FLAT BUNDLES ON AFFINE MANIFOLDS

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Abstract. We review some recent results in the theory of affine manifolds and bundles on them. Donaldson–Uhlenbeck–Yau type correspondences for flat vector bundles and principal bundles are shown. We also consider flat Higgs bundles and flat pairs on affine manifolds. A bijective correspondence between polystable flat Higgs bundles and solutions of the Yang–Mills–Higgs equation in the context of affine manifolds is shown. Also shown, in the context of affine manifolds, is a bijective correspondence between polystable flat pairs and solutions of the vortex equation.

1. Introduction

An affine manifold is a smooth real manifold M equipped with a flat torsion–free connection D on its tangent bundle. It is well–known (see, e. g., [Sh07]) that an n–dimensional real manifold M is an affine manifold if and only if M admits an atlas such that all the transition functions are affine maps of the form

$$x \longmapsto Ax + b\,, \quad \text{where } A \in \mathrm{GL}(n\,,\mathbb{R}) \quad \text{and} \quad b \in \mathbb{R}^n\,. \tag{1.1}$$

If M is an affine manifold, then the total space of its tangent bundle TM can be given a complex structure in a canonical way. More precisely, given an atlas of M with transition functions as in (1.1), let $\{x^i\}$ be the corresponding coordinates defined on an open subset $U \subset M$; such coordinates $\{x^i\}$ are called *local affine coordinates.* Write y^i for the fiber coordinates corresponding to the local trivialization of the tangent bundle TM over U given by $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$, meaning that every tangent vector y is written as

$$y = \sum_{i=1}^n y^i \frac{\partial}{\partial x^i}\,.$$

Then on the open subset $TU \subset TM$, we have the holomorphic coordinate functions

$$z^i := x^i + \sqrt{-1}\, y^i\,.$$

The complex structure on TM obtained this way will be denoted by $M^{\mathbb{C}}$. The zero section of $M^{\mathbb{C}} = TM \longrightarrow M$ makes M a totally real submanifold of TM.

Let M be an affine manifold. A Riemannian metric g on M is called an *affine Kähler* (or *Hessian*) metric if in a neighborhood of each point of M, there are affine coordinates

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2000 *Mathematics Subject Classification.* 53C07, 57N16.

Key words and phrases. Affine manifold, stability, Hermitian–Einstein connection, Higgs bundle, vortex equation.

$\{x^i\}$ and a real potential function ϕ such that g is given by

$$\sum_{i,j=1}^{n} g_{ij}\, dx^i\, dx^j \;=\; \sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial x^i \partial x^j}\, dx^i\, dx^j\, .$$

Every Riemannian metric g on M extends to a Hermitian metric $g^{\mathbb{C}}$ on $M^{\mathbb{C}}$. The metric g is an affine Kähler metric if and only if $g^{\mathbb{C}}$ is a Kähler metric. Cheng and Yau [CY82] proved the existence of affine Kähler–Einstein metrics on an affine manifold under suitable conditions. (See also [De89] for related results.)

An affine manifold M is called *special* if it admits a volume form (meaning a non–vanishing top–degree form) which is covariant constant with respect to the flat connection D on TM. An affine manifold M is special if and only if it admits an atlas with transition functions as in (1.1) satisfying the additional condition that $A \in \mathrm{SL}(n\,,\mathbb{R})$. Special affine manifolds form an important class of affine manifolds. In particular, there is a famous conjecture of Markus (see [Ma62]) stating that a compact affine manifold is special if and only if the flat connection D on TM is complete. Special affine manifolds also play a role in the Strominger–Yau–Zaslow conjecture; see [SYZ96]. Cheng and Yau showed that on a closed special affine manifold, an affine Kähler metric (if it exists) can be deformed to a flat metric by adding the Hessian of a smooth function; see [CY82].

