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A Connection between Finite- and Infinite-player Games

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Abstract We establish a link between certain finite-player games and their corresponding nonatomic games (NGs). In anonymous games, opponents' type and action profiles can be treated as empirical probability distributions. Thus, it is natural to consider the limiting regime where player types are randomly sampled from a given nonatomic distribution and the number of players approaches infinity. It is in this regime that we obtain our key results. These results demonstrate the asymptotic rationality of action plans obtained from the limiting NGs in the corresponding finite-player games.

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1 Introduction

We explore the following link between nonatomic games (NGs) and their finite-player counterparts: in a certain sense, a given equilibrium of an NG can be plugged back to the game's close-by finite-player games and still be approximate equilibria. Our focus is not on establishing equilibrium existence; rather, we take the existence question as settled in the affirmative, and then move on to tackle the aforementioned connection.

NGs are often easier to analyze than their finite counterparts, because in them, the action of an individual player has no impact on payoffs of the other players. Therefore, they are often used as proxies of real competitive systems

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in economic studies. For instance, Aumann (1964) established the equivalence between an exchange economy's core and its set of competitive equilibria when there is a continuum of nonatomic traders. Many other researchers used NGs as approximations of reality; see, e.g., Reny and Perry (2006). Our results hence has the potential to aid in the search of approximate equilibria for complex competitive situations.

It is quite clear that one of the original motivations for studying NGs lies in the hope that solutions for the idealized situation with a continuum of bit players can offer hints on solutions for actual, messier, situations with many big players. This research thus offers an alternative justification on why NGs may still be viewed as limiting cases of their finite-player counterparts.

Our finding is lower hemi-continuity in nature: we show the asymptotic rationality of an equilibrium developed for the NG setting when it is plugged back to finite-player settings. Rather than being pre-determined, our finite-game sequence is randomly generated in that player types are independently sampled from the given NG-defining type distribution. The "distributional" LLN naturally plays a prominent role in our derivation. We have also gone beyond single-period, strategic-form games. Indeed, we study two cases:

(I) a single-period (S) case, and

(II) a multi-period (M) case, where there is also a publicly known state of nature whose evolution is impacted by all players.

Systematic research on games involving a continuum of nonatomic players started with Schmeidler (1973). When the action space is finite, Schmeidler established the existence of pure equilibria when the game becomes anonymous (so that opponents affect a given player through "what was done collectively", but not in "who did what"). A finite-player counterpart to Schmeidler was obtained by Rashid (1983), who did not resort to any relationship between finite- and infinite-player games. Mas-Colell (1984) achieved a result similar to Schmeidler's using different representations of games and equilibria. Khan and Sun (1995) generalized Schmeidler's result to the case with a countable compact metric action space.

Green (1984) and Housman (1988) introduced different frameworks under which finite games and their NG counterparts can be studied simultaneously. Khan, Rath, and Sun (1997) identified a certain limit to which Schmeidler's result can be extended. Carmona (2004) showed that every pure equilibrium of an anonymous NG is in some sense a limit point of a sequence of approximate pure equilibria of suitably chosen anonymous games. Carmona (2009) showed that a property possessed by an equilibrium of a certain NG may be enjoyed by some approximate equilibria of its finite-player counterparts. Carmona and Podczeck (2009) pointed out that many existing pure-equilibrium existence results for anonymous NGs are roughly equivalent, and that they are roughly equivalent to the existence of approximate equilibria for certain large finite games.

Among works that studied convergences of finite games to their NG counterparts, many set as their goals the proof of some kind of convergence of equilibria. For instance, Green (1984) arrived to the upper hemi-continuity of equilibrium correspondences with respect to a space of games, including both finite- and infinite-player ones. Works touching upon the suitability of a game's equilibrium to nearby games include Housman (1988), Carmona (2004), and Al-Najjar (2008).

For multi-period games, Green (1980), Sabourian (1990), and Al-Najjar and Smorodinsky (2001) showed that equilibria for large games are nearly myopic. We showed something almost reciprocal–that myopic profiles can be nearly equilibrium for certain large games; that is, when the profiles are generated from equilibria of an NG correspondent to the given large games. Like us, Fudenberg and Levine (1986) worked on the lower hemi-continuity side of multi-period games. But their games had various differentiability properties. Our lower hemi-continuity approach is especially motivated by the observation that, it is even more pronounced in multi-period settings, that the idealized NGs are easier to analyze than their actual finite-player counterparts; hence, one is tempted to employ, in the latter, optimal actions found for the former and hope to obtain near-optimal results.

There is a sizable literature on LLN in game-theoretic contexts. But they mostly concentrated on providing remedies for the failure, in the usual sense, of LLN on a continuum of samples (Feldman and Gilles, 1985, and Judd 1985); see, e.g., Páscoa (1998) and Al-Najjar (2004). Khan and Sun (1999) also achieved asymptotic results for finite games. But theirs are about the existence of ϵ-equilibria with progressively smaller ϵ as the number of players increases, not about how NG equilibria would behave in finite-player settings.

The remainder of the paper is organized as follows. We present case (S) in Section 2, and in Section 3, we discuss the connection between our main result to related works and show through a price-competition example how the result can be useful. We deal with case (M) in Section 4, and conclude the paper in Section 5.

2 The Single-period Model

This investigation of the single-period model will also provide the key idea to be used in our study of the multi-period case.

2.1 Notation

Given a separable metric space A, we use d_A to denote its metric, $\mathcal{B}(A)$ its Borel σ-field, and $P(A)$ the set of all probability measures on the measurable space $(A, \mathcal{B}(A))$. The space $P(A)$ is metrized by the Prohorov metric ρ_A, and the resultant metric space is separable too. More specifically, for any distributions $\pi, \pi' \in P(A)$,

$$\rho_A(\pi, \pi') = \inf(\epsilon > 0 \mid \pi'(K^\epsilon) + \epsilon \geq \pi(K) \ \text{ for all closed } K \in \mathcal{B}(A)), \tag{1}$$

where

$$K^\epsilon = \{a \in A \mid d_A(a, a') < \epsilon \text{ for some } a' \in K\}. \tag{2}$$

For any $a \in A$, we use 1_a to denote the singleton probability measure with $1_a(\{a\}) = 1$. Given $a = (a_1, ..., a_n) \in A^n$, we use 1_a to denote $\sum_{m=1}^n 1_{a_m}/n$. We also use $P_n(A)$ to denote the space of probability measures of the type 1_a for $a \in A^n$.

We may iteratively define $(A^n, \mathcal{B}^n(A)) = (A^{n-1} \times A, \mathcal{B}^{n-1}(A) \times \mathcal{B}(A))$. Also, let A^∞ be the set of all sequences of the type $a = (a_1, a_2, ...)$ with each $a_n \in A$, and let $\mathcal{B}^\infty(A)$ be the smallest σ-field that contains cylinders $C_n(A')$, where for $n = 1, 2, ...$ and set $A' \in \mathcal{B}(A)$,

$$C_n(A') = \{(a_1, ..., a_{n-1}, a_n, a_{n+1}, ...) \in A^\infty \mid \\ a_n \in A' \text{ and for } m \neq n,\ a_m \in A\}. \tag{3}$$

Given $\pi \in P(A)$, we use π^n to denote the product measure on $(A^n, \mathcal{B}^n(A))$ and π^∞ to denote the product measure on $(A^\infty, \mathcal{B}^\infty(A))$, such that, for any $n = 1, 2, ...$ and $A_1, ..., A_n \in \mathcal{B}(A)$,

$$\pi^\infty(C_1(A_1) \cap \cdots \cap C_n(A_n)) = \pi(A_1) \times \cdots \times \pi(A_n). \tag{4}$$

Given $a = (a_1, a_2, ...) \in A^\infty$, we use a^n to denote the first-n cutoff $(a_1, ..., a_n)$. Given $a = (a_1, ..., a_n) \in A^n$ and $m = 1, 2, ..., n$, we use a_{-m} to represent $(a_1, ..., a_{m-1}, a_{m+1}, ..., a_n)$. For $a = (a_1, a_2, ...) \in A^\infty$, note that a^n_{-m} now stands for $(a_1, ..., a_{m-1}, a_{m+1}, ..., a_n)$.

