1}, \tilde{\mathbf{U}}_{T-1}(\mathbf{x}_{[2,T-1]})), \\ \tilde{Z}_T(\mathbf{x}_{[2,T]}) & = \alpha^{T-1} \cdot c_T^0(\mathbf{x}_T). \end{cases} \tag{138}$$

For the shock-driven system, we have considered time-consistent coherent and Markov risk measures that are generated from the form

$$M_{[1,T-1]}(\cdot,\cdot) = (M_t(\mathbf{x},\mathbf{u})|t = 1,2,...,T-1,\ (\mathbf{x},\mathbf{u}) \in \text{graph}(U_t)),$$

where each $M_t(\mathbf{x},\mathbf{u})$ is a convex subset of $\mathbb{M}_t^q \equiv \mathbb{M}^q(\hat{D},\mathcal{B}(\hat{D}),R_t)$ that is closed under the weak topology. Under each policy $\mathbf{U} \in \mathcal{U}$, there would emerge a time-consistent coherent risk measure $\rho^{\mathbf{U}} = (\rho_t^{\mathbf{U}_{[1,t]}}|t = 1,2,...,T-1)$, with each $\rho_t^{\mathbf{U}_{[1,t]}}$ definable by (15) and (16).

For the state-driven system, Ruszczyński [21] considered the form

$$\tilde{A}_{[1,T-1]}(\cdot,\cdot) = (\tilde{A}_t(\mathbf{x},\tilde{m})|t = 1,2,...,T-1,\ \mathbf{x} \in \hat{X},\ \tilde{m} \in \mathbb{M}_t^q),$$

where each $\tilde{A}_t(\mathbf{x},\tilde{m})$ is a convex subset of $\mathbb{M}_t^q$ that is closed under the weak topology. Then, each policy $\tilde{\mathbf{U}} \in \tilde{\mathcal{U}}$ would yield a time-consistent coherent risk measure $\tilde{\rho}^{\tilde{\mathbf{U}}} = (\tilde{\rho}_t^{\tilde{\mathbf{U}}_t}|t = 1,2,...,T-1)$. For any adapted sequence $(\tilde{W}_t|t = 2,3,...,T)$, the latter would imply

$$\begin{aligned}
&[\tilde{\rho}_t^{\tilde{\mathbf{U}}_t}(\tilde{W}_{t+1})](\mathbf{x}_{[1,t]})\\
&\qquad = \sup_{\tilde{m}\in\tilde{A}_t(\mathbf{x}_t,\tilde{Q}_t(\mathbf{x}_t,\tilde{\mathbf{U}}_t(\mathbf{x}_{[2,t]})))} \int_{\hat{X}} \tilde{W}_{t+1}(\mathbf{x}_{[1,t+1]}) \cdot \tilde{m}(\mathbf{x}_{t+1}) \cdot \tilde{P}_{t+1}(d\mathbf{x}_{t+1}),
\end{aligned} \tag{139}$$

for $t = 1,2,...,T-1$.

For $t = 2,3,...,T$, let $\tilde{\mathbf{X}}_t$ be the generic random variable satisfying $\tilde{\mathbf{X}}(\tilde{\omega}) = \mathbf{x}_t$ for any $\tilde{\omega} = \mathbf{x}_{[2,T]} \in \tilde{\Omega}$. From (138) and (139), we can obtain the state-driven system's answer to the shock-driven system's (17):

$$\begin{aligned}
\tilde{\rho}^{\tilde{\mathbf{U}}}(\tilde{Z}^{\tilde{\mathbf{U}}}) &= \tilde{Z}_1^{\tilde{\mathbf{U}}_1} + \tilde{\rho}_1^{\tilde{\mathbf{U}}_1}(\tilde{Z}_2^{\tilde{\mathbf{U}}_2} + \tilde{\rho}_2^{\tilde{\mathbf{U}}_2}(\tilde{Z}_3^{\tilde{\mathbf{U}}_3} + \cdots + \tilde{\rho}_{T-2}^{\tilde{\mathbf{U}}_{T-2}}(\tilde{Z}_{T-1}^{\tilde{\mathbf{U}}_{T-1}} + \tilde{\rho}_{T-1}^{\tilde{\mathbf{U}}_{T-1}}(\tilde{Z}_T))\cdots))\\
&= \tilde{c}_1(\mathbf{x}_1,\tilde{\mathbf{U}}_1) + \tilde{\rho}_1^{\tilde{\mathbf{U}}_1}(\alpha\cdot\tilde{c}_2(\tilde{\mathbf{X}}_2,\tilde{\mathbf{U}}_2(\tilde{\mathbf{X}}_2)) + \tilde{\rho}_2^{\tilde{\mathbf{U}}_2}(\alpha^2\cdot\tilde{c}_3(\tilde{\mathbf{X}}_3,\tilde{\mathbf{U}}_3(\tilde{\mathbf{X}}_{[2,3]})) + \cdots\\
&\quad + \tilde{\rho}_{T-2}^{\tilde{\mathbf{U}}_{T-2}}(\alpha^{T-2}\cdot\tilde{c}_{T-1}(\tilde{\mathbf{X}}_{T-1},\tilde{\mathbf{U}}_{T-1}(\tilde{\mathbf{X}}_{[2,T-1]})) + \tilde{\rho}_{T-1}^{\tilde{\mathbf{U}}_{T-1}}(\alpha^{T-1}\cdot c_T^0(\tilde{\mathbf{X}}_T)))\cdots)).
\end{aligned} \tag{140}$$

Like the shock-driven system's (19), the ultimate quest here is solving

$$\tilde{J}^* = \inf_{\tilde{\mathbf{U}}\in\tilde{\mathcal{U}}} \tilde{\rho}^{\tilde{\mathbf{U}}}(\tilde{Z}^{\tilde{\mathbf{U}}}). \tag{141}$$

Ruszczyński [21] showed that a Markov policy could be found for (141) by solving a DP.

7.2 Similarities and Differences

There is a rough correspondence between the two systems. For $\pi \in \mathbb{P}(\hat{D})$ and measurable mapping $\mathbf{s}$ from $\hat{D}$ to $\hat{X}$, we define $p \circ \mathbf{s}^{-1} \in \mathbb{P}(\hat{X})$, so that

$$(\pi \circ \mathbf{s}^{-1})(\hat{X}') = \pi(\mathbf{s}^{-1}(\hat{X}')) = \pi(\{\delta \in \hat{D} \mid \mathbf{s}(\delta) \in \hat{X}'\}), \qquad \forall \hat{X}' \in \mathcal{B}(\hat{X}). \tag{142}$$

Let us focus on a particular $t = 1, 2, ..., T-1$. For the shock-driven system, note the dynamics

$$\mathbf{X}_{t+1} = \mathbf{s}_{t+1}(\mathbf{x}_t, \mathbf{u}_t, \boldsymbol{\Delta}_t). \tag{143}$$

So given $(\mathbf{x}_t, \mathbf{u}_t) \in \text{graph}(U_t)$, the default distribution of X_{t+1} is $R_t \circ [\mathbf{s}_t(\mathbf{x}_t, \mathbf{u}_t, \cdot)]^{-1}$. For the state-driven system, the default distribution is given by $\int \tilde{Q}_t(\mathbf{x}_t, \mathbf{u}_t) \cdot d\tilde{P}_{t+1}$. Therefore, when $R_t \circ [\mathbf{s}_{t+1}(\mathbf{x}_t, \mathbf{u}_t, \cdot)]^{-1}$ is absolutely continuous in $\tilde{P}_{t+1}$ at every $(\mathbf{x}_t, \mathbf{u}_t) \in \text{graph}(U_t)$, the two systems will be equivalent in their period-t-to-period-$(t+1)$ transitions when

$$\tilde{Q}_t(\mathbf{x}_t, \mathbf{u}_t) = \frac{d(R_t \circ [\mathbf{s}_{t+1}(\mathbf{x}_t, \mathbf{u}_t, \cdot)]^{-1})}{d\tilde{P}_{t+1}}, \tag{144}$$

where the derivative is in the Radon-Nikodym sense.

Consider $m \in \mathbb{M}_t^q$. Since $\int m \cdot dR_t$ is absolutely continuous in R_t, we know that $[\int m \cdot dR_t] \circ [\mathbf{s}_{t+1}(\mathbf{x}_t, \mathbf{u}_t, \cdot)]^{-1}$ is absolutely continuous in $R_t \circ [\mathbf{s}_{t+1}(\mathbf{x}_t, \mathbf{u}_t, \cdot)]^{-1}$. Thus, we can define $\tilde{k}_t(m, \mathbf{x}_t, \mathbf{u}_t)$ so that

$$\tilde{k}_t(m, \mathbf{x}_t, \mathbf{u}_t) = \frac{d([\int m \cdot dR_t] \circ [\mathbf{s}_{t+1}(\mathbf{x}_t, \mathbf{u}_t, \cdot)]^{-1})}{d(R_t \circ [\mathbf{s}_{t+1}(\mathbf{x}_t, \mathbf{u}_t, \cdot)]^{-1})}. \tag{145}$$

Now, to equate $[\int m \cdot dR_t] \circ [\mathbf{s}_{t+1}(\mathbf{x}_t, \mathbf{u}_t, \cdot)]^{-1}$ stemmed from the shock-driven system with some $\int \tilde{m}_t(\mathbf{x}_t, \mathbf{u}_t) \cdot d\tilde{P}_{t+1}$ suitable for the state-driven system, we only have to let

$$\tilde{m}_t(\mathbf{x}_t, \mathbf{u}_t) = \frac{d([\int m \cdot dR_t] \circ [\mathbf{s}_{t+1}(\mathbf{x}_t, \mathbf{u}_t, \cdot)]^{-1})}{d\tilde{P}_{t+1}}, \tag{146}$$

which equals $\tilde{k}_t(m, \mathbf{x}_t, \mathbf{u}_t) \cdot \tilde{Q}_t(\mathbf{x}_t, \mathbf{u}_t)$ by (144) and (145).

With this understanding, we see that risk set $M_t(\mathbf{x}_t, \mathbf{u}_t)$ for the shock-driven system will correspond to $\tilde{K}_t(\mathbf{x}_t, \mathbf{u}_t) \cdot \tilde{Q}_t(\mathbf{x}_t, \mathbf{u}_t)$ in the state-driven system, with

$$\tilde{K}_t(\mathbf{x}_t, \mathbf{u}_t) = \{\tilde{k}_t(m, \mathbf{x}_t, \mathbf{u}_t) | m \in M_t(\mathbf{x}_t, \mathbf{u}_t)\}. \tag{147}$$

Note that the dependence of the $\tilde{k}_t(m, \mathbf{x}_t, \mathbf{u}_t)$ defined by (145) on $(\mathbf{x}_t, \mathbf{u}_t)$ is not superfluous. For $\pi = \int m \cdot dR_t$, $\kappa = R_t$, $f = \mathbf{s}_t(\mathbf{x}_t, \mathbf{u}_t, \cdot)$, and $g = \mathbf{s}_t(\mathbf{x}'_t, \mathbf{u}'_t, \cdot)$, we have an example with

$$\frac{d(\pi \circ f^{-1})}{d(\kappa \circ f^{-1})} \neq \frac{d(\pi \circ g^{-1})}{d(\kappa \circ g^{-1})}. \tag{148}$$

Indeed, consider the binary case, with $\pi = (\pi_0, \pi_1)$, $\kappa = (\kappa_0, \kappa_1)$, $f(a) = a$, and $g(a) = 1 - a$. Then, $d(\pi \circ f^{-1})/d(\kappa \circ f^{-1}) = (\pi_0/\kappa_0, \pi_1/\kappa_1)$ and $d(\pi \circ g^{-1})/d(\kappa \circ g^{-1}) = (\pi_1/\kappa_1, \pi_0/\kappa_0)$. We will have (148) as long as $\pi_0/\kappa_0 \neq \pi_1/\kappa_1$. This means that our form $\tilde{K}_t(\mathbf{x}_t, \mathbf{u}_t) \cdot \tilde{Q}_t(\mathbf{x}_t, \mathbf{u}_t)$ is fundamentally different from the state-driven system's $\tilde{A}_t(\mathbf{x}_t, \tilde{Q}_t(\mathbf{x}_t, \mathbf{u}_t))$.

