Graph Ricci Flow and Applications in Network Analysis and Learning

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Graph Data and Complex Networks

Complex networks in nature: social networks, biological networks, the Internet, WWW, mobility data.

- Small world phenomena
- Power law degree distribution
- Community structures (clustered, closely knit groups).
Analyzing Graph Data and Complex Networks

Understand a single network:
- Community detection.
- Graph learning (label propagation & prediction)
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Our project: use geometric tools, Ollivier Ricci curvature flow, to analyze complex networks.
Curvature in Geometry

Sphere: positive curvature; Plane: zero curvature; Hyperbolic plane: negative curvature.
Sectional Curvature in Geometry

Consider a tangent vector $v = xy$ and another tangent vector $w_x$ at $x$. Transport $w_x$ along $v$ to be a tangent vector $w_y$ at $y$. If $|x'y'| < |xy|$, then sectional curvature is positive.
Ollivier Ricci Curvature

Take the analog: for an edge $xy$, consider the “distances” from $x$’s neighbors to $y$’s neighbors and compare it with the length of $xy$.

How to compute the “distances” between two neighborhoods? Use the optimal transport distance.
Ollivier Ricci Curvature

Definition (Ollivier)

Let \((X, d)\) be a metric space and let \(m_1, m_2\) be two probability measures on \(X\). For any two distinct points \(x, y \in X\), the (Ollivier-) Ricci curvature along \(xy\) is defined as

\[
\kappa(x, y) := 1 - \frac{W_1(m_x, m_y)}{d(x, y)},
\]

where \(m_x \ (m_y)\) is a probability distribution defined on \(x \ (y)\) and its neighbors, \(W_1(\mu_1, \mu_2)\) is the \(L_1\) optimal transportation distance between two probability measure \(\mu_1\) and \(\mu_2\) on \(X\):

\[
W_1(\mu_1, \mu_2) := \inf_{\psi \in \Pi(\mu_1, \mu_2)} \int_{(u, v)} d(u, v)\psi(u, v)
\]
Examples

Zero curvature: 2D grid.
Examples

Negative curvature: tree: $\kappa(x, y) = 1/d_x + 1/d_y - 1$, $d_x$ is degree of $x$. 
Examples

Positive curvature: complete graph.
Curvature Distribution

Negatively curved edges are like “backbones”, maintaining the connectivity of clusters, in which edges are mostly positively curved.
Ricci Flow on Manifold vs. on Networks

Hamilton introduced Ricci flow, a curvature guided process.
Ricci Flow Metric

Intuition: flatten the network – shrink an edge if it is within a well connected community; stretch an edge if otherwise, s.t., the network curvature is uniform everywhere.

\[ d_{i+1}(x, y) = (d_i(x, y) - \varepsilon \cdot \kappa_i(x, y) \cdot d_i(x, y)) \cdot N \]
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Distribution on a node \( x \):
- Uniform distribution.
- \( \exp(-d(x, x_i)^p) \), for a constant \( p \).
Theory on Discrete Ricci Flow

Q: Does Ricci flow converge? Does it generate a unique solution?

- Classical manifold setting: contributes to the proof of the Poincare conjecture.
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Q: Does Ricci flow converge? Does it generate a unique solution?
- Classical manifold setting: contributes to the proof of the Poincare conjecture.
- Discrete curvature on graphs: largely unknown.

Ollivier Ricci flow:
- Analysis of a very special case. [Ni, Lin, Luo, Gao, 2019]
- Continuous flow, assumption that the edge $uv$ is the shortest path from $u$ to $v$. [Bai, Lin, Lu, Wang, Yau, 2021]
Applications of Discrete Ricci Flow

- Community detection
- Network alignment
- Graph neural network
Community Detection: Karate Club Network
Community Detection: Facebook Ego Network

792 friends and 14025 edges. The colors represent 24 different friend circles (communities).
Community Detection: Brain Connectome Network

Brain network from resting-state (rs-fMRI) data, where edges with cross-correlation less than a threshold are removed.
Cutoff Threshold vs Modularity

Adjusted Rand index (ARI) on Lancichinetti-Fortunato-Radicch (LFR) benchmark network (community size ∼ power law).

LFR: 1000 Nodes, 9539 Edges, 30 Communities, μ=0.4
Performance Comparison

Adjusted Rand index (ARI) on Lancichinetti-Fortunato-Radicch (LFR) benchmark network (community size $\sim$ power law).

![Graph showing performance comparison between different algorithms.](image-url)
Ricci Flow Metric: Quantify the Network Distance

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Robustness of Ricci Flow Metric: Remove Two Edges

Left: Spectral embedding; Right: Tutte/Spring embedding.
Robustness: Remove Two Edges

Left: Hop count; Right: our metric.
Evaluation on Resilience

Randomly remove 10 edges in a random regular graph.

Random Regular (1000 nodes, 6000 edges) with 10 edges removed

Histogram of RF Metric with OTD
Histogram of Spectral
Histogram of Spring
Histogram of Hop Count
Histogram of RF Metric with ATD

Percentage in Bin (log)

Shortest Path Stretch Ratio

-2.0 -1.5 -1.0 -0.5 0 0.5 1.0
Graph Isomorphism

Given a pair of graphs $G_1, G_2$, find a one-to-one correspondence of the vertices in $G_1$ to vertices in $G_2$ such that $(u, v)$ is an edge in $G_1$ if and only if their corresponding nodes $f(u), f(v)$ are connected in $G_2$. 
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![Graph Isomorphism Diagram]
Our Solution: A Geometric Approach

How to align two sets of points in the plane, assuming that some landmarks \( \ell_i \) are already aligned?

Any point \( p \) can be represented by the barycentric coordinates \((d_1, d_2, d_3)\), \(d_i\) is distance to \( \ell_i \).

If the barycentric coordinates of \( p \) and \( p' \) are similar, we match \( p \) and \( p' \).
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$\ell_1$ $d_1$ $d_2$ $\ell_2$ $p = (d_1, d_2, d_3)$

$\ell_1$ $d_1'$ $d_2'$ $\ell_2$ $p' = (d_1', d_2', d_3')$
Evaluation on Matching Performance

- Randomly remove one node in a random regular graph w/ degree 12.
- Right: remove randomly 10 edges in a protein protein network.
Graph Neural Network

Graph Neural Network for node classification: given labels of a subset of nodes, predict the labels of the rest.

- Graph topology $G = (V, E)$
- Vertex features $H$

Vulnerability: Removal/insertion of fake edges can dramatically hurt model performance.
Robust Graph Neural Network through Resampling

- Recover the underlying metric of $G$ using Ricci flow.
- Re-sample an ensemble of graphs for training.
Robust Graph Neural Network through Resampling
Robustness to Graph Topology Attacks

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Robustness to Graph Topology Attacks

Is the gain coming from graph ensembling or from the choice of metrics?

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Cora: sparse
Polblogs: dense, diameter = 4.
Conclusion and Discussion

- Classical geometric notions for discrete graph analysis.
- Network embedding: Euclidean, hyperbolic, hybrid?
- Network evolution: why?
- More applications due to robustness of the metric?
Acknowledgement

- Chien-Chun Ni, Yu-Yao Lin, Jie Gao, Xianfeng Gu, Network Alignment by Discrete Ollivier-Ricci Flow, Symposium on Graph Drawing and Network Visualization (GD’18).
- Ze Ye, Chien-Chun Ni, Tengfei Ma, Chao Chen, Jie Gao, Ricci-GNN: Defending Against Structural Attacks Through a Geometric Approach, under submission.
Github Code

- https://github.com/saibalmars/GraphRicciCurvature

Questions and comments?