Given separable metric spaces A and B, we use $M(A, B)$ to represent all measurable functions from A to B, i.e., functions $y : A \to B$ such that, for any $B' \in \mathcal{B}(B)$, the set $y^{-1}(B') = \{a \in A \mid y(a) \in B'\}$ is a member of $\mathcal{B}(A)$. Given $\pi \in P(A)$, one should understand πy^{-1} as a member of $P(B)$, such that, for any $B' \in \mathcal{B}(B)$, it follows that $\pi y^{-1}(B') = \pi(y^{-1}(B'))$.

2.2 Formulation

We assume that player types form a separable metric space T. Players, regardless of their types, have a common separable metric action space X. Let R stand for the real line. For any $n = 2, 3, ...$ and any $t \in T^n$, we define an anonymous n-person game $\Phi_n(1_t)$ with player-type distribution 1_t. In this game, the player set is $\{1, 2, ..., n\}$; under strategy profile $x = (x_1, ..., x_n) \in X^n$, the payoff to player m is $f_n(t_m, x_m, 1_{x_{-m}})$, where $f_n : T \times X \times P_{n-1}(X) \to R$ is a given payoff function. The anonymity of $\Phi_n(1_t)$ is reflected in that a player's payoff depends on himself only through his own type and action, and on other players only through what they did in aggregation.

We can apply usual equilibrium concepts to this game. For instance, for $\epsilon \geq 0$, we say $x^* = (x^*_1, ..., x^*_n) \in X^n$ forms an ϵ-Nash equilibrium for this game, if and only if, for any $m = 1, 2, ..., n$ and any $x_m \in X$,

$$f_n(t_m, x^*_m, 1_{x^*_{-m}}) \geq f_n(t_m, x_m, 1_{x^*_{-m}}) - \epsilon. \tag{5}$$

We let some function $f : T \times X \times P(X) \to R$ be used in our NGs. Given nonatomic type distribution $p \in P(T)$, we define our NG $\Phi(p)$ so that,

$f(t, x, py^{-1})$ is the payoff to a type-t player, who takes action x and faces the multitude with type distribution p and action profile $y = (y(t) \mid t \in T) \in M(T, X)$. Note that py^{-1} is the action distribution every player prepares to face, when it is anticipated that a type-t player will take action $y(t)$.

We say that $x^* \in M(T, X)$ forms an NG equilibrium for $\Phi(p)$, if and only if for any $t \in T$ and any $x \in X$,

$$f(t, x^*(t), p(x^*)^{-1}) \geq f(t, x, p(x^*)^{-1}). \tag{6}$$

The above reflects that $x^*(t)$ is the best response for a type-t player to action distribution $p(x^*)^{-1}$ generated by the type distribution p and action plan x^*. So, x^* is an equilibrium if and only if the anticipated action profile offers a best response to itself for every player type. For simplicity, we have foregone the qualification of (6) with the "p-almost surely" statement. We can do this because, if there was a p-null set of players violating (6), we could always, for each such player t, change $x^*(t)$ to the best response to the opponent-action distribution $p(x^*)^{-1}$, which would remain intact for modifications performed on a p-null set.

2.3 A Convergence Result

Let a nonatomic type distribution $p \in P(T)$ be given. For any $n = 1, 2, ...$, we may treat the projection map $(t_1, t_2, ...) \to t_n$ in $M(T^\infty, T)$ as a random variable Θ_n that is defined between the base probability space $(T^\infty, \mathcal{B}(T^\infty), p^\infty)$ and range measurable space $(T, \mathcal{B}(T))$. Now, $\Phi_n(1_{\Theta^n})$, where $\Theta^n = (\Theta_1, ..., \Theta_n)$, stands for the randomly generated game such that, for any $T' \in \mathcal{B}^n(T)$, there is a $p^n(T')$ chance that the realized game will be $\Phi_n(1_t)$ for some $t = (t_1, ..., t_n) \in T'$. We are to investigate the sense in which the sequence of randomly generated games $(\Phi_n(1_{\Theta^n}) \mid n = 2, 3, ...)$ converges to the NG $\Phi(p)$.

Let us introduce the following assumptions:

(S1) The payoff function $f(t_1, x_1, \cdot)$ is equi-continuous in the following fashion: for any $\epsilon > 0$, there exists $\delta > 0$, such that, for any $t_1 \in T$, $x_1 \in X$, and $r, s \in P(X)$ with $\rho_X(r, s) < \delta$,

$$\mid f(t_1, x_1, r) - f(t_1, x_1, s) \mid < \epsilon.$$

(S2) The payoff function f_n converges to f uniformly. That is, for any $\epsilon > 0$, there exists $N = 2, 3, ...$, such that, for any $n \geq N$, $t_1 \in T$, $x_1 \in X$, and $r \in P_{n-1}(X)$,

$$\mid f_n(t_1, x_1, r) - f(t_1, x_1, r) \mid < \epsilon.$$

Our single-period result says that an equilibrium of the NG $\Phi(p)$ can almost surely provide an ϵ-Nash equilibrium for all randomly generated games $\Phi_n(1_{\Theta^n})$ for n's greater than a certain threshold N, and we can let ϵ approach 0 when N goes to $+\infty$.

Theorem 1 *Suppose $p \in P(T)$ is a nonatomic type distribution and $x^* \in M(T,X)$ is an equilibrium of the NG $\Phi(p)$. Then, under assumptions (S1) and (S2), x^* will be an asymptotic Nash equilibrium for the randomly generated game sequence $(\Phi_n(1_{\Theta^n}) \mid n = 2,3,...)$ in the following sense: there exists $T' \in \mathcal{B}^\infty(T)$ with $p^\infty(T') = 1$, such that, for any $t = (t_1, t_2, ...) \in T'$ and $\epsilon > 0$, there exists $N = 2,3,...,$ so that for any $n \geq N$ and $x = (x_1, ..., x_n) \in X^n$,*

$$\max_{m=1}^{n}\{f_n(t_m, x_m, 1_{t^n_{-m}}(x^*)^{-1}) - f_n(t_m, x^*(t_m), 1_{t^n_{-m}}(x^*)^{-1})\} < \epsilon.$$

This theorem says that, when player types are independently drawn from distribution p and the number of players is huge, a player may as well adopt the type-specific equilibrium action initially intended for the p-defined NG. When all other players do the same, the player will be virtually guaranteed to gain not much less than the maximum he could have accomplished. This action plan is striking in that, it does not even depend on opponent types. The primary reason is that, when there are enough players, the type distribution of other players will be vastly predictable. In Appendices B and C, we show that the claim made in Theorem 1 need not be true when either (S1) or (S2) is slightly violated.