Of course, $\tilde{A}_t(\mathbf{x}_t, \mathbf{u}_t, \tilde{Q}_t(\mathbf{x}_t, \mathbf{u}_t))$, a form which we suspect to also work for the state-driven system, would cover our form. But then, the gap between (17) and (140) seems too wide to be bridgeable by a unified treatment of both systems in general risk-averse situations. For instance, when grouping terms after period 1, (17) can be written as

$$\rho^{\mathbf{U}}(Z^{\mathbf{U}}) = \rho_1^{\mathbf{U}_1}(c_1(\mathbf{x}_1, \mathbf{U}_1, \mathbf{\Delta}_1) + \phi(\mathbf{x}_1, \mathbf{U}_{[1,T-1]}, \mathbf{\Delta}_1)), \tag{149}$$

while (140) can be written as

$$\tilde{\rho}^{\tilde{\mathbf{U}}}(\tilde{Z}^{\tilde{\mathbf{U}}}) = \tilde{c}_1(\mathbf{x}_1, \tilde{\mathbf{U}}_1) + \tilde{\rho}_1^{\tilde{\mathbf{U}}_1}(\tilde{\phi}(\tilde{\mathbf{X}}_2, \mathbf{U}_{[2,T-1]})). \tag{150}$$

Due to the nonlinearity of the risk measures in general, the difference between (149) and (150) do not appear to be reconcilable.

Admittedly, the risk-neutral special case is an exception. When each $M_t(\mathbf{x}, \mathbf{u}) = \{1\}$ in the shock-driven system, Theorem 1 says that the DP solving (19) is $v_T(\mathbf{x}_T) = c_T^0(\mathbf{x}_T)$, and for $t = T-1, T-2, ..., 1$,

$$v_t(\mathbf{x}_t) = \inf_{\mathbf{u}_t \in U_t(\mathbf{x}_t)} \int_{\hat{D}} [c_t(\mathbf{x}_t, \mathbf{u}_t, \delta_t) + \alpha \cdot v_{t+1}(\mathbf{s}_{t+1}(\mathbf{x}_t, \mathbf{u}_t, \delta_t))] \cdot R_t(d\delta_t). \tag{151}$$

When each $\tilde{A}_t(\mathbf{x}, \tilde{m}) = \{\tilde{m}\}$ in the state-driven system, Theorem 2 of Ruszczyński [21] says that the DP solving (141) is $v_T(\mathbf{x}_T) = c_T^0(\mathbf{x}_T)$, and for $t = T-1, T-2, ..., 1$,

$$v_t(\mathbf{x}_t) = \inf_{\mathbf{u}_t \in U_t(\mathbf{x}_t)} \left[\tilde{c}_t(\mathbf{x}_t, \mathbf{u}_t) + \alpha \cdot \int_{\hat{X}} v_{t+1}(\mathbf{x}_{t+1}) \cdot \tilde{Q}_t(\mathbf{x}_{t+1}|\mathbf{x}_t, \mathbf{u}_t) \cdot \tilde{P}_{t+1}(d\mathbf{x}_{t+1})\right]. \tag{152}$$

Under the transformations (144) and

$$\tilde{c}_t(\mathbf{x}_t, \mathbf{u}_t) = \int_{\hat{D}} c_t(\mathbf{x}_t, \mathbf{u}_t, \delta_t) \cdot R_t(d\delta_t), \tag{153}$$

we may see that (151) and (152) are indeed identical.

Therefore, even though the equivalence between the shock-driven and state-driven systems are well known in the risk neutral setting, the two systems should not be taken as the same when risk aversion is involved.

8 Concluding Remarks

Our study of time-consistent coherent and Markov risk measures in shock-driven systems with IPC applications in mind has borne some fruits. First, we have proven the suitability of DP formulations in general. Second, for risk-averse versions of pure inventory and joint inventory-price control problems, we have achieved policy characterizations that are comparable to their risk-neutral counterparts. Third, lattice-theoretic structures of convex density sets have been shown to be responsible for monotone trends in pricing. Lastly, we have found that a set-based optimism order between risk measures would lead to intuitive comparative statics trends on ordering and pricing policies.

In the IPC context, we have so far not treated cases involving lost sales, nonzero lead times, multiple ordering stages, or unpredictable supplies. Beyond IPC, it remains unknown whether results for the general shock-driven system are applicable to other situations. Notably, Theorem 1 allows the risk sets $M_t(\mathbf{x},\mathbf{u})$ to be dependent on both state and control, and yet this feature remains unexploited in our IPC study.

A notion more general than the coherent risk measure is the convex risk measure proposed by Föllmer and Schied [12]. Such a measure $\rho(\cdot)$ is not restricted by the positive homogeneity axiom that $\rho(\beta Z) = \beta \cdot \rho(Z)$ for $\beta \geq 0$. However, it does not appear suitable for our multi-period problem. First, without positive homogeneity, the risk of a discounted cost term will not necessarily be the discounted value of the risk associated with the cost itself. This will make the proof of Theorem 1 collapse when there is a strict discount factor $\alpha \in [0,1)$.

Even if we focus on the un-discounted case with $\alpha = 1$, there is a hurdle associated with interpretation. To explain this well, we need to delve into the dual representation. According to Ruszczyński and Shapiro [22], the convex $\rho(\cdot)$ is representable by some convex function $\Phi^{(\cdot)}$ of densities on top of the convex subset M of densities known for a coherent risk measure:

$$\rho(Z) = \sup_{m \in M} \{\mathbb{E}^m[Z] - \Phi^m\}, \qquad \text{for any random variable } Z. \tag{154}$$

We will get back the coherent case when Φ^m stays at zero. For our multi-period setting, a time-consistent convex and Markov risk measure might be obtained through two generators: the already-adopted convex-set assembly $(M_t(\mathbf{x},\mathbf{u}) | t = 1, 2, ..., T-1,\ (\mathbf{x},\mathbf{u}) \in \text{graph}(U_t))$ and the new convex-function assembly $(\Phi_t^m(\mathbf{x},\mathbf{u}) | t = 1, 2, ..., T-1,\ (\mathbf{x},\mathbf{u}) \in \text{graph}(U_t),\ m \in \mathbb{M}_t^q)$. In our IPC application with generators independent states or actions, this translates into generators $(M_t | t = 1, 2, ..., T-1)$ and $(\Phi_t^m | t = 1, 2, ..., T-1,\ m \in \mathbb{M}_t^q)$. Then, over the current coherent-risk setup, we need to adapt (45) to

$$q_t(y,z) = \sup_{m \in M_t} \{w_t^m(y,z) - \Phi_t^m\}. \tag{155}$$

For M_t, we have earlier interpreted it as the collection of demand densities perceivable by the firm. However, there does not seem to be a satisfactory explanation for each Φ_t^m, except that it behaves as a premium for taking the current density m.

On the other hand, when shrugging aside the above difficulties, we do have a few concrete results with the convex risk measure. In the general shock-driven setting, when for each t the continuity of $\Phi_t^{(\cdot)}(\cdot,\cdot)$ on graph$(U_t) \times \mathbb{M}_t^q$ is assumed, we will indeed have a modified Theorem 1 at $\alpha = 1$, with $\mathbb{E}^m[(c_t + v_{t+1} \circ \mathbf{s}_{t+1})(\mathbf{x}, \mathbf{u}, \boldsymbol{\Delta}_t)]$ substituted by

$$\mathbb{E}^m[(c_t + v_{t+1} \circ \mathbf{s}_{t+1})(\mathbf{x}, \mathbf{u}, \boldsymbol{\Delta}_t)] - \Phi_t^m(\mathbf{x}, \mathbf{u}). \tag{156}$$

For IPC applications, since the newly added Φ_t^m term does not interfere with structural derivations concerning arguments other than m, we can see that Proposition 2 on the (s, S, A, p)-nature of ordering policies will stay intact. The same can be said for Proposition 3 on the monotonicity of the demand lever $z_t^*(y)$ and Proposition 4 on the monotonicity of $y - z_t^*(y)$ under additive demand, both on the condition that each M_t is a lattice.

Proposition 8 on the comparison of two measures 1 and 2 is preservable on the following condition imposed on the Φ_t^{im}'s: for any $m^1, m^2 \in \mathbb{M}^q(\Re, \mathcal{B}(\Re), R_t)$,

$$\Phi_t^{1m^1} + \Phi_t^{2m^2} \geq \Phi_t^{1,m^1 \wedge m^2} + \Phi_t^{2,m^1 \vee m^2}. \tag{157}$$

When $\Phi_t^{1m} = \Phi_t^{2m}$, this amounts to the submodularity of Φ_t^{im} in m. With (157) in place, (131) in the proposition's proof can be modified to

$$\begin{aligned}
&\tilde{q}_t^1(\upsilon, z^1) + \tilde{q}_t^2(\upsilon, z^2)\\
&\qquad = \sup\nolimits_{m \in M_t^1}\{\tilde{w}_t^{1m}(\upsilon, z^1) - \Phi_t^{1m}\} + \sup\nolimits_{m \in M_t^2}\{\tilde{w}_t^{2m}(\upsilon, z^2) - \Phi_t^{2m}\}\\
&\qquad \geq \tilde{w}_t^{1,m^1 \wedge m^2}(\upsilon, z^1) - \Phi_t^{1,m^1 \wedge m^2} + \tilde{w}_t^{2,m^1 \vee m^2}(\upsilon, z^2) - \Phi_t^{2,m^1 \vee m^2}\\
&\qquad \geq \tilde{w}_t^{1m^1}(\upsilon, z^2) - \Phi_t^{1m^1} + \tilde{w}_t^{2m^2}(\upsilon, z^1) - \Phi_t^{2m^2}\\
&\qquad \geq \sup\nolimits_{m \in M_t^1} \tilde{w}_t^{1m}(\upsilon, z^2) + \sup\nolimits_{m \in M_t^2} \tilde{w}_t^{2m}(\upsilon, z^1) - 2\epsilon\\
&\qquad = \tilde{q}_t^1(\upsilon, z^2) + \tilde{q}_t^2(\upsilon, z^1) - 2\epsilon.
\end{aligned} \tag{158}$$

We will then have the same conclusion on the monotonicity of ordering and pricing trends when $M_t^1 \leq M_t^2$ in the strong set order sense.

Appendices

A. Technical Details in Section 2.

Proof of Lemma 1: We first show that $[\rho_t(Z)](\cdot)$ is measurable in $\delta_{[1,t-1]}$. We can rewrite (11) as

$$[\rho_t(Z)](\delta_{[1,t-1]}) = \sup_{m\in\mathcal{M}_t(\delta_{[1,t-1]})} f(\delta_{[1,t-1]}, m), \tag{159}$$

where

$$f(\delta_{[1,t-1]}, m) = \int_{\hat{D}} Z(\delta_{[1,t-1]}, \delta_t)\cdot m(\delta_t)\cdot R_t(d\delta_t). \tag{160}$$

By the fact that $Z \in \mathbb{L}^{p+}_{[1,t]} \subset \mathbb{L}^{p}_{[1,t]}$ and Fubini's Theorem, we have from (160) that $f(\cdot, m)$ is measurable in $\delta_{[1,t-1]}$ at every m. At the same time, for any $m^1, m^2 \in \mathbb{M}^q_t$,

$$\mid f(\delta_{[1,t-1]}, m^1) - f(\delta_{[1,t-1]}, m^2) \mid \leq \parallel Z(\delta_{[1,t-1]}, \cdot) \parallel^p \cdot \parallel m^1 - m^2 \parallel^q . \tag{161}$$

Since $Z \in \mathbb{L}^{p+}_{[1,t]}$, we have $\parallel Z(\delta_{[1,t-1]}, \cdot) \parallel^p < +\infty$. Thus, $f(\delta_{[1,t-1]}, \cdot)$ is continuous from $\mathbb{M}^q_t$ to $\Re$. Due to the separability of $\mathbb{M}^q_t$ under the $\parallel \cdot \parallel^q$-induced metric, we can let $\mathcal{Q}$ be its countable dense subset. In view of the continuity just proved, (159) will amount to

$$[\rho_t(Z)](\delta_{[1,t-1]}) = \sup_{m\in\mathcal{M}_t(\delta_{[1,t-1]})\cap\mathcal{Q}} f(\delta_{[1,t-1]}, m). \tag{162}$$

Now define $f_n(\delta_{[1,t-1]}, m)$ for n belonging to $\mathbb{Z}$, the space of integers, so that

$$f_n(\delta_{[1,t-1]}, m) = \begin{cases} f(\delta_{[1,t-1]}, m) \vee n, & \text{when } \delta_{[1,t-1]} \in \mathcal{D}_{[1,t-1]}(m), \\ n, & \text{otherwise.} \end{cases} \tag{163}$$

Due to the measurability of $f(\cdot, m)$ and hypothesis (i) that $D_{[1,t-1]}(m) \in \mathcal{B}(\hat{D}^{t-1})$, we can deduce that $f_n(\cdot, m)$ is measurable for every $m \in \mathbb{M}^q_t$. Then, (162) and (163) will lead to

$$\begin{aligned} [\rho_t(Z)](\delta_{[1,t-1]}) &= \inf_{n\in\mathbb{Z}} \; \sup_{m\in\mathcal{M}_t(\delta_{[1,t-1]})\cap\mathcal{Q}} \; [f(\delta_{[1,t-1]}, m) \vee n] \\ &= \inf_{n\in\mathbb{Z}} \; \sup_{m\in\mathcal{Q}} \; f_n(\delta_{[1,t-1]}, m). \end{aligned} \tag{164}$$

From the countability of the sets $\mathbb{Z}$ and $\mathcal{Q}$, as well as the measurability of $f_n(\cdot, m)$, we can see that the left-hand side of (164) is measurable in $\delta_{[1,t-1]}$.

