Network Applications of Discrete Curvature and Ricci Flow

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Spring Workshop on DCG, April 20th, 2017
Wireless Networks

- Smart buildings, smart cities, smart communities: lots of wireless devices embedded in the physical space.
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Mobile Networks

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Social Networks

- Characterizing the social interactions.
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Outline

- Past work on wireless networks: Ricci flow on triangulations.
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- Recent work on Internet & Social networks: discrete graph curvatures.
Routing in A Distributed Network

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Geographical Greedy Routing

A message is sent to the neighbor closest to the destination.
Greedy Routing with Virtual Coordinates

Mapping the network into a ‘circular domain’ (all boundaries are circles). $\rightarrow$ Greedy routing has guaranteed delivery.

[Sarkar, Yin, G, Luo, Gu 2009]
Discrete Gaussian Curvature on a Triangulation

Discrete Gaussian Curvature defined by

\[ 2\pi - \sum \theta_i \]
Ricci flow: change the circle radius to meet the target curvature (zero in the interior, and $2\pi/K$ on a hole with $K$ nodes).
Circular Domain is Not Unique

Apply a mobius transformation (which maps circles to circles).

\[ m(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \]
Multi-Path Routing

Run greedy routing with different mobius transformation → a different path.

- Rapid response to failures.
- Multi-path routing with higher throughput.

[Jiang, Ban, Goswami, Zeng, G, Gu 2011]
Homotopic Routing

Can we find a path with the specified homotopic type?
Homotopic Routing

Can we find a path with the specified homotopic type?

Our idea: use universal covering space and hyperbolic embedding.
Homotopic Routing

- Cut a multi-hole domain open, embed it to a convex polygon in the hyperbolic plane.
- Greedy routing (using hyperbolic distance) guarantees delivery.
Homotopic Routing

- Use Mobius transformation to embed copies that could fill up the domain.
Homotopic Routing

- Greedy routing to the destination in different copies gives routes of different homotopy types.

[Zeng, Sarkar, Luo, Gu, G 2010]
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- Guaranteed delivery. ✓
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Greedy Routing with Virtual Coordinates

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- Routes of specified homotopy types. ✓
- Load balancing. ✓

Until we are tired of this setting.
What about other graphs?
Coarse Geometry: Gromov’s Curvature

It has been discovered (Baryshnikov, Narayan & Saniee) that the Internet topology has **negative curvature**, defined by Gromov’s **thin triangle property**.
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![Diagram of points a, b, c with distance δ]
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Internet is Negatively Curved

A variety of data sets, both AS-level and router level topologies, show \( \delta \)-hyperbolicity for small constant \( \delta \) (2 or 3).
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- Tree-like;
- There is a ‘core’ in which all shortest path visit.
- Congestion inside the core is high;
- But one can use preprocessing to speed up shortest path queries.
Global v.s. Local Curvature

Gromov hyperbolicity is a *global* measure. Here we examine *local* curvatures:

- Which edges are negatively curved?
- Local curvature and network congestion?
- Local curvature and node centrality?
Sectional Curvature in Geometry

Consider a tangent vector \( v = xy \) and another tangent vector \( w_x \) at \( x \). Transport \( w_x \) along \( v \) to be a tangent vector \( w_y \) at \( y \). If \( |x'y'| < |xy| \), then sectional curvature is positive.

Ricci curvature: averaging over all directions \( w \).
Discrete Ricci Curvature

Take the analog: for an edge $xy$, consider the distances from $x$’s neighbors to $y$’s neighbors and compare it with the length of $xy$. 

Issue: how to match $x$’s neighbors to $y$’s neighbors?