2.4 Technical Details

According to Parthasarathy (2005, Theorem 7.1, p. 53), the strong LLN applies to the empirical distribution under the Prohorov metric.

Proposition 1 *Given separable metric spaces A and B, suppose $\pi \in P(A)$ and $y \in M(A,B)$. Then, there exists $A' \in \mathcal{B}^\infty(A)$ with $\pi^\infty(A') = 1$, such that, for any $a = (a_1, a_2, ...) \in A'$ and $\epsilon > 0$, there exists $N = 2,3,...,$ so that for any $n \geq N$,*

$$\rho_B(1_{a^n}y^{-1}, \pi y^{-1}) < \epsilon.$$

For $a \in A$ and $\pi \in P_{n-1}(A)$, we use $(a,\pi)_n$ to represent the member of $P_n(A)$ that has an additional $1/n$ weight on the point a, but with probability masses in π being reduced to $(n-1)/n$ times of their original values. For $a \in A^n$ and $m = 1, ..., n$, we have $(a_m, 1_{a_{-m}})_n = 1_a$. The following is a useful observation, whose proof we have placed in Appendix A.

Proposition 2 *Let A be a separable metric space. Then, for any $n = 2,3,...,$ $a \in A$, and $\pi \in P_{n-1}(A)$, we have*

$$\rho_A((a,\pi)_n, \pi) \leq \frac{1}{n}.$$

Suppose we are given separable metric spaces A and B, $a \in A$, and $y \in M(A,B)$. If we let $y^n = (y(a_1), y(a_2), ..., y(a_n))$, then we may see that

$1_{a^n}y^{-1} = 1_{y^n}$ and $1_{a^n_{-m}}y^{-1} = 1_{y^n_{-m}}$ for any $m = 1, 2, ..., n$. Therefore, from Proposition 2, we have

$$\rho_B(1_{a^n_{-m}}y^{-1}, 1_{a^n}y^{-1}) = \rho_B(1_{y^n_{-m}}, 1_{y^n}) \leq \frac{1}{n}, \qquad \forall m = 1, 2, ..., n. \tag{7}$$

Combining Proposition 1 and (7), we may reach the following.

Proposition 3 *Given separable metric spaces A and B, suppose $\pi \in P(A)$ and $y \in M(A, B)$. Then, there exists $A' \in \mathcal{B}^\infty(A)$ with $\pi^\infty(A') = 1$, such that, for any $a = (a_1, a_2, ...) \in A'$ and $\epsilon > 0$, there exists $N = 2, 3, ...,$ so that for any $n \geq N$,*

$$\max_{m=1}^{n} \rho_B(1_{a^n_{-m}}y^{-1}, \pi y^{-1}) < \epsilon.$$

We are now in the position to prove Theorem 1 by using (S1), (S2), and Proposition 3.

Proof of Theorem 1: Due to (S1), we may identify a $\delta > 0$, so that, for any $n = 2, 3, ...,$

$$\max_{m=1}^{n} \mid f(t_m, x_m, 1_{t^n_{-m}}(x^*)^{-1}) - f(t_m, x_m, p(x^*)^{-1}) \mid < \frac{\epsilon}{4}, \tag{8}$$

as long as $t^n = (t_1, ..., t_n) \in T^n$ satisfies

$$\max_{m=1}^{n} \rho_X(1_{t^n_{-m}}(x^*)^{-1}, p(x^*)^{-1}) < \delta. \tag{9}$$

But by Proposition 3, we know the existence of $T' \in \mathcal{B}^\infty(A)$ with $p^\infty(T') = 1$, such that, for any $t = (t_1, t_2, ...) \in T'$, there exists some integer $N_0(t)$ for (9) to be true for any $n \geq N_0(t)$. On the other hand, due to (S2), we may identify an $N = 2, 3, ...,$ so that, for any $n \geq N$ and $t^n = (t_1, ..., t_n) \in T^n$,

$$\max_{m=1}^{n} \mid f_n(t_m, x_m, 1_{t^n_{-m}}(x^*)^{-1}) - f(t_m, x_m, 1_{t^n_{-m}}(x^*)^{-1}) \mid < \frac{\epsilon}{4}. \tag{10}$$

Also, because x^* is an NG equilibrium for $\Phi(p)$, we know that, for any $t \in T$ and $y \in X$,

$$f(t, x^*(t), p(x^*)^{-1}) \geq f(t, y, p(x^*)^{-1}). \tag{11}$$

Combining (8), (10), and (11), we see that, for any $t \in T'$, $n \geq N_0(t) \vee N$, $x \in X^n$, and $m = 1, ..., n$,

$$\begin{aligned} f_n(t_m, x^*(t_m), 1_{t^n_{-m}}(x^*)^{-1}) &> f(t_m, x^*(t_m), 1_{t^n_{-m}}(x^*)^{-1}) - \epsilon/4 \\ &> f(t_m, x^*(t_m), p(x^*)^{-1}) - \epsilon/2 \geq f(t_m, x_m, p(x^*)^{-1}) - \epsilon/2 \\ &> f(t_m, x_m, 1_{t^n_{-m}}(x^*)^{-1}) - 3\epsilon/4 > f_n(t_m, x_m, 1_{t^n_{-m}}(x^*)^{-1}) - \epsilon. \end{aligned} \tag{12}$$

We have thus obtained the desired result. □

3 Discussion

We first distinguish Theorem 1 with closely-related works in the literature, and then show how this result can be applied to an example involving price competition.

3.1 Closely-related Literature

Kalai (2004) used the tool of LLN to study semi-anonymous games, in which a player's payoff is affected by the empirical distribution of the type-action characters of his opponents. He showed that Bayesian equilibria of semi-anonymous Bayesian games possessing the uniform equi-continuity property are asymptotically *ex post Nash* and *extensively robust* (the reader may refer to his paper for exact definitions of these two concepts) as the number of players approaches infinity. In contrast, our starting point is an equilibrium of the limiting nonatomic game. We then go on to show that this equilibrium can guide players in randomly generated large games to their approximate equilibria.

Al-Najjar (2008) studied discrete large games on the basis of finitely additive probabilities. Due to LLN, almost every realization of a mixed-strategy equilibrium of such a game is a pure-strategy equilibrium. In addition, it was established that restrictions of an equilibrium of the discrete large game offer ϵ-equilibria for its finite counterparts. The last result is similar to our Theorem 1 in spirit. However, it is achieved on a pre-determined sequence of finite-player games.

When focusing on a certain space of games, Housman (1988) showed that an equilibrium at a game can be ϵ-close to a γ-equilibrium of a neighboring game, and both ϵ and γ can be made arbitrarily small when the two games are close enough. Our Theorem 1 is more specific in two aspects: First, we show that the closeness of randomly generated finite games with their corresponding nonatomic game is reflected in the former games' numbers of players. Second, we provide the implementation detail of NG equilibria to their finite-game counterparts to the effect that Housman's ϵ-distance in the strategy-profile space disappears.