We then show that $\rho_t(Z) \in \mathbb{L}^{p+}_{[1,t-1]}$. For any $t' = 0, 1, ..., t-2$ and $m \in \mathbb{M}^q_t$,

$$\begin{aligned} &\int_{\hat{D}^{t-1-t'}} \mid \mathbb{E}^m[Z(\delta_{[1,t']}, \delta_{[t'+1,t-1]}, \Delta_t)] \mid \cdot (\textstyle\prod_{s=t'+1}^{t-1} R_s)(d\delta_{[t'+1,t-1]}) \\ &\quad = \int_{\hat{D}^{t-t'}} \mid Z(\delta_{[1,t']}, \delta_{[t'+1,t]}) \cdot m(\delta_t) \mid \cdot (\textstyle\prod_{s=t'+1}^{t} R_s)(d\delta_{[t'+1,t]}) \\ &\quad \leq \parallel Z(\delta_{[1,t']}, \cdot) \parallel^p \cdot \parallel m \parallel^q . \end{aligned} \tag{165}$$

Note $\| Z(\delta_{[1,t']},\cdot) \|^p < +\infty$ due to Z's membership in $\mathbb{L}^{p+}_{[1,t]}$. Hence, by (11) and (165),

$$\begin{aligned}&\textstyle\int_{\hat{D}^{t-1-t'}} \mid [\rho_t(Z)](\delta_{[1,t']},\delta_{[t'+1,t-1]}) \mid \cdot (\prod_{s=t'+1}^{t-1} R_s)(d\delta_{[t'+1,t-1]}) \\ &\qquad \le \| Z(\delta_{[1,t']},\cdot) \|^p \cdot \sup\{\| m \|^q \mid m \in \bigcup_{\delta_{[1,t-1]}\in\hat{D}^{t-1}} \mathcal{M}_t(\delta_{[1,t-1]})\},\end{aligned} \tag{166}$$

which is finite by hypothesis (ii). Therefore, $\rho_t(Z) \in \mathbb{L}^{p+}_{[1,t]}$. ∎

Proof of Theorem 1: Due to (11) and the positive homogeneity property (i), we can rewrite (17) as

$$\begin{aligned}\rho^{\mathbf{U}}(Z^{\mathbf{U}}) &= \rho_1^{\mathbf{U}_1}(c_1(\mathbf{x}_1,\mathbf{U}_1,\boldsymbol{\Delta}_1) + \alpha\cdot\rho_2^{\mathbf{U}_{[1,2]}}(c_2(\mathbf{X}_2^{\mathbf{U}_1},\mathbf{U}_2,\boldsymbol{\Delta}_2) \\ &\quad + \cdots + \alpha\cdot\rho_{T-1}^{\mathbf{U}_{[1,T-1]}}(c_{T-1}(\mathbf{X}_{T-1}^{\mathbf{U}_{[1,T-2]}},\mathbf{U}_{T-1},\boldsymbol{\Delta}_{T-1}) + \alpha\cdot c_T^0(\mathbf{X}_T^{\mathbf{U}_{[1,T-1]}}))\cdots)).\end{aligned} \tag{167}$$

By the monotonicity property (ii), we can understand (19) as

$$\begin{aligned}J^* = \inf\nolimits_{\mathbf{U}\in\mathcal{U}} \rho^{\mathbf{U}}(Z^{\mathbf{U}}) &= \inf\nolimits_{\mathbf{U}_{[1,T-2]}\in\mathcal{U}_{[1,T-2]}} \rho_1^{\mathbf{U}_1}(c_1(\mathbf{x}_1,\mathbf{U}_1,\boldsymbol{\Delta}_1) \\ &\quad +\alpha\cdot\rho_2^{\mathbf{U}_{[1,2]}}(c_2(\mathbf{X}_2^{\mathbf{U}_1},\mathbf{U}_2,\boldsymbol{\Delta}_2) + \cdots + \alpha\cdot\inf\nolimits_{\mathbf{U}_{T-1}\in\mathcal{U}_{T-1}(\mathbf{U}_{[1,T-2]})} \\ &\qquad \rho_{T-1}^{\mathbf{U}_{[1,T-1]}}(c_{T-1}(\mathbf{X}_{T-1}^{\mathbf{U}_{[1,T-2]}},\mathbf{U}_{T-1},\boldsymbol{\Delta}_{T-1}) + \alpha\cdot c_T^0(\mathbf{X}_T^{\mathbf{U}_{[1,T-1]}}))\cdots)).\end{aligned} \tag{168}$$

The innermost optimization problem can be written as

$$J^*_{T-1}[\mathbf{U}_{[1,T-2]}] = \inf_{\mathbf{U}_{T-1}\in\mathcal{U}_{T-1}(\mathbf{U}_{[1,T-2]})} J_{T-1}^{\mathbf{U}_{T-1}}[\mathbf{U}_{[1,T-2]}], \tag{169}$$

where, according to (4),

$$J_{T-1}^{\mathbf{U}_{T-1}}[\mathbf{U}_{[1,T-2]}] = \rho_{T-1}^{\mathbf{U}_{[1,T-1]}}\left([c_{T-1}+\alpha\cdot c_T^0\circ\mathbf{s}_T](\mathbf{X}_{T-1}^{\mathbf{U}_{[1,T-2]}},\mathbf{U}_{T-1},\boldsymbol{\Delta}_{T-1})\right). \tag{170}$$

By $\rho_{T-1}^{\mathbf{U}_{[1,T-1]}}$'s definition through (15) and (16), we see that $J_{T-1}^{\mathbf{U}_{T-1}}[\mathbf{U}_{[1,T-2]}](\cdot)$ depends on the history $\delta_{[1,T-2]}$ only through $\mathbf{X}_{T-1}^{\mathbf{U}_{[1,T-2]}}(\delta_{[1,T-2]})$, and by (169), so does $J^*_{T-1}[\mathbf{U}_{[1,T-2]}](\cdot)$. At the same time, we know from arguments around (27) that any $\mathbf{u}_{T-1}\in\mathcal{U}^M_{T-1}$ would induce policy $\mathbf{u}_{T-1}(\mathbf{X}_{T-1}^{\mathbf{U}_{[1,T-2]}}(\cdot)) \in \mathcal{U}_{T-1}(\mathbf{U}_{[1,T-2]})$. Therefore, (169) will result with

$$\begin{aligned}&J^*_{T-1}[\mathbf{U}_{[1,T-2]}](\delta_{[1,T-2]}) \\ &\qquad \le \inf\nolimits_{\mathbf{u}_{T-1}\in\mathcal{U}^M_{T-1}} j_{T-1}\left(\mathbf{X}_{T-1}^{\mathbf{U}_{[1,T-2]}}(\delta_{[1,T-2]}),\mathbf{u}_{T-1}(\mathbf{X}_{T-1}^{\mathbf{U}_{[1,T-2]}}(\delta_{[1,T-2]}))\right),\end{aligned} \tag{171}$$

where, for $(\mathbf{x},\mathbf{u})\in\mathrm{graph}(U_{T-1})$,

$$j_{T-1}(\mathbf{x},\mathbf{u}) = \sup_{m\in M_{T-1}(\mathbf{x},\mathbf{u})} \mathbb{E}^m[q_{T-1}(\mathbf{x},\mathbf{u},\boldsymbol{\Delta}_{T-1})], \tag{172}$$

and for $(\mathbf{x},\mathbf{u})\in\mathrm{graph}(U_{T-1})$ and $\delta\in\hat{D}$,

$$q_{T-1}(\mathbf{x},\mathbf{u},\delta) = (c_{T-1}+c_T^0\circ\mathbf{s}_T)(\mathbf{x},\mathbf{u},\delta). \tag{173}$$

Let us consider the following alternative problem:

$$v_{T-1}(\mathbf{x}) = \inf_{\mathbf{u} \in U_{T-1}(\mathbf{x})} j_{T-1}(\mathbf{x}, \mathbf{u}). \tag{174}$$

As (174) is not concerned with measurability, we can conclude that

$$J^*_{T-1}[\mathbf{U}_{[1,T-2]}](\delta_{[1,T-2]}) \geq v_{T-1}\left(\mathbf{X}_{T-1}^{\mathbf{U}_{[1,T-2]}}(\delta_{[1,T-2]})\right). \tag{175}$$

As $U_{T-1}(\mathbf{x})$ is a closed subset of the compact $\hat{U}$, it is a compact set as well. Note also that $U_{T-1}(\cdot)$ is a measurable multi-function. According to Rockafellar and Wets [20] (Theorem 14.37), as long as $j_{T-1}(\mathbf{x}, \cdot)$ is lower semi-continuous, there would be a measurable function $\mathbf{u}_{T-1}(\cdot)$ that solves (174) at every $\mathbf{x} \in \hat{X}$. This would mean that

$$v_{T-1}(\mathbf{x}) = \inf_{\mathbf{u}_{T-1} \in \mathcal{U}^M_{T-1}} j_{T-1}(\mathbf{x}, \mathbf{u}_{T-1}(\mathbf{x})). \tag{176}$$

Combining (171), (175), and (176), we could then obtain

$$J^*_{T-1}[\mathbf{U}_{[1,T-2]}](\delta_{[1,T-2]}) = v_{T-1}\left(\mathbf{X}_{T-1}^{\mathbf{U}_{[1,T-2]}}(\delta_{[1,T-2]})\right). \tag{177}$$

With (28), we could also arrive at the recursive relation (29) as well as the equation (30) at $t = T-1$ from (172) to (174).

Now we set out to prove the lower semi-continuity of $j_{T-1}(\cdot, \cdot)$, which is stronger than the lower semi-continuity of $j_{T-1}(\mathbf{x}, \cdot)$ at any given $\mathbf{x} \in \hat{X}$. It shall be clear later that the stronger property is needed to facilitate our induction process.

First, we establish a few useful bounds. Due to Assumptions 2 to 4, we know there are positive constants A, B, C, and D, so that

$$\mid q_{T-1}(\mathbf{x}, \mathbf{u}, \delta) \mid \leq A \cdot \mid\mid \mathbf{x} \mid\mid + B \cdot \mid\mid \mathbf{u} \mid\mid + C \cdot \mid\mid \delta \mid\mid + D. \tag{178}$$

Also, Assumptions 6 and 8, (172), and (178) together lead to

$$\mid j_{T-1}(\mathbf{x}, \mathbf{u}) \mid \leq A' \cdot \mid\mid \mathbf{x} \mid\mid + B' \cdot \mid\mid \mathbf{u} \mid\mid + C'', \tag{179}$$

for some positive constants A', B', and C''. In view of Assumption 5 and (174),

$$\mid v_{T-1}(\mathbf{x}) \mid \leq A'' \cdot \mid\mid \mathbf{x} \mid\mid + B'', \tag{180}$$

for some positive constants A'' and B''. This bound is in the same form as Assumption 3.

Now, from the lower semi-continuity of $c_{T-1}(\cdot, \cdot, \delta)$ and $c^0_T(\cdot)$ as prescribed by Assumptions 9 and 10, as well as the continuity of $\mathbf{s}_T(\cdot, \cdot, \delta)$ as prescribed by Assumption 11, we know the lower semi-continuity of $q_{T-1}(\cdot, \cdot, \delta)$ on graph(U_{T-1}).