Cheng and Yau's result shows that compact special affine Kähler manifolds are all torus quotients. On the other hand, there are many examples of compact special affine manifolds which do not admit any affine Kähler metric; see for example [FGH81, FG83]. In order to work on a the full class of special affine manifolds, we must consider consider affine Gauduchon metrics, which generalize the Kähler condition.

In the context of affine manifolds, the right analogue of a holomorphic vector bundle over a complex manifold is a flat vector bundle. To explain this, let E be a smooth (real or complex) vector bundle over an affine manifold M. The pullback of E to $M^{\mathbb{C}}$ by the natural projection $M^{\mathbb{C}} = TM \longrightarrow M$ will be denoted by $E^{\mathbb{C}}$. The transition functions of $E^{\mathbb{C}}$ are obtained by extending the transition functions of E in a constant way along the fibers of TM. Such a transition function on $M^{\mathbb{C}}$ is holomorphic if and only if it is locally constant. Consequently, $E^{\mathbb{C}}$ is a holomorphic vector bundle over $M^{\mathbb{C}}$ if and only if E is a flat vector bundle over M.

Our aim here is to review some recent results in the theory of flat vector bundles and flat principal bundles over a compact connected special affine manifold equipped with an affine Gauduchon metric. More precisely, we will adapt various Donaldson–Uhlenbeck–Yau type correspondences, which relate stability conditions on a bundle to the existence of Hermitian–Einstein structures, from the complex case to the situation of affine manifolds. An important aspect of the proofs of these statements is that the above correspondence between flat vector bundles over an affine manifold M and holomorphic vector bundles over the complex manifold $M^{\mathbb{C}}$ ensures that local calculations can be done exactly in the same way as in the complex case. When integrating by parts on an n–dimensional compact special affine manifold, one uses the covariant constant volume form to convert a $2n$–form on $M^{\mathbb{C}}$ to an n–form on M, which can be integrated.

In Section 2, we will introduce the relevant notions and basic results that are needed in the proofs of the Donaldson–Uhlenbeck–Yau type theorems. In particular, we will give some details on the affine Dolbeault complex and integration by parts on affine manifolds. The Donaldson–Uhlenbeck–Yau type correspondences for flat vector bundles and flat principal bundles over affine manifolds will be given in Sections 3 and 4, respectively.

In Section 5, we will introduce Higgs fields on flat vector bundles and flat principal bundles over affine manifolds, and show that a flat Higgs bundle admits a Yang–Mills–Higgs metric if and only if it is polystable. If X is a compact connected Kähler manifold, then there is a natural bijective correspondence between the isomorphism classes of polystable Higgs vector bundles on X of vanishing Chern classes of positive degrees and the isomorphism classes of direct sums of irreducible flat connections on X [Hi87], [Do87b], [Co88], [Si92, p. 19, Theorem 1]. This correspondence breaks down for a compact complex manifold with Gauduchon metric, in fact even with a balanced metric [Bi11]. Therefore, we cannot expect the analog of the correspondence to hold for a general affine manifold.

In Section 6, we will introduce τ–stability and the τ–vortex equation (where τ is a real number) for pairs consisting of a flat vector bundle over an affine manifold and a flat non–zero section of it, and give a bijective correspondence between polystable flat pairs and solutions of the vortex equation.

There are other generalizations of the Donaldson–Uhlenbeck–Yau correspondence [LT95]. It would be interesting to try to establish these in the frame-work of affine manifolds.

Acknowledgement: We thank the two referees for helpful comments. The first author thanks ICMAT, Madrid, for hospitality while the work was carried out. The second author is partially supported by Simons Collaboration Grant for Mathematicians number 210124.

2. Preliminaries

2.1. **Affine Dolbeault complex.** Let $(M\,,D)$ be an affine manifold of dimension n, meaning that D is a flat torsion–free connection on the tangent bundle of M.