Carmona (2004) showed that a strategy profile is a pure equilibrium of an NG if and only if its restricted versions on any given sequence of finite games provide approximate pure equilibria for these games. Our Theorem 1 is certainly related to the "only if" part of this work. In Carmona, one requirement of the given game sequence implicitly places restrictions on ways in which player types can be chosen for the finite games and the payoff function of the NG. In contrast, we have let random sampling take care of player-type selections and explicitly stated our payoff-function requirement in (S1). Also, our (S2) is less stringent than Carmona's assumption, that finite-game payoff functions be restricted versions of their NG-counterpart.

3.2 A Pricing-game Example

For strictly positive constants $\bar{t}$ and $\bar{x}$, let type space $T = [0, \bar{t}]$ and action space $X = [0, \bar{x}]$. Also, let d_T and d_X be Euclidean distances. We will use type $t \in T$ to stand for a firm's cost and action $x \in X$ to stand for the price it charges. For any distribution on real numbers $q \in P(R)$, we shall use $F[q] =$

$(F[q](a) \mid a \in R)$ to denote q's distribution function: $F[q](a) = q((-\infty, a])$ for any $a \in R$.

We consider a family of price competition games. In the game $\Phi_n(1_t)$ for some $t \in T^n$, suppose the price profile of the n firms is $x = (x_1, ..., x_n) \in X^n$. Then, firm m will receive payoff $f_n(t_m, x_m, 1_{x_{-m}})$, where

$$\begin{aligned} f_n(t_m, x_m, 1_{x_{-m}}) &= (x_m - t_m) \cdot (\bar{a}_n - \bar{b}_n \cdot x_m + \bar{c}_n \cdot \textstyle\sum_{l \neq m} x_l/(n-1)) \\ &= (x_m - t_m) \cdot (\bar{a}_n - \bar{b}_n \cdot x_m + \bar{c}_n \cdot \textstyle\int_X y \cdot 1_{x_{-m}}(dy)) \\ &= (x_m - t_m) \cdot (\bar{a}_n - \bar{b}_n \cdot x_m + \bar{c}_n \cdot \textstyle\int_0^{\bar{x}} (1 - F[1_{x_{-m}}](z)) \cdot dz). \end{aligned} \tag{13}$$

The term $(x_m - t_m)$ represents the profit firm m can earn for every demand unit it satisfies, and the term $(\bar{a}_n - \bar{b}_n \cdot x_m + \bar{c}_n \cdot \sum_{l \neq m} x_l/(n-1))$ stands for the demand the firm can attract when opponents collectively charge the price vector x_{-m}.

In (13), $\bar{a}_n$, $\bar{b}_n$, and $\bar{c}_n$ are positive constants. We may interpret $\bar{a}_n$ as the potential market size presented to any firm, when the total number of firms is n. Also, $\bar{b}_n$ stands for the demand-reducing effect of a firm raising its own price, and $\bar{c}_n$ stands for the demand-boosting effect of any competing firm raising its price.

Correspondingly, we may define an NG $\Phi(p)$ for every nonatomic cost distribution $p \in P(T)$. In the game, when a t-costing firm decides to charge a price of y and faces an opponent cost distribution p, while its opponents behave in such a way that any t'-costing firm will charge the price of $x(t')$ for some $x \in M(T, X)$, the firm will earn $f(t, y, px^{-1})$, with

$$\begin{aligned} f(t, y, px^{-1}) &= (y - t) \cdot (\bar{a} - \bar{b} \cdot y + \bar{c} \cdot \textstyle\int_X z \cdot (px^{-1})(dz)) \\ &= (y - t) \cdot (\bar{a} - \bar{b} \cdot y + \bar{c} \cdot \textstyle\int_0^{\bar{x}} (1 - F[px^{-1}](z)) \cdot dz), \end{aligned} \tag{14}$$

for positive constants $\bar{a}$, $\bar{b}$, and $\bar{c}$. For the NG, we have the following fact, whose proof has been relegated to Appendix D.

Fact 1 *The game $\Phi(p)$ has an equilibrium x^* that is Lipschitz continuous with coefficient 1/2 and increasing.*

The current case satisfies (S1) with the corresponding δ equal to $(2\bar{b}/(\bar{c} \cdot (\bar{t} \vee \bar{x}) \cdot \bar{x})) \cdot \epsilon$. When $\lim_{n \to +\infty} \bar{a}_n = \bar{a}$, $\lim_{n \to +\infty} \bar{b}_n = \bar{b}$, and $\lim_{n \to +\infty} \bar{c}_n = \bar{c}$, (S2) will be satisfied too. Then, by Theorem 1, we may predict the increasingly superior performance of the x^*-induced action plan in the randomly generated game $\Phi_n(1_{\Theta^n})$, as the number of firms n increases. Even without the usual diagonal dominance assumption $\bar{c}_n < \bar{b}_n$, we have obtained that each firm's equilibrium price in the finite game should, in a rough sense, be increasing in its cost.

4 The Multi-period Model

Continuing after Section 2, we study a discounted, discrete-time model here. This model involves the additional features of publicly known state of nature and global shocks. The evolution of the state of nature is influenced by random global shocks which are commonly experienced by all players. For instance, we may think of the state of nature as an economic indicator like the Dow Jones Industrial Average and global shocks as major news releases.

4.1 Formulation

We consider a k-period model for some $k = 1, 2, ...$ We label periods reversely, so that period k is the first period, and period 1 the last. Let $W \equiv \{1, 2, ..., \bar{w}\}$ be the finite set of states of nature. For each $n = 2, 3, ...$, the payoff function f_n for each single period is now of the nature $T \times X \times P_{n-1}(X) \times W \to R$, with $f_n(t_m, x_m, 1_{x_{-m}} \mid w)$ for $t_m \in T$, $x \in X^n$, and $w \in W$ standing for the payoff to player m in any period, when the player's type is t_m, players' action profile is x, and the state of nature is w. We may similarly define the NG payoff function $f : T \times X \times P(X) \times W \to R$.

Let $(G, \mathcal{G}, \mu)$ be a probability space that represents random global shocks. For each $n = 2, 3, ...$, we define a transition probability function $q_n : W^2 \times P_n(X) \times G \to R$ that dictates the evolution of the state of nature. Given the current period's state of nature w^j, player action profile $x \in X^n$, and realized global shock $g \in G$, the probability for the next-period state of nature to be w^{j-1} is $q_n(w^{j-1} \mid w^j, 1_x, g)$. For any $x \in X^n$ and $g \in G$,

$$\begin{cases} q_n(w' \mid w, 1_x, g) \geq 0, & \forall w, w' \in W, \\ \sum_{w' \in W} q_n(w' \mid w, 1_x, g) = 1, & \forall w \in W. \end{cases} \tag{15}$$

Similarly, we define transition probability function $q : W^2 \times P(X) \times G \to R$ for the NG. Finally, we let some $\bar{\gamma} \in [0, 1]$ be our inter-period discount factor.