Let us fix $(\mathbf{x}, \mathbf{u}) \in \text{graph}(U_{T-1})$ and $\epsilon > 0$. From (172), we know there exists $m \in M_{T-1}(\mathbf{x}, \mathbf{u})$, so that

$$\mathbb{E}^m_{T-1}[q_{T-1}(\mathbf{x}, \mathbf{u}, \mathbf{\Delta}_{T-1})] \geq j_{T-1}(\mathbf{x}, \mathbf{u}) - \epsilon. \tag{181}$$

Suppose a sequence $((\mathbf{x}_k, \mathbf{u}_k) \mid k = 1, 2, ...)$ in $\text{graph}(U_{T-1})$ converges to $(\mathbf{x}, \mathbf{u})$. In view of (178), we may, without loss of generality, suppose the existence of positive constants C' and D', so that

$$| \, q_{T-1}(\mathbf{x}_k, \mathbf{u}_k, \delta) \, | \leq C' \cdot || \, \delta \, || + D', \qquad \forall k = 1, 2, ... \tag{182}$$

By m's membership in the bounded $M_t(\mathbf{x}, \mathbf{u})$ and Assumption 6, this leads to

$$\mathbb{E}^m_{T-1}[| \, q_{T-1}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{\Delta}_{T-1}) \, |] \leq C' \cdot \mathbb{E}^m_{T-1}[|| \, \mathbf{\Delta}_{T-1} \, ||] + D' < +\infty. \tag{183}$$

Thus, by Fatou's lemma, we have

$$\liminf_{k \to +\infty} \mathbb{E}^m_{T-1}[q_{T-1}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{\Delta}_{T-1})] \geq \mathbb{E}^m_{T-1}[\liminf_{k \to +\infty} q_{T-1}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{\Delta}_{T-1})]. \tag{184}$$

On the other hand, from the lower semi-continuity of $q_{T-1}(\cdot, \cdot, \delta)$, we have

$$\liminf_{k \to +\infty} q_{T-1}(\mathbf{x}_k, \mathbf{u}_k, \delta) \geq q_{T-1}(\mathbf{x}, \mathbf{u}, \delta), \qquad \forall \delta \in \hat{D}. \tag{185}$$

Combining (184) and (185), we can obtain

$$\liminf_{k \to +\infty} \mathbb{E}^m_{T-1}[q_{T-1}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{\Delta}_{T-1})] \geq \mathbb{E}^m_{T-1}[q_{T-1}(\mathbf{x}, \mathbf{u}, \mathbf{\Delta}_{T-1})]. \tag{186}$$

This amounts to the existence of some $K = 1, 2, ...$, such that

$$\mathbb{E}^m_{T-1}[q_{T-1}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{\Delta}_{T-1})] \geq \mathbb{E}^m_{T-1}[q_{T-1}(\mathbf{x}, \mathbf{u}, \mathbf{\Delta}_{T-1})] - \epsilon, \qquad \forall k = K, K+1, ... \tag{187}$$

On the other hand, by Assumption 6, (182), and Assumption 12 on the lower hemi-continuity of $M_{T-1}(\cdot, \cdot)$, we know the existence of some $K' = K, K+1, ...$, so that

$$\mathbb{E}^{m_k}_{T-1}[q_{T-1}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{\Delta}_{T-1})] \geq \mathbb{E}^m_{T-1}[q_{T-1}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{\Delta}_{T-1})] - \epsilon, \tag{188}$$

for some $m_k \in M_{T-1}(\mathbf{x}_k, \mathbf{u}_k)$ at each $k = K', K'+1, ...$ Combine (172), (181), (187), and (188), and we obtain, for $k = K', K'+1, ...,$

$$\begin{aligned} j_{T-1}(\mathbf{x}_k, \mathbf{u}_k) &\geq \mathbb{E}^{m_k}_{T-1}[q_{T-1}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{\Delta}_{T-1})] \geq \mathbb{E}^m_{T-1}[q_{T-1}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{\Delta}_{T-1})] - \epsilon \\ &\geq \mathbb{E}^m_{T-1}[q_{T-1}(\mathbf{x}, \mathbf{u}, \mathbf{\Delta}_{T-1})] - 2\epsilon \geq j_{T-1}(\mathbf{x}, \mathbf{u}) - 3\epsilon. \end{aligned} \tag{189}$$

As $\epsilon > 0$ is arbitrarily chosen, this means the lower semi-continuity of $j_{T-1}(\cdot, \cdot)$ on $\text{graph}(U_{T-1})$.

From (168), (169), and (177), we have

$$
\begin{aligned}
J^* = \inf\nolimits_{\mathbf{U}_{[1,T-2]}\in\mathcal{U}_{[1,T-2]}} &\rho_1^{\mathbf{U}_1}(c_1(\mathbf{x}_1,\mathbf{U}_1,\boldsymbol{\Delta}_1)+\alpha\cdot\rho_2^{\mathbf{U}_{[1,2]}}(c_2(\mathbf{X}_2^{\mathbf{U}_1},\mathbf{U}_2,\boldsymbol{\Delta}_2)\\
&+\cdots+\alpha\cdot\rho_{T-2}^{\mathbf{U}_{[1,T-2]}}(c_{T-2}(\mathbf{X}_{T-2}^{\mathbf{U}_{[1,T-3]}},\mathbf{U}_{T-2},\boldsymbol{\Delta}_{T-2})+\alpha\cdot v_{T-1}(\mathbf{X}_{T-1}^{\mathbf{U}_{[1,T-2]}}))\cdots)).
\end{aligned}
\tag{190}
$$

This is almost the same problem as that we faced at (168), except that T needs to be replaced by $T-1$, c_T^0 replaced by v_{T-1}, and Assumption 3 replaced by (180). We can follow similar logic used to derive (29) and (30) for $t = T-2$. Except for the lower semi-continuity of $v_{T-1}(\cdot)$, we have everything needed for the derivation.

Now we set out to derive this property. Let $\mathbf{x}\in\hat{X}$ be given and let $(\mathbf{x}_k \mid k = 1,2,...)$ be a sequence in $\hat{X}$ that converges to $\mathbf{x}$. In view of (174), for any $\epsilon > 0$ and each $k = 1,2,...$, there is some $\mathbf{u}_k \in U_{T-1}(\mathbf{x}_k) \subseteq \hat{U}$ so that

$$
v_{T-1}(\mathbf{x}_k) \geq j_{T-1}(\mathbf{x}_k,\mathbf{u}_k) - \epsilon. \tag{191}
$$

Let $((\mathbf{x}_{k_i},\mathbf{u}_{k_i}) \mid i = 1,2,...)$ be a sub-sequence of the sequence $((\mathbf{x}_k,\mathbf{u}_k) \mid k = 1,2,...)$ in $\mathrm{graph}(U_{T-1})$ that satisfies

$$
\liminf_{k\to+\infty} j_{T-1}(\mathbf{x}_k,\mathbf{u}_k) \geq j_{T-1}(\mathbf{x}_{k_i},\mathbf{u}_{k_i}) - \epsilon, \qquad \forall i = 1,2,.... \tag{192}
$$

Since $\hat{U}$ is compact, we can identify a sub-sequence $(\mathbf{u}_{k_{i_h}} \mid h = 1,2,...)$ of $(\mathbf{u}_{k_i} \mid i = 1,2,...)$ that converges to some $\mathbf{u}\in\hat{U}$. By the upper hemi-continuity of $U_{T-1}(\cdot)$ as prescribed by Assumption 1, we know

$$
\mathbf{u} \in U_{T-1}(\mathbf{x}); \tag{193}
$$

whereas, from the lower semi-continuity of $j_{T-1}(\cdot,\cdot)$ on $\mathrm{graph}(U_{T-1})$, we have

$$
\liminf_{h\to+\infty} j_{T-1}(\mathbf{x}_{k_{i_h}},\mathbf{u}_{k_{i_h}}) \geq j_{T-1}(\mathbf{x},\mathbf{u}). \tag{194}
$$

Note (192) would translate into

$$
\liminf_{k\to+\infty} j_{T-1}(\mathbf{x}_k,\mathbf{u}_k) \geq \liminf_{h\to+\infty} j_{T-1}(\mathbf{x}_{k_{i_h}},\mathbf{u}_{k_{i_h}}) - \epsilon. \tag{195}
$$

Combine (174), (191), (193), (194), and (195), and we obtain

$$
\begin{aligned}
\liminf\nolimits_{k\to+\infty} v_{T-1}(\mathbf{x}_k) &\geq \liminf\nolimits_{k\to+\infty} j_{T-1}(\mathbf{x}_k,\mathbf{u}_k) - \epsilon\\
&\geq \liminf\nolimits_{h\to+\infty} j_{T-1}(\mathbf{x}_{k_{i_h}},\mathbf{u}_{k_{i_h}}) - 2\epsilon \geq j_{T-1}(\mathbf{x},\mathbf{u}) - 2\epsilon \geq v_{T-1}(\mathbf{x}) - 2\epsilon.
\end{aligned}
\tag{196}
$$

From the arbitrariness of $\epsilon > 0$, we see that $v_{T-1}(\cdot)$ is lower semi-continuous on $\hat{X}$. We can now derive (29) and (30) for $t = T-2$; and, this process can go on with $t = T-3, T-4, ..., 1$. The theorem can thus be iteratively proved. ∎

B. Technical Details in Section 3.

Proof of Lemma 2: Let $\mathbf{w}^0, \mathbf{w}^1 \in \hat{W}$ and $\lambda \in [0,1]$ be given, and let $\mathbf{w}^\lambda$ denote $(1-\lambda)\cdot \mathbf{w}^0 + \lambda \cdot \mathbf{w}^1$. For any $\epsilon > 0$, we can, due to $g(\mathbf{w}^\lambda)$'s definition, pick $i \in I$ so that

$$f(\mathbf{w}^\lambda, i) \geq g(\mathbf{w}^\lambda) - \epsilon. \tag{197}$$

Due to $f(\cdot, i)$'s $\mathcal{C}$-convexity, this would entail

$$g(\mathbf{w}^\lambda) \leq f(\mathbf{w}^\lambda, i) + \epsilon \leq \mathcal{C}(f(\mathbf{w}^0, i), f(\mathbf{w}^1, i), \mathbf{w}^0, \mathbf{w}^1, \lambda) + \epsilon. \tag{198}$$

But by the definitions of $g(\mathbf{w}^0)$ and $g(\mathbf{w}^1)$ as well as the monotonicity of $\mathcal{C}$ in its first two arguments, this further implies that

$$g(\mathbf{w}^\lambda) \leq \mathcal{C}(g(\mathbf{w}^0), g(\mathbf{w}^1), \mathbf{w}^0, \mathbf{w}^1, \lambda) + \epsilon. \tag{199}$$

Since ϵ can be made arbitrarily small, we can conclude that $g(\cdot)$ is $\mathcal{C}$-convex on $\hat{W}$. ∎

Proof of Lemma 3: Let $\mathbf{w}^0, \mathbf{w}^1 \in \hat{W}$ and $\lambda \in [0,1]$ be given, and let $\mathbf{w}^\lambda$ denote $(1-\lambda)\cdot \mathbf{w}^0 + \lambda \cdot \mathbf{w}^1$. For each $\omega \in \Omega^0$, we have

$$f(\mathbf{w}^\lambda, \omega^0) \leq (1-\lambda)\cdot f(\mathbf{w}^0, \omega) + \lambda \cdot f(\mathbf{w}^1, \omega) + c(\mathbf{w}^0, \mathbf{w}^1, \lambda). \tag{200}$$

Integrate both sides over Ω^0 and we obtain

$$g(\mathbf{w}^\lambda) \leq (1-\lambda)\cdot g(\mathbf{w}^0) + \lambda \cdot g(\mathbf{w}^1) + c(\mathbf{w}^0, \mathbf{w}^1, \lambda), \tag{201}$$

the desired inequality. ∎

Proof of Lemma 4: We can check that all assumptions leading to Theorem 1, except Assumption 5, are satisfied. First, note that $Y_t(\cdot) = [\cdot, +\infty]$ is indeed closed; hence, Assumption 1 is true. Next, this setup satisfies Assumptions 2 to 4. Also, Assumption 6 for R_t is always there, while Assumptions 7, 8, and 12 are automatically true for the current state-control-independent M_t. Finally, due to the various compactness and continuity properties, Assumptions 9 to 11 can be validated.

Now we go through an induction process. By (28), $v_T(x) = c_T^0(x) = -\bar{c}_T \cdot x$ is with an increasing rate that is above $-\bar{c}_T$; also, $(\bar{c}_T + v_T)(x) = 0$ indeed has a positive increasing rate. Suppose for some $t = T-1, T-2, ..., 1$, we know that

(i) $v_{t+1}(\cdot)$ is with an increasing rate above $-(\sum_{\tau=t+1}^{T-1} \alpha^{\tau-t-1}\cdot \bar{b}_\tau + \alpha^{T-t-1}\cdot \bar{c}_T)$; and,

(ii) $(\bar{c}_{t+1} + v_{t+1})(x)$ has a positive increasing rate when x is large enough.