Define the bundle of $(p\,,q)$–forms on M by

$$\mathcal{A}^{p,q}\,:=\,\bigwedge\nolimits^{p} T^*M\otimes \bigwedge\nolimits^{q} T^*M\,.$$

Given local affine coordinates $\{x^i\}_{i=1}^n$ on M, we will denote the induced frame on $\mathcal{A}^{p,q}$ as

$$\left\{dz^{i_1}\wedge\cdots\wedge dz^{i_p}\otimes d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q}\right\},$$

where $z^i\,=\,x^i+\sqrt{-1}\,y^i$ are the complex coordinates on $M^{\mathbb{C}}$ defined above; note that $dz^i\,=\,d\bar{z}^i\,=\,dx^i$ on M. There is a natural restriction map from $(p\,,q)$–forms on the complex manifold $M^{\mathbb{C}}$ to $(p\,,q)$–forms on M given in local affine coordinates on an open subset $U\subset M$ by

$$\begin{aligned}&\sum \phi_{i_1,\ldots,i_p,j_1,\ldots,j_q}\,(dz^{i_1}\wedge\cdots\wedge dz^{i_p})\wedge(d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q})\\ \longmapsto &\sum \phi_{i_1,\ldots,i_p,j_1,\ldots,j_q}|_U\,(dz^{i_1}\wedge\cdots\wedge dz^{i_p})\otimes(d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q})\,,\end{aligned} \tag{2.1}$$

where $\phi_{i_1,\dots,i_p,j_1,\dots,j_q}$ are smooth functions on $TU \subset TM = M^{\mathbb{C}}$, U is considered as the zero section of $TU \longrightarrow U$, and the sums are taken over all $1 \leqslant i_1 < \cdots < i_p \leqslant n$ and $1 \leqslant j_1 < \cdots < j_q \leqslant n$.

One can define natural operators

$$\partial : \mathcal{A}^{p,q} \longrightarrow \mathcal{A}^{p+1,q} \quad \text{and}$$
$$\overline{\partial} : \mathcal{A}^{p,q} \longrightarrow \mathcal{A}^{p,q+1}$$

given in local affine coordinates by

$$\partial\big(\phi \otimes (d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q})\big) \;:=\; \frac{1}{2}\,(d\phi) \otimes (d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q})$$

if ϕ is a p–form, respectively by

$$\overline{\partial}\big((dz^{i_1} \wedge \cdots \wedge dz^{i_p}) \otimes \psi\big) \;:=\; (-1)^p\frac{1}{2}\,(dz^{i_1} \wedge \cdots \wedge dz^{i_p}) \otimes (d\psi)$$

if ψ is a q–form. These operators are the restrictions of the corresponding operators on $M^{\mathbb{C}}$ with respect to the restriction map given in (2.1).

Similarly, there is a wedge product defined by

$$(\phi_1 \otimes \psi_1) \wedge (\phi_2 \otimes \psi_2) \;:=\; (-1)^{q_1 p_2}\,(\phi_1 \wedge \phi_2) \otimes (\psi_1 \wedge \psi_2)$$

if $\phi_i \otimes \psi_i$ are forms of type $(p_i\,, q_i)$, $i = 1, 2$, and a conjugation map from $(p\,, q)$–forms to $(q\,, p)$–forms given by

$$\overline{\phi \otimes \psi} \;:=\; (-1)^{pq}\,\overline{\psi} \otimes \overline{\phi}$$

if ϕ is a p–form and ψ is a q–form; as above, they are the restrictions of the corresponding operations on $M^{\mathbb{C}}$.

A smooth Riemannian metric g on M gives rise to a natural $(1\,, 1)$–form expressed in local affine coordinates as

$$\omega_g \;=\; \sum_{i,j=1}^{n} g_{ij}\, dz^i \otimes d\bar{z}^j\,;$$

it is the restriction of the corresponding $(1\,, 1)$–form on $M^{\mathbb{C}}$ given by the extension of g to $M^{\mathbb{C}}$. The metric g is called an *affine Gauduchon metric* if

$$\partial\overline{\partial}(\omega_g^{n-1}) \;=\; 0$$

(recall that $n = \dim M$). By [Lo09, Theorem 5], on a compact connected special affine manifold, every conformal class of Riemannian metrics contains an affine Gauduchon metric, which is unique up to a positive scalar.