Let an initial state of nature $w^k \in W$ be given. For any $n = 2, 3, ...$ and $t \in T^n$, the action profile of the k-period, n-player game $\Phi_n^k(1_t, w^k)$ can be described by some $x = (x^k, x^{k-1}, ..., x^1) \in X^n \times ((X^n)^W)^{k-1}$, with $x^k = x^k(w^k) = (x_1^k(w^k), ..., x_n^k(w^k))$, while for $j = k-1, ..., 1$, $x^j = (x^j(w^j) \mid w^j \in W)$ and $x^j(w^j) = (x_1^j(w^j), ..., x_n^j(w^j))$. Here, each $x_m^j(w^j)$ is the action to be taken by player m in period j in the face of a state of nature w^j. Meanwhile, for a given nonatomic type distribution $p \in P(T)$, the action profile of the k-period NG $\Phi^k(p, w^k)$ can be described by some $x = (x^k, x^{k-1}, ..., x^1) \in M(T, X) \times ((M(T, X))^W)^{k-1}$, with $x^k = x^k(w^k) = (x^k(t, w^k) \mid t \in T)$, while for $j = k-1, ..., 1$, $x^j = (x^j(w^j) \mid w^j \in W)$ and $x^j(w^j) = (x^j(t, w^j) \mid t \in T)$. Here, each $x^j(t, w^j)$ is the action to be taken by the type-t player in period j in the face of a state of nature w^j.

In the finite game $\Phi_n^k(1_t, w^k)$, suppose the player action profile $x \in X^n \times ((X^n)^W)^{k-1}$ is given. Then, the total, discounted, expected payoff to player m with initial state w^k will be recursively determined by

$$\begin{aligned} h_n^k(t_m, x_m, 1_{x_{-m}} \mid w^k) &= f_n(t_m, x_m^k(w^k), 1_{x_{-m}^k(w^k)} \mid w^k) \\ &+ \bar{\gamma} \cdot \sum_{w^{k-1} \in W} \int_G h_n^{k-1}(t_m, x_m^{[k-1,1]}(w^{k-1}), 1_{x_{-m}^{[k-1,1]}(w^{k-1})} \mid w^{k-1}) \times \\ &\quad \times q_n(w^{k-1} \mid w^k, 1_{x^k(w^k)}, g) \cdot \mu(dg). \end{aligned} \tag{16}$$

A few words have to be said on the notational convention taken for (16). First, we have extended our game to period 0, by letting $h_n^0(t, w^0) = 0$ for any $t \in T$ and $w^0 \in W$. Secondly, when y is a tuple of members in X^{n-1}, we let 1_y mean a tuple of corresponding empirical distributions in $P_{n-1}(X)$. This applies to both the $1_{x_{-m}}$ on the left-hand side and the $1_{x_{-m}^{[k-1,1]}(w^{k-1})}$ on the right-hand side. Thirdly, each $x^{[j1]}(w^j)$ iteratively means $(x^j(w^j), x^{[j-1,1]})$, where $x^{[j-1,1]} = (x^{[j-1,1]}(w^{j-1}) \mid w^{j-1} \in W)$. Note that $x = x^{[k1]}(w^k) = (x^k(w^k), x^{[k-1,1]})$ and $x_{(-)m} = x_{(-)m}^{[k1]}(w^k) = (x_{(-)m}^k(w^k), x_{(-)m}^{[k-1,1]})$. By (16), players' actions in period k and the realized global shock together affect the next-period state of the nature w^{k-1}, which in turn influences each player's future payoffs.

In the NG $\Phi^k(p, w^k)$, suppose $x \in M(T,X) \times ((M(T,X))^W)^{k-1}$ is given as a player action profile. Then, a type-t player with initial state w^k will receive the following total, discounted, expected payoff:

$$\begin{array}{r}
h^k(t, x(t), px^{-1} \mid w^k) = f(t, x^k(t, w^k), p(x^k(w^k))^{-1} \mid w^k) + \bar{\gamma} \times \\
\times \sum_{w^{k-1} \in W} \int_G h^{k-1}(t, x^{[k-1,1]}(t, w^{k-1}), p(x^{[k-1,1]}(w^{k-1}))^{-1} \mid \\
w^{k-1}) \cdot q(w^{k-1} \mid w^k, p(x^k(w^k))^{-1}, g) \cdot \mu(dg).
\end{array} \tag{17}$$

Let us add a few more words on our notational convention. Again, we have extended our NG to period 0, by letting $h^0(t, w^0) = 0$ for any $t \in T$ and $w^0 \in W$. Also, when y is a tuple of functions in $M(T, X)$, we let py^{-1} mean a tuple of distributions in $P(X)$ that result from applying p to the functions. This applies to both the px^{-1} on the left-hand side and the $p(x^{[k-1,1]}(w^{k-1}))^{-1}$ on the right-hand side.

4.2 Perfect Equilibria

For both finite games and NGs, we as usual let an equilibrium refer to an action plan with the following property: no player can deviate from it in any single period under any circumstance to gain anything, provided that all other players are to follow this plan.

For any $\epsilon > 0$, we say that $x^* \in X^n \times ((X^n)^W)^{k-1}$ is an ϵ-perfect equilibrium for the game $\Phi_n^k(1_t, w^k)$, if and only if, for $j = k$ or any $(j, w^j) \in \{k-1, ..., 1\} \times W$, $m = 1, ..., n$, and $x_m \in X$,

$$\begin{array}{l}
h_n^j(t_m, (x^*)_m^{[j1]}(w^j), 1_{(x^*)_{-m}^{[j1]}(w^j)} \mid w^j) \\
\quad \geq h_n^j(t_m, (x_m, (x^*)_m^{[j-1,1]}), 1_{(x^*)_{-m}^{[j1]}(w^j)} \mid w^j) - \epsilon,
\end{array} \tag{18}$$

which, by (16), amounts to

$$\begin{aligned}
&f_n(t_m, (x^*)^j_m(w^j), 1_{(x^*)^j_{-m}(w^j)} \mid w^j)\\
&+\bar{\gamma} \cdot \textstyle\sum_{w^{j-1}\in W} \int_G h^{j-1}_n(t_m, (x^*)^{[j-1,1]}_m(w^{j-1}), 1_{(x^*)^{[j-1,1]}_{-m}(w^{j-1})} \mid w^{j-1})\times\\
&\qquad \times q_n(w^{j-1} \mid w^j, 1_{(x^*)^j(w^j)}, g) \cdot \mu(dg)\\
&\geq f_n(t_m, x_m, 1_{(x^*)^j_{-m}(w^j)} \mid w^j)\\
&+\bar{\gamma} \cdot \textstyle\sum_{w^{j-1}\in W} \int_G h^{j-1}_n(t_m, (x^*)^{[j-1,1]}_m(w^{j-1}), 1_{(x^*)^{[j-1,1]}_{-m}(w^{j-1})} \mid w^{j-1})\times\\
&\qquad \times q_n(w^{j-1} \mid w^j, 1_{(x_m,(x^*)^j_{-m}(w^j))}, g) \cdot \mu(dg) - \epsilon.
\end{aligned} \tag{19}$$

We say that $x^* \in M(T,X) \times ((M(T,X))^W)^{k-1}$ is a perfect equilibrium for the NG $\Phi^k(p,w^k)$, if and only if, for $j=k$ or any $(j,w^j) \in \{k-1,...,1\} \times W$, $t \in T$, and $x \in X$,

$$\begin{aligned}
&h^j(t, (x^*)^{[j1]}(w^j)(t), p((x^*)^{[j1]}(w^j))^{-1} \mid w^j)\\
&\quad \geq h^j(t, (x, (x^*)^{[j-1,1]}(t)), p((x^*)^{[j1]}(w^j))^{-1} \mid w^j),
\end{aligned} \tag{20}$$

which, by (17), amounts to

$$f(t, (x^*)^j(t,w^j), p((x^*)^j(w^j))^{-1} \mid w^j) \geq f(t, x, p((x^*)^j(w^j))^{-1} \mid w^j). \tag{21}$$

In comparison to (6), we may see that x^* is a perfect equilibrium for the multi-period NG $\Phi^k(p,w^k)$ if and only if each $(x^*)^j(w^j)$ is an equilibrium for the single-period NG $\Phi(p \mid w^j)$ characterized by the nonatomic distribution p and payoff function $f(\cdot \mid w^j)$.