Let $\Delta y \in \Re^+$ be arbitrary. From (38) and (39), we obtain

$$u_t(y + \Delta y) - u_t(y) = (\bar{c}_t - \alpha \bar{c}_{t+1}) \cdot \Delta y + \sup_{m \in M_t} o_t^m(y + \Delta y) - \sup_{m' \in M_t} o_t^{m'}(y), \tag{202}$$

where

$$o_t^m(y) = \mathbb{E}^m[(\tilde{h}_t + \alpha \cdot \bar{c}_{t+1} + \alpha \cdot v_{t+1})(y - \Delta_t)]. \tag{203}$$

For any y and $\epsilon > 0$, let $m_t(y, \epsilon) \in M_t$ be such that

$$o_t^{m_t(y,\epsilon)}(y) \geq \sup_{m' \in M_t} o_t^{m'}(y) - \epsilon. \tag{204}$$

It is certainly true that

$$\sup_{m \in M_t} o_t^m(y + \Delta y) \geq o_t^{m_t(y,\epsilon)}(y + \Delta y). \tag{205}$$

Hence, from (202), we obtain

$$u_t(y + \Delta y) - u_t(y) \geq (\bar{c}_t - \alpha \bar{c}_{t+1}) \cdot \Delta y + \mathbb{E}^{m_t(y,\epsilon)}[j_t(y + \Delta y - \Delta_t) - j_t(y - \Delta_t)] - \epsilon, \tag{206}$$

where

$$j_t(z) = \bar{h}_t \cdot z^+ + \bar{b}_t \cdot z^- + \alpha \bar{c}_{t+1} \cdot z + \alpha \cdot v_{t+1}(z). \tag{207}$$

Due to induction hypothesis (i), we see that $j_t(\cdot)$ is with an increasing rate above $\alpha \bar{c}_{t+1} - \sum_{\tau=t}^{T-1} \alpha^{\tau - t} \cdot \bar{b}_\tau - \alpha^{T-t} \cdot \bar{c}_T$. In view of (206) and the arbitrariness of ϵ, we see that $u_t(\cdot)$ has an increasing rate above $\bar{c}_t - (\sum_{\tau=t}^{T-1} \alpha^{\tau-t} \cdot \bar{b}_\tau + \alpha^{T-t} \cdot \bar{c}_T)$.

Suppose x and $\Delta x \geq 0$ are arbitrary. First consider the case where $u_t(x + \Delta x) \leq \bar{k}_t + \min_{y \in [x + \Delta x, +\infty]} u_t(y)$. From (37), we have

$$\begin{cases} v_t(x + \Delta x) & = -\bar{c}_t \cdot (x + \Delta x) + u_t(x + \Delta x), \\ v_t(x) & \leq -\bar{c}_t \cdot x + u_t(x). \end{cases} \tag{208}$$

Hence,

$$v_t(x + \Delta x) - v_t(x) \geq -\bar{c}_t \cdot \Delta x + u_t(x + \Delta x) - u_t(x). \tag{209}$$

Next consider the case where $u_t(x + \Delta x) > \bar{k}_t + \min_{y \in [x + \Delta x, +\infty]} u_t(y)$. We can identify $y^* \in (x + \Delta x, +\infty]$ so that

$$u_t(y^*) = \min_{y \in [x + \Delta x, +\infty]} u_t(y), \tag{210}$$

which renders

$$u_t(x + \Delta x) \wedge [\bar{k}_t + \min_{y \in [x + \Delta x, +\infty]} u_t(y)] = \bar{k}_t + \min_{y \in [x + \Delta x, +\infty]} u_t(y) \geq \bar{k}_t + u_t(y^*). \tag{211}$$

Note that $y^* - \Delta x \in [x, +\infty]$. Thus,

$$u_t(y^* - \Delta x) \geq \min_{y' \in [x,+\infty]} u_t(y'), \tag{212}$$

and hence

$$u_t(x) \wedge [\bar{k}_t + \min_{y' \in [x,+\infty]} u_t(y')] \leq \bar{k}_t + u_t(y^* - \Delta x). \tag{213}$$

Combining (37), (211), and (213), we obtain

$$v_t(x + \Delta x) - v_t(x) \geq -\bar{c}_t \cdot \Delta x + u_t(y^*) - u_t(y^* - \Delta x). \tag{214}$$

The above (209) for the first case and (214) for the second case, along with the increasing rates known for $u_t(\cdot)$, would together help to show that $v_t(\cdot)$ is with an increasing rate above $-(\sum_{\tau=t}^{T-1} \alpha^{\tau-t} \cdot \bar{b}_\tau + \alpha^{T-t} \cdot \bar{c}_T)$ and that $(\bar{c}_t + v_t)(x)$ has a positive increasing rate when x is large enough.

When we combine induction hypothesis (ii) with (207), we see that $j_t(z)$ increases in z at a rate above $\bar{h}_t$ when z is large enough, say $z \geq z_t^0$ for some constant z_t^0. For convenience, let $\bar{l}_t = \alpha\bar{c}_{t+1} - \sum_{\tau=t}^{T-1} \alpha^{\tau-t} \cdot \bar{b}_\tau - \alpha^{T-t} \cdot \bar{c}_T$. Earlier, we have shown that $j_t(\cdot)$'s increasing rate is always above $\bar{l}_t$. Suppose $\bar{h}_t \leq \bar{l}_t$, then from (206), we have

$$u_t(y + \Delta y) - u_t(y) \geq (\bar{c}_t + \bar{h}_t - \alpha\bar{c}_{t+1}) \cdot \Delta y - \epsilon, \tag{215}$$

which, due to the arbitrariness of ϵ and IPC Assumption 1, leads to the strict positivity of $u_t(\cdot)$'s increasing rate. On the other hand, suppose $\bar{h}_t > \bar{l}_t$. By IPC Assumption 2, we can let $\delta_t \geq 0$ be such that, for any $m \in M_t$,

$$\mathbb{E}^m[\mathbf{1}(\Delta_t > \delta_t)] < \frac{\epsilon}{\bar{h}_t - \bar{l}_t}, \tag{216}$$

where

$$\epsilon = \frac{\bar{c}_t + \bar{h}_t - \alpha\bar{c}_{t+1}}{3}, \tag{217}$$

whose strict positivity is ensured by IPC Assumption 1. Then from (206), we obtain, for any $y \geq z_t^0 + \delta_t$,

$$\begin{aligned}
&u_t(y + \Delta y) - u_t(y) - (\bar{c}_t - \alpha\bar{c}_{t+1}) \cdot \Delta y + \epsilon \\
&\quad \geq \mathbb{E}^{m_t(y,\epsilon)}[\{j_t(y + \Delta y - \Delta_t) - j_t(y - \Delta_t)\} \cdot \mathbf{1}(\Delta_t \leq \delta_t)] \\
&\qquad + \mathbb{E}^{m_t(y,\epsilon)}[\{j_t(y + \Delta y - \Delta_t) - j_t(y - \Delta_t)\} \cdot \mathbf{1}(\Delta_t > \delta_t)] \\
&\quad \geq \bar{h}_t \cdot \mathbb{E}^{m_t(y,\epsilon)}[\mathbf{1}(\Delta_t \leq \delta_t)] + \bar{l}_t \cdot \mathbb{E}^{m_t(y,\epsilon)}[\mathbf{1}(\Delta_t > \delta_t)] \\
&\quad = \bar{l}_t + (\bar{h}_t - \bar{l}_t) \cdot \mathbb{E}^{m_t(y,\epsilon)}[\mathbf{1}(\Delta_t \leq \delta_t)] \\
&\quad \geq \bar{l}_t + (\bar{h}_t - \bar{l}_t) \cdot (1 - \epsilon/(\bar{h}_t - \bar{l}_t)) = \bar{h}_t - \epsilon.
\end{aligned} \tag{218}$$

Due to (217) and IPC Assumption 1, we see that $u_t(y)$ is with a strictly positive increasing rate above $(2-\alpha)\cdot(\bar{c}_t+\bar{h}_t-\alpha\bar{c}_{t+1})/3$ when $y \geq z_t^0+\delta_t$. When we repeat arguments (208) to (214) for $x \geq z_t^0+\delta_t$, we can get that $(\bar{c}_t+v_t)(x)$ has a positive increasing rate when x is large enough.

In view of (37) and the strict positive increasing rate of $u_t(y)$ at large y values, we can see that $Y_t(x)$, ostensibly $[x,+\infty]$, is effectively $[x, x\vee \overline{y}_t]$ for some constant $\overline{y}_t$. Thus, the period-t version of (180) in the proof of Theorem 1 remains valid. So the induction from period $t+1$ to period t during that proof can be carried out without any hindrance.

The above constitutes an inductive step from period $t+1$ to period t. We can complete the induction process by letting t go from $T-1$ down to 1. When at the last point, Theorem 1's conclusion will also have been reached. ∎

Proof of Lemma 5: Just as in the proof of Lemma 4, we can check the satisfaction of Assumptions 1 to 12 except Assumption 5. Now we go through an induction process. By (28), $v_T(x)=c_T^0(x)=-\bar{c}_T\cdot x$ is with an increasing rate that is above $-\bar{c}_T$; also, $(\bar{c}_T+v_T)(x)=0$ indeed has a positive increasing rate. Suppose for some $t=T-1,T-2,...,1$, we know that

(i) $v_{t+1}(\cdot)$ is with an increasing rate above $-(\sum_{\tau=t+1}^{T-1}\alpha^{\tau-t-1}\cdot\bar{b}_\tau+\alpha^{T-t-1}\cdot\bar{c}_T)$; and,

(ii) $(\bar{c}_{t+1}+v_{t+1})(x)$ has a positive increasing rate when x is large enough.

Let $\Delta y\in\Re^+$ be arbitrary. From (45) and (46), we obtain

$$q_t(y+\Delta y,z)-q_t(y,z)=-\alpha\bar{c}_{t+1}\cdot\Delta y+\sup_{m\in M_t}o_t^m(y+\Delta y,z)-\sup_{m'\in M_t}o_t^{m'}(y,z), \tag{219}$$

where

$$o_t^m(y,z)=\mathbb{E}^m[(\tilde{h}_t+\alpha\cdot\bar{c}_{t+1}+\alpha\cdot v_{t+1})(y-z-\Theta_t)]. \tag{220}$$

For any (y,z) and $\epsilon>0$, let $m_t(y,z,\epsilon)\in M_t$ be such that

$$o_t^{m_t(y,z,\epsilon)}(y,z)\geq\sup_{m'\in M_t}o_t^{m'}(y,z)-\epsilon. \tag{221}$$

It is certainly true that

$$\sup_{m\in M_t}o_t^m(y+\Delta y,z)\geq o_t^{m_t(y,z,\epsilon)}(y+\Delta y,z). \tag{222}$$

Hence, from (219), we obtain

$$\begin{aligned}q_t(y+\Delta y,z)-q_t(y,z)\geq&-\alpha\bar{c}_{t+1}\cdot\Delta y\\&+\mathbb{E}^{m_t(y,z,\epsilon)}[j_t(y+\Delta y-z-\Theta_t)-j_t(y-z-\Theta_t)]-\epsilon,\end{aligned} \tag{223}$$

where $j_t(\cdot)$ is the same as that defined in (207). Due to induction hypothesis (i), we see that $j_t(\cdot)$ is with an increasing rate above $\alpha\bar{c}_{t+1}-\sum_{\tau=t}^{T-1}\alpha^{\tau-t}\cdot\bar{b}_\tau-\alpha^{T-t}\cdot\bar{c}_T$. In view of (223) and the arbitrariness of ϵ, we see that $q_t(\cdot, z)$ has an increasing rate above $-(\sum_{\tau=t}^{T-1} \alpha^{\tau-t} \cdot \bar{b}_\tau + \alpha^{T-t} \cdot \bar{c}_T)$. From (44), we have

$$u_t(y + \Delta y) - u_t(y) = \bar{c}_t \cdot \Delta y + \min_{z \in [\underline{z}_t, \overline{z}_t]} q_t(y + \Delta y, z) - \min_{z \in [\underline{z}_t, \overline{z}_t]} q_t(y, z). \tag{224}$$

Using logic similar to the one used above, we can show that $u_t(\cdot)$ has an increasing rate above $\bar{c}_t - (\sum_{\tau=t}^{T-1} \alpha^{\tau-t} \cdot \bar{b}_\tau + \alpha^{T-t} \cdot \bar{c}_T)$. Following the same reasoning as employed from (208) to (214), we can then establish that $v_t(\cdot)$ as defined by (37) has an increasing rate above $-(\sum_{\tau=t}^{T-1} \alpha^{\tau-t} \cdot \bar{b}_\tau + \alpha^{T-t} \cdot \bar{c}_T)$.