Assign uniform distribution $1, 2$ on $x$ and $y$’s neighbors. Use optimal transportation distance (earth-mover distance) from $1$ to $2$: the matching that minimizes the total transport distance.
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- Issue: how to match $x$’s neighbors to $y$’s neighbors?
- Assign uniform distribution $\mu_1$, $\mu_2$ on $x$’ and $y$’s neighbors.
- Use optimal transportation distance (earth-mover distance) from $\mu_1$ to $\mu_2$: the matching that minimizes the total transport distance.
Discrete Ricci Curvature

Definition (Ollivier, Lin & Yau)

Let \((X, d)\) be a metric space and let \(m_1, m_2\) be two probability measures on \(X\). For any two distinct points \(x, y \in X\), the (Ollivier-) Ricci curvature along \(xy\) is defined as

\[
\kappa(x, y) := 1 - \frac{W_1(m_x, m_y)}{d(x, y)},
\]

where \(m_x \ (m_y)\) is a probability distribution defined on \(x \ (y)\) and its neighbors, \(W_1(\mu_1, \mu_2)\) is the \(L_1\) optimal transportation distance between two probability measure \(\mu_1\) and \(\mu_2\) on \(X\):

\[
W_1(\mu_1, \mu_2) := \inf_{\psi \in \mathcal{P}(\mu_1, \mu_2)} \int_{(u,v)} d(u, v) d\psi(u, v)
\]
Examples

Zero curvature: 2D grid.
Examples

Negative curvature: tree: $\kappa(x, y) = 1/d_x + 1/d_y - 1$, $d_x$ is degree of $x$. 

![Diagram of a tree with nodes and edges labeled with values -0.167, 0.33]
Examples

Positive curvature: complete graph.
## Data Sets

<table>
<thead>
<tr>
<th>Data Set</th>
<th>Node</th>
<th>Edge</th>
<th>MaxDeg</th>
<th>AvgDeg</th>
<th>Diam</th>
<th>Mean SPL</th>
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<td>18.99</td>
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RocketFuel data set: router level network.

<table>
<thead>
<tr>
<th>RocketFuel</th>
<th>Node</th>
<th>Edge</th>
<th>MaxDeg</th>
<th>AvgDeg</th>
<th>Diam</th>
<th>Mean SPL</th>
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<td>11</td>
<td>6.95</td>
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<tr>
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<td>1156</td>
<td>90</td>
<td>2.23</td>
<td>14</td>
<td>5.27</td>
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<tr>
<td>AS: 3967</td>
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<td>2071</td>
<td>75</td>
<td>4.63</td>
<td>13</td>
<td>5.94</td>
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<tr>
<td>AS: 1221</td>
<td>2998</td>
<td>3789</td>
<td>106</td>
<td>2.53</td>
<td>12</td>
<td>5.53</td>
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</table>
Curvature Distribution

Negatively curved edges are like “backbones”, maintaining the connectivity of clusters, in which edges are mostly positively curved.
Curvature Distribution
Network Connectivity

Negatively curved edges are well connected. Adding edges with increasing/decreasing curvature: few/many connected components.

Exodus (US)

<table>
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<tbody>
<tr>
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<td>50</td>
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<tr>
<td>100</td>
</tr>
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</table>

Ricci Curvature, $\alpha=0.5$

-1.0 -0.5 0 0.5

![Graph showing network connectivity with Ricci Curvature, $\alpha=0.5$.](image)
Robustness v.s. Vulnerability

Removing edges with increasing curvature: size of largest connected component drops quickly.

Graph showing the ratio of edges removed against the largest connected component for the Exodus (US) network, comparing the removal by Ricci curvature and random order.
Model Networks

Erdos Renyi Graph, Watts Strogatz Graph, Random Regular Graph, Configuration Model, Preferential Attachment Model, Hyperbolic Grid $H(3, 7)$
Network Connectivity Behavior

Small world property is not relevant; graph hyperbolicity and power law degree distribution appear to be more relevant.
Graph Isomorphism

Given a pair of graphs $G_1, G_2$, find a one-to-one correspondence of the vertices in $G_1$ to vertices in $G_2$ such that $(u, v)$ is an edge in $G_1$ if and only if their corresponding nodes $f(u), f(v)$ are connected in $G_2$. 
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Graph Isomorphism

One of the most fundamental problems in theoretical computer science.