As long as x^* is in existence, we may always choose one in which $(x^*)^j$ is independent of the period index j. Also, the equilibrium set of $\Phi^k(p,w^k)$ does not depend on the transition probability function q or the global-shock distribution μ. These simplistic features are emblematic of the current NG, in which no individual player has any discernable impact on the evolution of the publicly observable state of nature. This observation is consistent with the literature on large multi-period games, as Green (1980), Sabourian (1990), and Al-Najjar and Smorodinsky (2001) all showed that, in large multi-period games, most players would behave nearly myopically in equilibrium.

4.3 A Convergence Result

Besides letting $\mid f \mid$ be bounded by some constant $\bar{M}$, we introduce the following assumptions:

(M1) The payoff function $f(t_1, x_1, \cdot \mid w)$ is equi-continuous in the following fashion: for any $\epsilon > 0$, there exists $\delta > 0$, such that, for any $t_1 \in T$, $x_1 \in X$, $r, s \in P(X)$ with $\rho_X(r,s) < \delta$, and $w \in W$,

$$\mid f(t_1, x_1, r \mid w) - f(t_1, x_1, s \mid w) \mid < \epsilon.$$

(M2) The payoff function f_n converges to f uniformly. That is, for any $\epsilon > 0$, there exists $N = 2, 3, ...$, such that, for any $n \geq N$, $t_1 \in T$, $x_1 \in X$, $r \in P_{n-1}(X)$, and $w \in W$,

$$| f_n(t_1, x_1, r \mid w) - f(t_1, x_1, r \mid w) | < \epsilon.$$

(M3) The transition probability function $q(w' \mid w, \cdot, g)$ is equi-continuous in the following fashion: for any $\epsilon > 0$, there exist $\delta > 0$, such that, for any $w, w' \in W$, $r, s \in P(X)$ with $\rho_X(r, s) < \delta$, and $g \in G$,

$$| q(w' \mid w, r, g) - q(w' \mid w, s, g) | < \epsilon.$$

(M4) The transition probability function q_n converges to q uniformly. That is, for any $\epsilon > 0$, there exists $N = 2, 3, ...$, such that, for any $n \geq N$, $w, w' \in W$, $r \in P_n(X)$, and $g \in G$,

$$| q_n(w' \mid w, r, g) - q(w' \mid w, r, g) | < \epsilon.$$

Due to the finiteness of the global-shock space W, (M1) is merely (S1) for $f(\cdot \mid w)$ at every $w \in W$, and (M2) is merely (S2) for $(f_n(\cdot \mid w) \mid n = 2, 3, ...)$ and $f(\cdot \mid w)$ at every $w \in W$.

The following is a multi-period version of Theorem 1.

Theorem 2 *Let nonatomic type distribution $p \in P(T)$, integer $k = 1, 2, ...$, and initial state of nature $w^k \in W$ be given. Suppose $x^* \in M(T, X) \times ((M(T, X))^W)^{k-1}$ is a perfect equilibrium for the NG game $\Phi^k(p, w^k)$. Then, under assumptions (M1) to (M4), x^* will be an asymptotic perfect equilibrium for the random game sequence $(\Phi_n^k(1_{\Theta^n}, w^k) \mid n = 2, 3, ...)$ in the following sense: there exists $T' \in \mathcal{B}^\infty(T)$ with $p^\infty(T') = 1$, such that, for any $t = (t_1, t_2, ...) \in T'$ and $\epsilon > 0$, there exists $N = 2, 3, ...$, so that for any $n \geq N$, $j = k$, or $j = k-1, ..., 1$ and $w^j \in W$, and $x \in X^n$,*

$$\begin{aligned}\max_{m=1}^n \{ & h_n^j(t_m, (x_m, (x^*)^{[j-1,1]}(t_m)), 1_{t_{-m}^n}((x^*)^{[j1]}(w^j))^{-1} \mid w^j) \\ & - h_n^j(t_m, (x^*)^{[j1]}(w^j)(t_m), 1_{t_{-m}^n}((x^*)^{[j1]}(w^j))^{-1} \mid w^j)\} < \epsilon.\end{aligned}$$

Here we offer an intuitive explanation to the theorem. When there are a huge number of players, and player types are randomly drawn from known distributions, each player knows not only that the environment made up of distributions of other players' types and actions is quite predictable, but also that his current action will not significantly impact the environment he is going to face in the future. Therefore, adopting the myopic action plan suitable for the environment at the moment will serve the player remarkably well.

4.4 The Proof

The proof of Theorem 2 relies on two observations: (1) As the NG equilibrium is also equilibrium for each single period under a particular state, a large finite game will not lose much in any single period by adopting the NG equilibrium.

(2) When the number of players is huge, the action deviation of one player will not much alter the evolution course of the state.

Proof of Theorem 2: By (21), (M1), and (M2), we know that Theorem 1 applies for every $w \in W$. Due to this and the finiteness of W, there exists $T' \in \mathcal{B}^\infty(T)$ with $p^\infty(T') = 1$, such that, for any $t = (t_1, t_2, ...) \in T'$ and $\epsilon > 0$, there exists some $N_{12}(t, \epsilon) = 2, 3, ...$, so that for any $n \geq N_{12}(t, \epsilon)$, $x \in X^n$, $m = 1, 2, ..., n$, $j = 1, 2, ..., k-1$ and $w^j \in W$ or $j = k$,

$$\Delta F < \frac{\epsilon}{2}, \tag{22}$$

where

$$\begin{aligned} \Delta F = f_n(t_m, x_m, 1_{t^n_{-m}}((x^*)^j(w^j))^{-1} \mid w^j) \\ -f_n(t_m, (x^*)^j(t_m, w^j), 1_{t^n_{-m}}((x^*)^j(w^j))^{-1} \mid w^j). \end{aligned} \tag{23}$$

We shall see that this T' is the one we should be looking for. For any particular $\epsilon > 0$, we may denote the corresponding δ in (M3) by $\delta_3(\epsilon)$ and the corresponding N in (M4) by $N_4(\epsilon)$.