Capitalizing on induction hypothesis (ii) and IPC Assumptions 1 and 4, we can take almost the same steps as those from (215) to (218) to demonstrate that $(\bar{c}_t + v_t)(x)$ has a positive increasing rate when x is large enough. The only change needed is that the earlier Δ_t be substituted by the current $z + \Theta_t$, which is bounded by $\overline{z}_t + \Theta_t$. The rest of the proof constitutes a repetition of the last two paragraphs of Lemma 4's proof. ∎

C. Technical Details in Section 5.

Proof of Lemma 13: By (86) to (89), we obtain

$$\begin{aligned} v_1(x) - v_0(x) &= u_1(x) \wedge [\bar{k} + \min\nolimits_{y \in [x, +\infty]} u_1(y)] \geq \min\nolimits_{y \in [x, +\infty]} u_1(y) \\ &= \min\nolimits_{y \in [x, +\infty]} \{(1-\alpha) \cdot \bar{c} \cdot y + \sup\nolimits_{m \in M} \mathbb{E}^m[\tilde{h}(y - \Delta)]\} + \alpha \bar{c} \cdot \sup\nolimits_{m \in M} \mathbb{E}^m[\Delta]. \end{aligned} \tag{225}$$

Due to the positivity of Δ, $\bar{c}$, and $\tilde{h}(\cdot)$, as well as the fact that α is between 0 and 1, this leads to

$$v_1(x) - v_0(x) \geq \min_{y \in [x, +\infty]} \{(1-\alpha) \cdot \bar{c} \cdot y\} = (1-\alpha) \cdot \bar{c} \cdot \min_{y \in [x, +\infty]} y = (1-\alpha) \cdot \bar{c} \cdot x. \tag{226}$$

Thus, (90) is true for $t = 1$. Suppose (90) is true for $t = T-1$ for some $T = 2, 3, ...$ That is,

$$v_{T-1}(x) - v_{T-2}(x) \geq \alpha^{T-2} \cdot (1-\alpha) \cdot \bar{c} \cdot (x - (T-2) \cdot \overline{\delta}). \tag{227}$$

Now, from (89), we get

$$w_T^m(y) - w_{T-1}^m(y) = \alpha \cdot \mathbb{E}^m[(v_{T-1} - v_{T-2})(y - \Delta)], \tag{228}$$

which by (227), is greater than $\alpha^{T-1} \cdot (1-\alpha) \cdot \bar{c} \cdot (y - \mathbb{E}^m[\Delta] - (T-2) \cdot \overline{\delta})$. Due to (88) and IPC Assumption 6, this results in

$$\begin{aligned} u_T(y) - u_{T-1}(y) &= \sup\nolimits_{m \in M} w_T^m(y) - \sup\nolimits_{m' \in M} w_{T-1}^{m'}(y) \\ &\geq \sup\nolimits_{m \in M} \{w_{T-1}^m(y) + (\alpha^{T-1} - \alpha^T) \cdot \bar{c} \cdot (y - \mathbb{E}^m[\Delta])\} - \sup\nolimits_{m' \in M} w_{T-1}^{m'}(y) \\ &\quad -(\alpha^{T-1} - \alpha^T) \cdot (T-2) \cdot \overline{\delta} \\ &\geq (\alpha^{T-1} - \alpha^T) \cdot \bar{c} \cdot (y - \sup\nolimits_{m \in M} \mathbb{E}^m[\Delta]) - (\alpha^{T-1} - \alpha^T) \cdot (T-2) \cdot \overline{\delta} \\ &= \alpha^{T-1} \cdot (1-\alpha) \cdot \bar{c} \cdot (y - (T-1) \cdot \overline{\delta}). \end{aligned} \tag{229}$$

Thus, by (87), we obtain

$$
\begin{aligned}
&v_T(x) - v_{T-1}(x) \\
&\quad = u_T(x) \wedge [\bar{k} + \min_{y\in[x,+\infty]} u_T(y)] - u_{T-1}(x) \wedge [\bar{k} + \min_{y'\in[x,+\infty]} u_{T-1}(y')] \\
&\quad \geq u_T(x) \wedge [\bar{k} + \min_{y\in[x,+\infty]} u_T(y)] - [u_T(x) - \alpha^{T-1} \cdot (1-\alpha) \cdot \bar{c} \cdot x] \wedge \\
&\qquad\qquad \wedge [\bar{k} + \min_{y'\in[x,+\infty]} \{u_T(y') - \alpha^{T-1} \cdot (1-\alpha) \cdot \bar{c} \cdot y'\}] \\
&\qquad\quad - \alpha^{T-1} \cdot (1-\alpha) \cdot (T-1) \cdot \bar{c} \cdot \overline{\delta} \\
&\quad \geq \alpha^{T-1} \cdot (1-\alpha) \cdot \bar{c} \cdot (y^*_T(x) - (T-1) \cdot \overline{\delta}) \\
&\quad \geq \alpha^{T-1} \cdot (1-\alpha) \cdot \bar{c} \cdot (x - (T-1) \cdot \overline{\delta}).
\end{aligned}
\tag{230}
$$

Therefore, (90) is true for $t = T$. We have thus completed the induction process. ∎

Proof of Lemma 14: For convenience, let

$$A_t = (1-\alpha^t) \cdot (\bar{c} + \frac{\bar{h}}{1-\alpha}) - \alpha^t \cdot \bar{k}, \qquad \forall t = 1, 2, ..., \tag{231}$$

and

$$B = \bar{b} + \alpha \cdot \bar{k} - (1-\alpha) \cdot \bar{c}. \tag{232}$$

Note $\bar{c} > 0$ or $\bar{h} > 0$. As $\alpha \in [0, 1)$ and $\bar{k} \geq 0$, it is easy to see that A_t as defined in (231) is increasing in t; also, there must exist a positive integer $\bar{t}$ so that

$$A_{\bar{t}} > 0, \qquad \forall t \geq \bar{t}. \tag{233}$$

Hence, as long as $\epsilon \in (0, 1]$ is small enough, we can ensure that

$$(1-\epsilon) \cdot A_{\bar{t}} > \epsilon \cdot B, \tag{234}$$

regardless of the sign of B; for instance, we can let $\epsilon = 1$ when $B \leq 0$.

For a fixed $T = \bar{t}+1, \bar{t}+2, ...$, let sequence $m = (m_1, m_2, ..., m_{T-1}) \in M^{T-1}$ be arbitrarily given. Consider a $(T-1)$-period problem. Suppose the firm starts period 1 with inventory level x_1, and suppose the demand levels it experiences in periods $1, 2, ..., T-1$ are $\Delta_1, \Delta_2, ..., \Delta_{T-1}$, where each Δ_t is distributed according to the probability measure $\int m_t \cdot dR$ and different Δ_t and $\Delta_{t'}$ levels are mutually independent. We can let stopping time τ be such that $x_1 - \sum_{t=1}^{\tau} \Delta_t \geq 0$ and yet $\tau + 1 = T$ or $x_1 - \sum_{t=1}^{\tau+1} \Delta_t < 0$. We certainly have $\tau = 0$ when $\Delta_1 > x_1$ and $\tau = T-1$ when $x_1 - \sum_{t=1}^{T-1} \Delta_t \geq 0$.

Note that IPC Assumption 6 implies a time-invariant version of IPC Assumption 2. Due to this, as long as x_1 is large enough, we can have

$$\sup_{m'\in M} \mathbb{E}^{m'} \left[\mathbf{1}(\Delta > \frac{x_1}{\bar{t}}) \right] < \frac{\epsilon}{\bar{t}}. \tag{235}$$

Let P^m be the product measure $\prod_{t=1}^{T-1}(\int m_t \cdot dR)$ on the measurable space $((\Re^+)^{T-1}, \mathcal{B}((\Re^+)^{T-1}))$. Now,

$$\begin{aligned}
&\sup_{m\in M^{T-1}} P^m[0 \le \tau \le \bar{t}-1] \\
&\quad = \sup_{m_1,\dots,m_{\bar{t}}\in M} \mathbb{E}^{m_1}[\cdots[\mathbb{E}^{m_{\bar{t}}}[\mathbf{1}(\Delta_1+\cdots+\Delta_{\bar{t}} > x_1)]]\cdots] \\
&\quad \le \sup_{m_1,\dots,m_{\bar{t}}\in M} \mathbb{E}^{m_1}[\cdots[\mathbb{E}^{m_{\bar{t}}}[\mathbf{1}(\Delta_1, \text{ or } \cdots, \text{or } \Delta_{\bar{t}} > x_1/\bar{t})]]\cdots] \\
&\quad \le \sup_{m_1,\dots,m_{\bar{t}}\in M}\{\mathbb{E}^{m_1}[\mathbf{1}(\Delta_1 > x_1/\bar{t})]+\cdots+\mathbb{E}^{m_{\bar{t}}}[\mathbf{1}(\Delta_{\bar{t}} > x_1/\bar{t})]\} \\
&\quad = \sup_{m_1\in M} \mathbb{E}^{m_1}[\mathbf{1}(\Delta_1 > x_1/\bar{t})]+\cdots+\sup_{m_{\bar{t}}\in M}\mathbb{E}^{m_{\bar{t}}}[\mathbf{1}(\Delta_{\bar{t}} > x_1/\bar{t})],
\end{aligned} \tag{236}$$

which is in turn smaller than ϵ due to (235).

Let us consider the timing of ordering for an extra item. When $\tau = T-1$, the order would cost at least $\bar{c}+\bar{h}+\alpha\cdot\bar{h}+\cdots+\alpha^{T-2}\cdot\bar{h}-\alpha^{T-1}\cdot\bar{c} = (1-\alpha^{T-1})\cdot(\bar{c}+\bar{h}/(1-\alpha)) \ge A_{T-1}$. When $\tau = 1,2,\dots,\bar{t}-1,\bar{t},\dots,T-2$, that of occurring in period 1 would cost more than that of occurring in period $\tau+1$ by at least $\bar{c}+\bar{h}+\alpha\cdot\bar{h}+\cdots+\alpha^{\tau-1}\cdot\bar{h}-\alpha^{\tau}\cdot(\bar{k}+\bar{c}) = (1-\alpha^{\tau})\cdot(\bar{c}+\bar{h}/(1-\alpha))-\alpha^{\tau}\cdot\bar{k} \ge A_{\tau}$. When $\tau=0$, that of occurring in period 1 could save from that of occurring in period 2 by at most $\bar{b}+\alpha\cdot(\bar{k}+\bar{c})-\bar{c} = \bar{b}-(1-\alpha)\cdot\bar{c}+\alpha\cdot\bar{k} = B$.

From (231) and (232), we may see

$$A_{T-1} \ge A_{T-2} \ge \cdots \ge A_{\bar{t}} \ge A_{\bar{t}-1} \ge \cdots \ge A_1 \ge -B. \tag{237}$$

Then, the expected cost ΔC^m of ordering an extra item in period 1 rather than at an optimal time satisfies

$$\Delta C^m \ge A_{\bar{t}} \cdot P^m[\bar{t} \le \tau \le T-1] + (-B)\cdot P^m[0 \le \tau \le \bar{t}-1], \tag{238}$$

which by (236), results with

$$\Delta C^m \ge A_{\bar{t}}\cdot(1-\epsilon) - B\cdot\epsilon > 0, \tag{239}$$

where the last inequality is due to (234).

This shows it is not optimal to order when the starting inventory x_1 is high enough in a t-period problem when $t = \bar{t}, \bar{t}+1, \dots$. From (231) to (235), we see that the criterion of "high enough" can be set by $x_1 \ge \overline{y}'_{\bar{t}}$ for some constant $\overline{y}'_{\bar{t}}$. From Lemma 4, we already know the existence of a similar level $\overline{y}_t$ for period 1 in a t-period problem, when $t = 1,2,\dots,\bar{t}-1$. We can simply set $\overline{y} = (\max_{t=1}^{\bar{t}-1}\overline{y}_t \vee \overline{y}'_{\bar{t}})^+$. This $\overline{y}$ would be an upper bound for x_1 beyond which ordering is not necessary for any t-period problem, at $t = 1,2,\dots$. But the order-up-to level in period 1 in a t-period problem is simply a solution to (87). ∎

Proof of Lemma 15: By (86) to (89), we obtain

$$v_1(x) - v_0(x) \le u_1(x) = (1-\alpha)\cdot\bar{c}\cdot x + \sup_{m\in M}\mathbb{E}^m[\tilde{h}(x-\Delta)] + \alpha\bar{c}\cdot\sup_{m\in M}\mathbb{E}^m[\Delta]. \tag{240}$$

This, by $\tilde{h}(x-\delta) \leq (\bar{h} \vee \bar{b}) \cdot (\mid x \mid +\delta)$, IPC Assumption 6, and other obvious facts, results with

$$v_1(x) - v_0(x) \leq A \cdot \mid x \mid + B \cdot \overline{\delta}. \tag{241}$$

Thus, (91) is true for $t = 1$. Suppose (91) is true for $t = T - 1$ for some $T = 2, 3, ...$ That is,

$$v_{T-1}(x) - v_{T-2}(x) \leq \alpha^{T-2} \cdot (A \cdot \mid x \mid + (A \cdot (T-2) + B) \cdot \overline{\delta} + A \cdot (T-2) \cdot \overline{y}). \tag{242}$$