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- Many practical algorithms: e.g., NAUTY.
- Subgraph isomorphism is NP-complete.
- **Approximate graph isomorphism**: find the best correspondence between vertices in $G_1$ and $G_2$. 

Applications of Graph Isomorphism

- Understanding network evolution.
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- Understanding network evolution.
- Match two social networks on the same population – de-anonymization.
Quantify the ‘Position’ of a Node

Select well positioned nodes as *landmarks* and define the position of a node wrt landmarks.
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- Internet: Measure the delay;
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Our Idea: Edge Weights Generated by Ricci Flow

Given a graph $G$ in which $d(x, y)$ is the weight of the edge $xy$ and $\kappa(x, y)$ is the discrete Ricci curvature, we run

/* Ricci flow

\[ d_{i+1}(x, y) = d_i(x, y) - \varepsilon \cdot \kappa_i(x, y) \cdot d_i(x, y) \]

/* Normalization

\[ d_i(x, y) \leftarrow d_i(x, y) \cdot \frac{|E|}{\sum_{xy \in E} d_i(x, y)} \]

Until convergence.
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**Ricci Flow Metric:** Shortest path metric with edge weights computed above.

[Ni, Lin, G, Gu under submission]
Ricci Flow Metric

\[(d(x, y) - \kappa(x, y) \cdot d(x, y)) \cdot N \approx d(x, y)\]
Ricci Flow Metric

\[(d(x, y) - \kappa(x, y) \cdot d(x, y)) \cdot N \approx d(x, y)\]

\[\frac{T(x, y)}{d(x, y)} \approx \frac{1}{N}.\]

If \((x, y)\) is removed, there are likely alternative paths from \(x\) to \(y\) that give similar distance. \(\rightarrow\) Much better robustness under edge insertion/deletion.
Compute the Correspondence

- Identify $k$ landmarks $\ell_1, \ldots, \ell_k$ on $G_1$ and their correspondences in $G_2$. 

Run discrete Ricci flow on $G_1$ and $G_2$ respectively. Each node $u$ computes the distance vectors $d(u; \ell_1), d(u; \ell_k)$. Form a bipartite graph $H$ with $X/Y$ as vertices in $G_1/G_2$. The edge weight $(u_2 G_1; u_0 2 G_2)$ is the $\ell_2$ norm of their distance vectors. Run min-cost matching on $H$. 

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Evaluations

- Edge weights generated by Ricci flow;
- Spectral embedding;
- Spring embedding;
- Hop count.
Evaluations

- Edge weights generated by Ricci flow;
- Spectral embedding;
- Spring embedding;
- Hop count.

- Internet AS graphs.
- Email graphs.
- Protein-protein interaction graph.
Randomly remove some edges.

Accuracy

# Landmark

G(n, p) (1000 Nodes, 4979 Edges) / Edge removal

RF Metric

Spectral

Spring

Hop Count
Email networks

Randomly remove some edges.

![Email Network (1133 Nodes, 5451 Edges) / Edge removal](chart)

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<th># Landmark</th>
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</table>
Protein-Protein networks

Randomly remove some edges.

![Graph showing accuracy vs. number of landmarks for different edge removal metrics.](image-url)
Conclusion and On-going Work

Geometric approaches for network analysis has great potential.
Conclusion and On-going Work

Geometric approaches for network analysis has great potential.

- Speed up the computation.
- Find good approximation.
- More applications.
Acknowledgement

- David Gu (Stony Brook), Feng Luo (Rutgers), Emil Saucan (Technion).
- Xiaomeng Ban, Mayank Goswami (CUNY), Yu-Yao Lin, Siming Li, Ruirui Jiang, Chien-Chun Ni, Rik Sarkar (Edinburgh), Rui Shi, Xiaotian Yin, Xiaokang Yu, Wei Zeng (Florida).
- https://www3.cs.stonybrook.edu/~jgao
- Questions and comments?