Suppose $t = (t_1, t_2, ...) \in T'$ and $\epsilon > 0$ are given. Define N so that

$$N = N_{12}(t, \epsilon) \vee (2 \cdot \lceil \frac{1}{\delta_3(\epsilon/(6k\bar{M}\bar{w}))} \rceil + 1) \vee N_4(\frac{\epsilon}{6k\bar{M}\bar{w}}), \tag{24}$$

where $\bar{M}$ is the bound for f and $\bar{w}$ the cardinality of W. Let $n \geq N$, $x \in X^n$, $m = 1, 2, ..., n$, $j = 1, ..., k-1$ and $w^j \in W$ or $j = k$ be given. For convenience, we let $y \in X^n \times ((X^n)^W)^{k-1}$ stand for x^*'s realization at $t^n = (t_1, t_2, ..., t_n)$. That is, each $y^j_m(w^j) = (x^*)^j(t_m, w^j)$. From Proposition 2 and (24), we have

$$\begin{aligned} \rho_X(1_{(x_m, y^j_{-m}(w^j))}, 1_{y^j(w^j)}) \leq \rho_X(1_{(x_m, y^j_{-m}(w^j))}, 1_{y^j_{-m}(w^j)}) \\ +\rho_X(1_{y^j_{-m}(w^j)}, 1_{y^j(w^j)}) < \delta_3(\epsilon/(6k\bar{M}\bar{w})). \end{aligned} \tag{25}$$

By (M3), this leads to, for any $w^{j-1} \in W$ and $g \in G$,

$$\mid q(w^{j-1} \mid w^j, 1_{(x_m, y^j_{-m}(w^j))}, g) - q(w^{j-1} \mid w^j, 1_{y^j(w^j)}, g) \mid < \frac{\epsilon}{6k\bar{M}\bar{w}}. \tag{26}$$

From (M4) and (24), we also have, for any $w^{j-1} \in W$ and $g \in G$,

$$\mid q_n(w^{j-1} \mid w^j, r, g) - q(w^{j-1} \mid w^j, r, g) \mid < \frac{\epsilon}{6k\bar{M}\bar{w}}, \qquad \forall r \in P_n(X). \tag{27}$$

For any $w^j \in W$ and $g \in G$, define $\Delta Z(w^{j-1}, g)$, so that

$$\begin{aligned} &\Delta Z(w^{j-1}, g) \\ &= \mid q_n(w^{j-1} \mid w^j, 1_{(x_m, y^j_{-m}(w^j))}, g) - q_n(w^{j-1} \mid w^j, 1_{y^j(w^j)}, g) \mid . \end{aligned} \tag{28}$$

Combining (26) and (27), we have

$$\begin{aligned} &\Delta Z(w^{j-1}, g) \\ &\leq \mid q_n(w^{j-1} \mid w^j, 1_{(x_m, y^j_{-m}(w^j))}, g) - q(w^{j-1} \mid w^j, 1_{(x_m, y^j_{-m}(w^j))}, g) \mid \\ &\quad + \mid q(w^{j-1} \mid w^j, 1_{(x_m, y^j_{-m}(w^j))}, g) - q(w^{j-1} \mid w^j, 1_{y^j(w^j)}, g) \mid \\ &\quad + \mid q(w^{j-1} \mid w^j, 1_{y^j(w^j)}, g) - q_n(w^{j-1} \mid w^j, 1_{y^j(w^j)}, g) \mid < \epsilon/(2k\bar{M}\bar{w}). \end{aligned} \tag{29}$$

Consider ΔV, which satisfies

$$\begin{aligned}\Delta V = &\textstyle\sum_{w^{j-1}\in W}\int_G \mid h_n^{j-1}(t_m, y_m^{[j-1,1]}(w^{j-1}), 1_{y_{-m}^{[j-1,1]}(w^{j-1})} \mid w^{j-1}) \mid \times \\ &\times \Delta Z(w^{j-1}, g)\cdot \mu(dg).\end{aligned} \tag{30}$$

Since each $\mid h_n^j(\cdot) \mid$ is bounded by $k\bar{M}$, we may see from (29) that

$$\Delta V < \frac{\epsilon}{2}. \tag{31}$$

On the other hand, we may see from (16) that

$$\begin{aligned}&h_n^j(t_m, (x_m, (x^*)^{[j-1,1]}(t_m)), 1_{t_{-m}^n}((x^*)^{[j1]}(w^j))^{-1} \mid w^j) \\ &\quad -h_n^j(t_m, (x^*)^{[j1]}(w^j)(t_m), 1_{t_{-m}^n}((x^*)^{[j1]}(w^j))^{-1} \mid w^j) \\ &= \Delta F \\ &\quad +\bar{\gamma}\cdot\textstyle\sum_{w^{j-1}\in W}\int_G h_n^{j-1}(t_m, y_m^{[j-1,1]}(w^{j-1}), 1_{y_{-m}^{[j-1,1]}(w^{j-1})} \mid w^{j-1})\times \\ &\quad\times[q_n(w^{j-1} \mid w^j, 1_{(x_m, y_{-m}^j(w^j))}, g) - q_n(w^{j-1} \mid w^j, 1_{y^j(w^j)}, g)]\cdot\mu(dg) \\ &\le \Delta F + \bar{\gamma}\cdot\Delta V,\end{aligned} \tag{32}$$

which, in view of (22), (24), and (31), is less than ϵ. □

5 Concluding Remarks

Leveraging convergence results on empirical distributions, we have demonstrated the existence of a link between finite-player games and infinite-player NGs. Under proper limiting regimes, we have shown that equilibria of NGs can be used in finite-player settings to produce asymptotically rational results. Through the filtering-out of the influence of each individual player's action on the commonly-encountered environment, an NG is very often easier to handle than its finite-player counterpart. Our results therefore offer a justification for the study of NGs in situations where the number of involved parties, though huge, is finite.

In a nutshell, we have shown the importance of random sampling of player types to the link between large finite games and their NG counterparts. In the future, we may still consider bringing the same idea to more general equilibrium concepts and more involved games, such as mixed equilibria and semi-anonymous games. It may even be possible that known results currently cast in NG formats, such as the core-equilibrium equivalence result of Aumann (1964), may find their finite-party counterparts through using the present idea.

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Appendices

A. Proof of Proposition 2: Let $K \in \mathcal{B}(A)$ be closed. If $a \notin K$, then

$$(a,\pi)_n(K) \leq \pi(K) \leq (a,\pi)_n(K) + \frac{1}{n}; \tag{33}$$

if $a \in K$, then

$$(a,\pi)_n(K) - \frac{1}{n} \leq \pi(K) \leq (a,\pi)_n(K). \tag{34}$$

Hence, it is always true that

$$\mid (a,\pi)_n(K) - \pi(K) \mid \leq \frac{1}{n}. \tag{35}$$

In view of (1) and (2), we have

$$\rho_A((a,\pi)_n, \pi) \leq \frac{1}{n}. \tag{36}$$

We have thus completed the proof. □

B. Necessity of Assumption (S1): We consider an example where the type space $T = [0,1)$, the action space $X = \{+1,-1\}$, both d_T and d_X are Euclidean, and payoff functions satisfy the following:

$$f_1(t,x,r) = f_2(t,x,r) = \cdots = f(t,x,r) = \frac{x}{(1-t)^2} \cdot [r(\{+1\}) - r(\{-1\})]. \tag{37}$$

Note that $\mathcal{B}(X)$ coincides with the discrete 2^X. We can check that (S2) is satisfied. On the other hand, we can verify that $f(t,x,\cdot)$ is continuous, but not uniformly in a t-equal fashion. That is, (S1) is slightly violated.

Consider the strategy profile $x = (x(t) \mid t \in T) \in M(T,X)$, where

$$x(t) = \begin{cases} +1, & t \in \bigcup_{k=0}^{+\infty} [1-1/2^k, 1-1/2^{k+1}-1/2^{k+2}) \\ & \qquad = [0,1/4) \cup [1/2,5/8) \cup \cdots, \\ -1, & t \in \bigcup_{k=0}^{+\infty} [1-1/2^{k+1}-1/2^{k+2}, 1-1/2^{k+1}) \\ & \qquad = [1/4,1/2) \cup [5/8,3/4) \cup \cdots. \end{cases} \tag{38}$$

In the NG involving the uniform type distribution p, this x leads to the px^{-1} with $(px^{-1})(\{+1\}) = (px^{-1})(\{-1\}) = 1/2$. By (37), we know that all players under this action distribution will be indifferent between the two action choices. Hence, x is an NG equilibrium.