Note (228) is again true. Through IPC Assumption 6 and (242), this leads to

$$\begin{aligned} w_T^m(y) - w_{T-1}^m(y) \quad &\leq \alpha^{T-1} \cdot A \cdot \mathbb{E}^m[\mid y - \Delta \mid] + \alpha^{T-1} \cdot (A \cdot (T-2) + B) \cdot \overline{\delta} \\ &\quad + \alpha^{T-1} \cdot A \cdot (T-2) \cdot \overline{y} \\ &\leq \alpha^{T-1} \cdot (A \cdot \mid y \mid + (A \cdot (T-1) + B) \cdot \overline{\delta} + A \cdot (T-2) \cdot \overline{y}). \end{aligned} \tag{243}$$

Note this upper bound is m-independent. Thus, by (88), we have

$$\begin{aligned} u_T(y) - u_{T-1}(y) \quad &= \sup_{m \in M} w_T^m(y) - \sup_{m' \in M} w_{T-1}^{m'}(y) \\ &\leq \alpha^{T-1} \cdot (A \cdot \mid y \mid + (A \cdot (T-1) + B) \cdot \overline{\delta} + A \cdot (T-2) \cdot \overline{y}). \end{aligned} \tag{244}$$

Now with (87), we obtain

$$\begin{aligned} & v_T(x) - v_{T-1}(x) \\ & \quad = u_T(x) \wedge [\bar{k} + \min_{y \in [x,+\infty]} u_T(y)] - u_{T-1}(x) \wedge [\bar{k} + \min_{y' \in [x,+\infty]} u_{T-1}(y')] \\ & \quad \leq [u_{T-1}(x) + \alpha^{T-1} \cdot A \cdot \mid x \mid] \wedge [\bar{k} + \min_{y \in [x,+\infty]} \{u_{T-1}(y) + \alpha^{T-1} \cdot A \cdot \mid y \mid\}] \\ & \qquad - u_{T-1}(x) \wedge [\bar{k} + \min_{y' \in [x,+\infty]} u_{T-1}(y')] \\ & \qquad + \alpha^{T-1} \cdot ((A \cdot (T-1) + B) \cdot \overline{\delta} + A \cdot (T-2) \cdot \overline{y}) \\ & \quad \leq \alpha^{T-1} \cdot (A \cdot \mid y_{T-1}^*(x) \mid + (A \cdot (T-1) + B) \cdot \overline{\delta} + A \cdot (T-2) \cdot \overline{y}). \end{aligned} \tag{245}$$

Due to Lemma 14, this then leads to

$$\begin{aligned} v_T(x) - v_{T-1}(x) \quad &\leq \alpha^{T-1} \cdot (A \cdot \mid x \vee \overline{y} \mid + (A \cdot (T-1) + B) \cdot \overline{\delta} + A \cdot (T-2) \cdot \overline{y}) \\ &\leq \alpha^{T-1} \cdot (A \cdot \mid x \mid + (A \cdot (T-1) + B) \cdot \overline{\delta} + A \cdot (T-1) \cdot \overline{y}). \end{aligned} \tag{246}$$

Therefore, (91) is true for $t = T$. We have thus completed the induction process. ∎

Proof of Lemma 16: We can combine Lemmas 13 and 15 into that, for some positive constants A and B,

$$\mid v_{t+1}(x) - v_t(x) \mid \leq \alpha^t \cdot (A \cdot \mid x \mid + t \cdot B), \qquad \forall t = 0, 1, ... \tag{247}$$

Thus, for $t, t' = 1, 2, ...$ satisfying $t < t'$,

$$\begin{aligned} \mid v_t(x) - v_{t'}(x) \mid &\leq A \cdot \mid x \mid \cdot \sum_{t''=t}^{t'-1} \alpha^{t''} + B \cdot \sum_{t''=t}^{t'-1} t'' \alpha^{t''} \\ &\leq A \cdot \mid x \mid \cdot \sum_{t''=t}^{+\infty} \alpha^{t''} + B \cdot \sum_{t''=t}^{+\infty} t'' \alpha^{t''} \\ &= \alpha^t \cdot [A \cdot \mid x \mid + ((t-1) + 1/(1-\alpha)) \cdot B]/(1-\alpha). \end{aligned} \tag{248}$$

As $t \to +\infty$, the right-hand side uniformly converges to 0 in any bounded x-region. Therefore, $\{v_t(x) \mid t = 0, 1, ...\}$ is a Cauchy sequence with uniform convergence in any bounded x-region. Thus, there exists some $v(x)$ that validates the conclusion of the proposition. Especially, from (248), we can see that (93) is true. From (248) again, we have

$$\mid v_1(x) - v(x) \mid \leq \alpha \cdot [A \cdot \mid x \mid + B/(1-\alpha)]/(1-\alpha). \tag{249}$$

This, when combined with (86), as well as (90) and (91) at $t = 1$, would lead to (92). ∎

Proof of Lemma 17: In view of (89), (94), and Lemma 16, we can use the dominated convergence theorem to deduce the convergence of $w_t^m(y)$ to $w^m(y)$. Due to IPC Assumption 6 and (93), the convergence is uniform in $m \in M$ and in any bounded y-region. With (88) and (95), we have

$$u_t(y) - u(y) = \sup_{m \in M} w_t^m(y) - \sup_{m' \in M} w^{m'}(y) \leq \sup_{m \in M} (w_t^m(y) - w^m(y)), \tag{250}$$

and symmetrically an opposite inequality. Combined with the convergence result on $w_t^m(y)$ to $w^m(y)$, this leads to the convergence of $u_t(y)$ to $u(y)$, with it being uniform in any bounded y-region.

From the above, we see that $\min_{y \in [x, x \vee \overline{y}]} u_t(y)$ would converge to $\min_{y \in [x, x \vee \overline{y}]} u(y)$. On the other hand, Lemma 14 says that $\min_{y \in [x, +\infty]} u_t(y) = \min_{y \in [x, x \vee \overline{y}]} u_t(y)$ for any t, and implies that $u_t(y) \geq u_t(\overline{y})$ for any t and $y \geq \overline{y}$. By the convergence of $u_t(y)$ to $u(y)$, we have $u(y) \geq u(\overline{y})$ for $y \geq \overline{y}$ too. Thus, $\min_{y \in [x, +\infty]} u(y) = \min_{y \in [x, x \vee \overline{y}]} u(y)$. Therefore, $\min_{y \in [x, +\infty]} u_t(y)$ converges to $\min_{y \in [x, +\infty]} u(y)$. This, in combination with (87) and Lemma 16, would result with (96). ∎

Proof of Lemma 18: By (86), (87), and (97) to (99), we obtain

$$\begin{aligned}
&v_1(x) - v_0(x) = u_1(x) \wedge [\bar{k} + \min_{y \in [x, +\infty]} u_1(y)] \geq \min_{y \in [x, +\infty]} u_1(y) \\
&\quad = \min_{y \in [x, +\infty]} \{\bar{c} \cdot y + \min_{z \in [\underline{z}, \overline{z}]} \sup_{m \in M} w_1^m(y, z)\} \\
&\quad = \min_{y \in [x, +\infty]} \{(1-\alpha) \cdot \bar{c} \cdot y + \min_{z \in [\underline{z}, \overline{z}]} \sup_{m \in M} \\
&\qquad\qquad \{\alpha \bar{c} \cdot \mathbb{E}^m[z + \Theta] - \tilde{r}^m(z) + \mathbb{E}^m[\tilde{h}(y - z - \Theta)]\}\} \\
&\quad \geq \min_{y \in [x, +\infty]} \{(1-\alpha) \cdot \bar{c} \cdot y\} + \min_{z \in [\underline{z}, \overline{z}]} \sup_{m \in M} \{\alpha \bar{c} \cdot \mathbb{E}^m[z + \Theta] - \tilde{r}^m(z)\},
\end{aligned} \tag{251}$$

where the last inequality is due to the positivity of $\tilde{h}(\cdot)$. Due to IPC Assumption 8,

$$v_1(x) - v_0(x) \geq (1-\alpha) \cdot \bar{c} \cdot x - \overline{r}. \tag{252}$$

Thus, (100) is true for $t = 1$. Suppose (100) is true for $t = T - 1$ with $T = 2, 3, ...$ That is,

$$v_{T-1}(x) - v_{T-2}(x) \geq \alpha^{T-2} \cdot (1-\alpha) \cdot \bar{c} \cdot (x - (T-2) \cdot \overline{\delta}) - \alpha^{T-2} \overline{r}. \tag{253}$$

Now, from (99), we get

$$w_T^m(y,z) - w_{T-1}^m(y,z) = \alpha \cdot \mathbb{E}^m[(v_{T-1} - v_{T-2})(y - z - \Theta)], \tag{254}$$

which by (253), is greater than $\alpha^{T-1} \cdot (1-\alpha) \cdot \bar{c} \cdot (y - \mathbb{E}^m[z+\Theta] - (T-2) \cdot \overline{\delta}) - \alpha^{T-1}\overline{r}$. Due to (98) and Assumption 7, this results in

$$\begin{array}{l} q_T(y,z) - q_{T-1}(y,z) = \sup_{m \in M} w_T^m(y,z) - \sup_{m' \in M} w_{T-1}^{m'}(y,z) \\ \quad \geq \sup_{m \in M}\{w_{T-1}^m(y,z) + (\alpha^{T-1} - \alpha^T) \cdot \bar{c} \cdot (y - \mathbb{E}^m[z+\Theta])\} \\ \qquad - \sup_{m' \in M} w_{T-1}^{m'}(y,z) - (\alpha^{T-1} - \alpha^T) \cdot (T-2) \cdot \overline{\delta} - \alpha^{T-1}\overline{r} \\ \quad \geq (\alpha^{T-1} - \alpha^T) \cdot \bar{c} \cdot (y - \sup_{m \in M} \mathbb{E}^m[\Gamma\overline{z} + \Theta]) \\ \qquad -(\alpha^{T-1} - \alpha^T) \cdot (T-2) \cdot \overline{\delta} - \alpha^{T-1}\overline{r} \\ \quad = \alpha^{T-1} \cdot (1-\alpha) \cdot \bar{c} \cdot (y - (T-1) \cdot \overline{\delta}) - \alpha^{T-1}\overline{r}. \end{array} \tag{255}$$

Note the above right-hand side is z-independent. Thus, with (97), we have

$$\begin{array}{ll} u_T(y) - u_{T-1}(y) & = \min_{z \in [\underline{z},\overline{z}]} q_T(y,z) - \min_{z' \in [\underline{z},\overline{z}]} q_{T-1}(y,z') \\ & \geq \alpha^{T-1} \cdot (1-\alpha) \cdot \bar{c} \cdot (y - (T-1) \cdot \overline{\delta}) - \alpha^{T-1}\overline{r}. \end{array} \tag{256}$$

Then, due to (87), we have

$$\begin{array}{l} v_T(x) - v_{T-1}(x) \\ \quad = u_T(x) \wedge [\bar{k} + \min_{y \in [x,+\infty]} u_T(y)] - u_{T-1}(x) \wedge [\bar{k} + \min_{y' \in [x,+\infty]} u_{T-1}(y')] \\ \quad \geq u_T(x) \wedge [\bar{k} + \min_{y \in [x,+\infty]} u_T(y)] - [u_T(x) - \alpha^{T-1} \cdot (1-\alpha) \cdot \bar{c} \cdot x] \wedge \\ \qquad\qquad \wedge[\bar{k} + \min_{y' \in [x,+\infty]}\{u_T(y') - \alpha^{T-1} \cdot (1-\alpha) \cdot \bar{c} \cdot y'\}] \\ \qquad -\alpha^{T-1} \cdot (1-\alpha) \cdot (T-1) \cdot \bar{c} \cdot \overline{\delta} - \alpha^{T-1}\overline{r} \\ \quad \geq \alpha^{T-1} \cdot (1-\alpha) \cdot \bar{c} \cdot (y_T^*(x) - (T-1) \cdot \overline{\delta}) - \alpha^{T-1}\overline{r} \\ \quad \geq \alpha^{T-1} \cdot (1-\alpha) \cdot \bar{c} \cdot (x - (T-1) \cdot \overline{\delta}) - \alpha^{T-1}\overline{r}. \end{array} \tag{257}$$

Therefore, (100) is true for $t = T$. We have thus completed the induction process. ∎

Proof of lemma 19: Define A_t's and B following the earlier (231) and (232), respectively. It is again known that A_t is increasing in t, and that there is a positive integer $\bar{t}$ that satisfies (233). When $\epsilon \in (0,1]$ is small enough, (234) would be true.

For a fixed $T = \bar{t}+1, \bar{t}+2, ...,$ let sequence $m = (m_1, m_2, ..., m_{T-1}) \in M^{T-1}$ be arbitrarily given. Consider a $(T-1)$-period problem. Suppose the firm starts period 1 with inventory level x_1, and suppose the demand parameters it experiences in periods $1, 2, ..., T-1$ are $\Theta_1, \Theta_2, ..., \Theta_{T-1}$, where each Θ_t is distributed according to the probability measure $\int m_t \cdot dR$ and different Θ_t's are mutually independent. We can let stopping time τ be such that

$x_1 - \sum_{t=1}^{\tau}(z+\Theta_t) \geq 0$ and yet $\tau+1=T$ or $x_1 - \sum_{t=1}^{\tau+1}(\overline{z}+\Theta_t) < 0$. We certainly have $\tau = 0$ when $\overline{z}+\Theta_1 > x_1$ and $\tau = T-1$ when $x_1 - \sum_{t=1}^{T-1}(\overline{z}+\Theta_t) \geq 0$.

Note that Assumption 7 implies a time-invariant version of IPC Assumption 4. Due to this, as long as x_1 is large enough, we can have

$$\sup_{m'\in M} \mathbb{E}^{m'}[\mathbf{1}(\overline{z}+\Theta > \frac{x_1}{\overline{t}})] < \frac{\epsilon}{\overline{t}}. \tag{258}$$

Let P^m be the product measure $\prod_{t=1}^{T-1}(\int m_t \cdot dR)$ on the measurable space $\Re^{T-1}, \mathcal{B}(\Re^{T-1}))$. Let $\Delta_t(z) = z+\Theta_t$. Then,

$$\begin{aligned}
&\sup_{m\in M^{T-1}} P^m[0 \leq \tau \leq \overline{t}-1] \\
&\quad = \sup_{m_1,\ldots,m_{\overline{t}}\in M} \mathbb{E}^{m_1}[\cdots[\mathbb{E}^{m_{\overline{t}}}[\mathbf{1}(\Delta_1(\overline{z})+\cdots+\Delta_{\overline{t}}(\overline{z}) > x_1)]]\cdots] \\
&\quad \leq \sup_{m_1,\ldots,m_{\overline{t}}\in M} \mathbb{E}^{m_1}[\cdots[\mathbb{E}^{m_{\overline{t}}}[\mathbf{1}(\Delta_1(\overline{z}), \text{ or } \cdots, \text{ or } \Delta_{\overline{t}}(\overline{z}) > x_1/\overline{t})]]\cdots] \\
&\quad \leq \sup_{m_1,\ldots,m_{\overline{t}}\in M} \{\mathbb{E}^{m_1}[\mathbf{1}(\Delta_1(\overline{z}) > x_1/\overline{t})] + \cdots + \mathbb{E}^{m_{\overline{t}}}[\mathbf{1}(\Delta_{\overline{t}}(\overline{z}) > x_1/\overline{t})]\} \\
&\quad = \sup_{m_1\in M} \mathbb{E}^{m_1}[\mathbf{1}(\Delta_1(\overline{z}) > x_1/\overline{t})] + \cdots + \sup_{m_{\overline{t}}\in M} \mathbb{E}^{m_{\overline{t}}}[\mathbf{1}(\Delta_{\overline{t}}(\overline{z}) > x_1/\overline{t})],
\end{aligned} \tag{259}$$

which is in turn smaller than ϵ due to (258).

Let us consider the timing of ordering for an extra item against the backdrop of any fixed pricing strategy. Note the revenue the firm can earn from sales does not hinge on the timing of ordering. When $\tau = T-1$, the order would cost at least $\overline{c}+\overline{h}+\alpha\cdot\overline{h}+\cdots+\alpha^{T-2}\cdot\overline{h} - \alpha^{T-1}\cdot\overline{c} = (1-\alpha^{T-1})\cdot(\overline{c}+\overline{h}/(1-\alpha)) \geq A_{T-1}$. When $\tau = 1,2,...,\overline{t}-1,\overline{t},...,T-2$, that of occurring in period 1 would cost more than that of occurring in period $\tau+1$ by at least $\overline{c}+\overline{h}+\alpha\cdot\overline{h}+\cdots+\alpha^{\tau-1}\cdot\overline{h}-\alpha^{\tau}\cdot(\overline{k}+\overline{c}) = (1-\alpha^{\tau})\cdot(\overline{c}+\overline{h}/(1-\alpha)) - \alpha^{\tau}\cdot\overline{k} \geq A_{\tau}$. When $\tau = 0$, that of occurring in period 1 could save from that of occurring in period 2 by at most $\overline{b}+\alpha\cdot(\overline{k}+\overline{c})-\overline{c} = \overline{b}-(1-\alpha)\cdot\overline{c}+\alpha\cdot\overline{k} = B$.

It would then follow that (237) to (238) are all true. This shows it is not optimal to order when the starting inventory x_1 is high enough in a t-period problem when $t = \overline{t}, \overline{t}+1, \ldots$. From (231) to (258), we see that the criterion of "high enough" can be set by $x_1 \geq \overline{y}'_{\overline{t}}$ for some constant $\overline{y}'_{\overline{t}}$. From Lemma 5, we already know the existence of a similar level $\overline{y}_t$ for period 1 in a t-period problem, when $t = 1,2,...,\overline{t}-1$. We can simply set $\overline{y} = (\max_{t=1}^{\overline{t}-1}\overline{y}_t \vee \overline{y}'_{\overline{t}})^+$. This $\overline{y}$ would be an upper bound for x_1 beyond which ordering is not necessary for any t-period problem, at $t = 1,2,\ldots$. But the order-up-to level in period 1 in a t-period problem is simply a solution to (87). ■

Proof of Lemma 20: By (86), (87), and (97) to (99), we obtain

$$\begin{aligned}
v_1(x) - v_0(x) \leq u_1(x) = (1-\alpha)\cdot\overline{c}\cdot x \\
+\min_{z\in[\underline{z},\overline{z}]}\sup_{m\in M}\{\alpha\overline{c}\cdot\mathbb{E}^m[z+\Theta] - \tilde{r}^m(z) + \mathbb{E}^m[\tilde{h}(x-z-\Theta)]\}
\end{aligned} \tag{260}$$

which, by $\tilde{h}(x-\delta) \leq (\bar{h} \vee \bar{b}) \cdot (\mid x \mid +\delta)$ and Assumption 7, results with

$$\begin{array}{l} v_1(x) - v_0(x) \leq (1-\alpha) \cdot \bar{c} \cdot x + \min_{z \in [\underline{z}, \bar{z}]} \sup_{m \in M} \{\alpha \bar{c} \cdot \mathbb{E}^m[z + \Theta] - \tilde{r}^m(z)\} \\ \qquad + \alpha \cdot (\bar{h} \vee \bar{b}) \cdot (\mid x \mid + \overline{\delta}) \leq A \cdot \mid x \mid + B \cdot \overline{\delta}. \end{array} \tag{261}$$

Thus, (101) is true for $t = 1$. Suppose (101) is true for $t = T-1$ for some $T = 2, 3, ...$ That is,

$$v_{T-1}(x) - v_{T-2}(x) \leq \alpha^{T-2} \cdot (A \cdot \mid x \mid + (A \cdot (T-2) + B) \cdot \overline{\delta} + A \cdot (T-2) \cdot \overline{y}). \tag{262}$$

Note (254) is again true. Through Assumption 7 and (262), this leads to

$$\begin{array}{l} w_T^m(y,z) - w_{T-1}^m(y,z) \leq \alpha^{T-1} \cdot A \cdot \mathbb{E}^m[\mid y - z - \Theta \mid] \\ \qquad + \alpha^{T-1} \cdot (A \cdot (T-2) + B) \cdot \overline{\delta} + \alpha^{T-1} \cdot A \cdot (T-2) \cdot \overline{y} \\ \quad \leq \alpha^{T-1} \cdot (A \cdot \mid y \mid + (A \cdot (T-1) + B) \cdot \overline{\delta} + A \cdot (T-2) \cdot \overline{y}). \end{array} \tag{263}$$

Note this upper bound is both m- and z-independent. Thus, by (97) and (98), we have

$$\begin{array}{ll} u_T(y) - u_{T-1}(y) & = \min_{z \in [\underline{z}, \bar{z}]} \sup_{m \in M} w_T^m(y,z) - \min_{z' \in [\underline{z}, \bar{z}]} \sup_{m' \in M} w_{T-1}^{m'}(y,z') \\ & \leq \alpha^{T-1} \cdot (A \cdot \mid y \mid + (A \cdot (T-1) + B) \cdot \overline{\delta} + A \cdot (T-2) \cdot \overline{y}). \end{array} \tag{264}$$

Now with (87), we obtain (245) like earlier, which due to Lemma 19, results in the same (246) as earlier. Therefore, (101) is true for $t = T$. We have thus completed the induction process. ∎

Proof of Lemma 21: We can use Lemmas 18 and 20 in almost the same manner as that in which the proof of Lemma 16 used Lemmas 13 and 15. Due to the slight difference between Lemmas 13 and 18, we only have to update (247) in the proof of Lemma 16 to

$$\mid v_{t+1}(x) - v_t(x) \mid \leq \alpha^t \cdot (A \cdot \mid x \mid + t \cdot B + C), \qquad \forall t = 0, 1, ... \tag{265}$$

with C being a newly introduced positive constant. But the remainder of the proof can proceed without any hindrance. ∎

Proof of Lemma 22: In view of (99), (104), and Lemma 21, we can use the dominated convergence theorem to deduce the convergence of $w_t^m(y,z)$ to $w^m(y,z)$. Due to (93) and Assumption 7, the convergence is uniform in $m \in M$, in $z \in [\underline{z}, \bar{z}]$, and in any bounded y-region. With (98) and (103), we have

$$q_t(y,z) - q(y,z) = \sup_{m \in M} w_t^m(y,z) - \sup_{m' \in M} w^{m'}(y,z) \leq \sup_{m \in M} (w_t^m(y,z) - w^m(y,z)), \tag{266}$$

and symmetrically an opposite inequality. Combined with the convergence result on $w_t^m(y,z)$ to $w^m(y,z)$, this leads to the convergence of $q_t(y,z)$ to $q(y,z)$, with it being uniform in $z \in [\underline{z}, \overline{z}]$, and in any bounded y-region. With (97) and (102), we have

$$u_t(y) - u(y) = \min_{z \in [\underline{z}, \overline{z}]} q_t(y,z) - \min_{z' \in [\underline{z}, \overline{z}]} q(y,z') \geq \min_{z \in [\underline{z}, \overline{z}]} (q_t(y,z) - q(y,z)), \tag{267}$$

and symmetrically an opposite inequality. Combined with the convergence result on $q_t(y,z)$ to $q(y,z)$, this leads to the convergence of $u_t(y)$ to $u(y)$, with it being uniform in any bounded y-region.

The rest of the proof can use Lemmas 19 and 21 in the fashion in which the corresponding portion of the proof of Lemma 17 used Lemmas 14 and 16. ∎

D. Technical Details in Section 6.

Proof of Theorem 2: From (28), we have $v_T^1(\mathbf{x}) = v_T^2(\mathbf{x}) = c_T^0(\mathbf{x})$ at every $\mathbf{x} \in \hat{X}$. Suppose for some $t = T-1, T-2, ..., 1$, we know $v_{t+1}^1(\mathbf{x}) \leq v_{t+1}^2(\mathbf{x})$ at every $\mathbf{x} \in \hat{X}$. Then, at every $(\mathbf{x}, \mathbf{u}) \in \text{graph}(U_t)$ and $m \in \mathbb{M}_t^q$, it will follow that

$$\mathbb{E}^m[(c_t + \alpha \cdot v_{t+1}^1 \circ \mathbf{s}_{t+1})(\mathbf{x}, \mathbf{u}, \mathbf{\Delta}_t)] \leq \mathbb{E}^m[(c_t + \alpha \cdot v_{t+1}^2 \circ \mathbf{s}_{t+1})(\mathbf{x}, \mathbf{u}, \mathbf{\Delta}_t)]. \tag{268}$$

By the fact that $M_t^1(\mathbf{x}, \mathbf{u}) \subseteq M_t^2(\mathbf{x}, \mathbf{u})$, this will lead to

$$\sup_{m \in M_t^1(\mathbf{x},\mathbf{u})} \mathbb{E}^m[(c_t + \alpha \cdot v_{t+1}^1 \circ \mathbf{s}_{t+1})(\mathbf{x}, \mathbf{u}, \mathbf{\Delta}_t)] \leq \sup_{m \in M_t^2(\mathbf{x},\mathbf{u})} \mathbb{E}^m[(c_t + \alpha \cdot v_{t+1}^2 \circ \mathbf{s}_{t+1})(\mathbf{x}, \mathbf{u}, \mathbf{\Delta}_t)]. \tag{269}$$

Due to (29), this will result in

$$v_t^1(\mathbf{x}) \leq v_t^2(\mathbf{x}), \qquad \forall \mathbf{x} \in \hat{X}. \tag{270}$$

We have thus completed the induction process. ∎

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