For $a \in (0,1)$, we use $\lfloor a \rfloor$ to represent the largest $1/2^k$, for $k = 1,2,...$, such that $1/2^k \leq a$. Let $T' \in \mathcal{B}^\infty(T)$ be the set of sequences $t = (t_1,t_2,...)$ satisfying the following: at an infinite number of n's, the $(2n)$-th type t_{2n} is in the interval $[1-\lfloor 1/\sqrt{2n} \rfloor, 1)$ and $x(t_{2n})$ is of the opposite sign of $k_n \equiv (1_{t^{2n}_{-2n}} x^{-1})(\{+1\}) - (1_{t^{2n}_{-2n}} x^{-1})(\{-1\})$, which, as $t^{2n}_{-2n} = (t_1,...,t_{2n-1})$ contains an odd number of components, satisfies $\mid k_n \mid \geq 1/(2n)$. When the latter occurs at a particular n, we may check through (37) that, in the game $\Phi_{2n}(1_{t^{2n}})$, the action $x(t_{2n})$ leaves player $2n$'s payoff at least 1 away from his best possible.

Let us find $p^\infty(T')$. Based on the above description, we have

$$T' = \bigcap_{l=1}^{+\infty} \bigcup_{n=l}^{+\infty} T_n, \tag{39}$$

where, for $n = 1, 2, ...,$

$$\begin{array}{l} T_n = \{(t_1, ..., t_{2n}, ...) \in T^\infty \mid t_{2n} \in [1 - \lfloor 1/\sqrt{2n} \rfloor, 1) \text{ and } x(t_{2n}) \text{ is} \\ \qquad \text{of the opposite sign of } (1_{t^{2n}_{-2n}} x^{-1})(\{+1\}) - (1_{t^{2n}_{-2n}} x^{-1})(\{-1\})\}. \end{array} \tag{40}$$

From the above, especially (38) and (40), we can check that

$$p^\infty(T_n) = \frac{1}{2} \cdot \lfloor \frac{1}{\sqrt{2n}} \rfloor \geq \frac{1}{4\sqrt{2n}}, \tag{41}$$

and hence,

$$\sum_{n=1}^{+\infty} p^\infty(T_n) = +\infty. \tag{42}$$

But all the T_n's are independent under p^∞. So, by (39) and the second Borel-Cantelli Lemma, we can decide that

$$p^\infty(T') = p^\infty(\bigcap_{l=1}^{+\infty} \bigcup_{n=l}^{+\infty} T_n) = 1. \tag{43}$$

Therefore, the conclusion of Theorem 1 will not be true.

C. Necessity of Assumption (S2): We consider an example where the type space $T = [0, 1)$, the action space $X = \{+1, -1\}$, and both d_T and d_X are Euclidean. We let payoff functions be r-independent. In particular, we let

$$f(t, +1, r) = 1, \qquad f(t, -1, r) = 0, \qquad \forall t \in [0, 1), \tag{44}$$

and for $n = 2, 3, ...,$

$$\begin{array}{ll} f_n(t, +1, r) = 1, & \forall t \in [0, 1), \\ f_n(t, -1, r) = 0, & \forall t \in [0, 1 - 1/n), \\ f_n(t, -1, r) = 2, & \forall t \in [1 - 1/n, 1). \end{array} \tag{45}$$

We can check that (S1) is satisfied. On the other hand, we can verify that f_n converges to f, but not in a t-uniform fashion. That is, (S2) is slightly violated.

Consider the strategy profile $x = (x(t) \mid t \in T) \in M(T, X)$ with $x(t) = +1$ for all $t \in T$. In any NG, including the one involving the uniform type distribution p, this x is clearly an NG equilibrium due to (44). Let $T' \in \mathcal{B}^\infty(T)$ be the set of sequences $t = (t_1, t_2, ...)$ satisfying the following: at an infinite number of n's, the n-th type t_n is in the interval $[1 - 1/n, 1)$. When the latter occurs at a particular n, we may check through (45) that, in the game $\Phi_n(1_{t^n})$, the action $x(t_n)$ leaves player n's payoff 1 away from his best possible.

Let us find $p^\infty(T')$. Based on the above description, we have

$$T' = \bigcap_{l=1}^{+\infty} \bigcup_{n=l}^{+\infty} T_n, \tag{46}$$

where, for $n = 1, 2, ...,$

$$T_n = \{(t_1, ..., t_n, ...) \in T^\infty \mid t_n \in [1 - \frac{1}{n}, 1)\}. \tag{47}$$

We can check that

$$p^\infty(T_n) = \frac{1}{n}, \tag{48}$$

and hence,

$$\sum_{n=1}^{+\infty} p^\infty(T_n) = +\infty. \tag{49}$$

But all the T_n's are independent under p^∞. So, by (46) and the second Borel-Cantelli Lemma, we have

$$p^\infty(T') = p^\infty(\bigcap_{l=1}^{+\infty} \bigcup_{n=l}^{+\infty} T_n) = 1. \tag{50}$$

Therefore, the conclusion of Theorem 1 will not be true.

D. Proof of Fact 1: We may verify that $f(t, y, px^{-1})$ is concave in y, and the t-costing firm's best price response $y^*(t, px^{-1})$ is determined by

$$y^*(t, px^{-1}) = \frac{\bar{a} + \bar{b} \cdot t + \bar{c} \cdot \int_0^{\bar{x}} (1 - F[px^{-1}](z)) \cdot dz}{2\bar{b}} \wedge \bar{x}. \tag{51}$$

It is apparent that $y^*(t, px^{-1})$, as a function of t, is Lipschitz continuous with coefficient $1/2$ and increasing. Let S be the set of all cost-to-price maps in X^T that are Lipschitz continuous with coefficient $1/2$ and increasing. Note that $S \subset M(T, X)$, as the inverse of any interval in X through any $x \in S$ is an interval in T. We may define function $B : S \to S$, so that $B(x) = (y^*(t, px^{-1}) \mid t \in T)$ for every $x \in S$.

We can define a partial order for S by admitting $x^1 \leq x^2$ for any $x^1, x^2 \in S$, if and only if $x^1(t) \leq x^2(t)$ for every $t \in T$. It is known that our S is a complete lattice, in the sense that, for any $S' \subset S$, both $\sup S'$ and $\inf S'$ are well defined and within S itself. For $x^1, x^2 \in S$ with $x^1 \leq x^2$, it is easy to check that $F[p(x^1)^{-1}](z) \geq F[p(x^2)^{-1}](z)$ for each fixed z, and hence $y^*(t, p(x^1)^{-1}) \leq y^*(t, p(x^2)^{-1})$ for every $t \in T$. This translates into that B is an increasing map from S to itself. By Tarski's (1955) Theorem, we know that B has a fixed point in S. That is, there exists an $x^* \in S$, such that $y^*(t, p(x^*)^{-1}) = x^*(t)$ for every $t \in T$. The latter means that, for any $t \in T$ and $x \in X$,

$$f(t, x^*(t), p(x^*)^{-1}) \geq f(t, x, p(x^*)^{-1}). \tag{52}$$

Therefore, this $x^* \in S$ is an equilibrium of $\Phi(p)$. $\square$

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