2.2. **Integration by parts.** The main difference between complex and affine manifolds is that on an n–dimensional complex manifold, an $(n\,, n)$–form is a top–degree form, which can be integrated, while on an n–dimensional affine manifold, an $(n\,, n)$–form is not a top–degree form. In this subsection, we will explain how to overcome this under the additional assumption that the affine manifold M is special.

Let $(M\,,D\,,\nu)$ be a special affine manifold, meaning that $(M\,,D)$ is an affine manifold as above and ν is a D–covariant constant volume form on M. The volume form ν induces natural maps

$$\mathcal{A}^{n,q} \longrightarrow \bigwedge^q T^*M\,, \quad \nu\otimes\chi \longmapsto (-1)^{\frac{n(n-1)}{2}}\chi\,,$$
$$\mathcal{A}^{p,n} \longrightarrow \bigwedge^p T^*M\,, \quad \chi\otimes\nu \longmapsto (-1)^{\frac{n(n-1)}{2}}\chi\,;$$

these maps are called *division by* ν. The choice of sign ensures that for every Riemannian metric g on M, the volume form

$$\frac{\omega_g^n}{\nu}$$

induces the same orientation as the volume form ν. If M is compact, an $(n\,,n)$–form χ on M can be integrated by considering the integral

$$\int_M \frac{\chi}{\nu}\,.$$

The fact that ν is covariant constant with respect to D ensures that the usual integration by parts formulas for $(p\,,q)$–forms still work on the affine manifold M. More precisely, we have the following proposition from [Lo09].

Proposition 1 ([Lo09, Proposition 3]). *Let $(M\,,D\,,\nu)$ be an n–dimensional special affine manifold. Then if χ is an $(n-1\,,n)$–form on M, we have*

$$\frac{\partial\chi}{\nu} = \frac{1}{2}\,d\left(\frac{\chi}{\nu}\right)\,.$$

Also, if χ is an $(n\,,n-1)$–form, we have

$$\frac{\overline{\partial}\chi}{\nu} = (-1)^n\,\frac{1}{2}\,d\left(\frac{\chi}{\nu}\right)\,.$$

3. Hermitian–Einstein metrics on flat vector bundles

Let $(M\,,D\,,\nu)$ be a compact connected special affine manifold equipped with an affine Gauduchon metric g. Let $(E\,,\nabla)$ be a flat complex vector bundle over M, meaning that E is a smooth complex vector bundle and ∇ is a flat connection on E. (In the following, unless otherwise stated, we will always be concerned with complex vector bundles.) As in the introduction, the pullback of E to $M^{\mathbb{C}}$ by the natural projection $M^{\mathbb{C}} = TM \longrightarrow M$ will be denoted by $E^{\mathbb{C}}$. The flat connection ∇ pulls back to a flat connection on $E^{\mathbb{C}}$. The flat vector bundle over $M^{\mathbb{C}}$ obtained this way can be considered as an extension to $M^{\mathbb{C}}$ of the flat vector bundle $(E\,,\nabla)$ on the zero section of TM.

A connection ∇ on a smooth vector bundle E is flat if and only if E admits *locally constant frames*, meaning locally defined smooth frames $\{s_\alpha\}$ satisfying $\nabla(s_\alpha) = 0$. Any locally constant frame of E over M extends to a locally constant frame of $E^{\mathbb{C}}$ over $M^{\mathbb{C}}$.

Let h be a Hermitian metric on E; it defines a Hermitian metric on $E^{\mathbb{C}}$. Let d^h be the Chern connection associated to this Hermitian metric on $E^{\mathbb{C}}$. Then d^h corresponds to a pair

$$(\partial^h\,,\overline{\partial})\;=\;(\partial^{h,\nabla}\,,\overline{\partial}^{\nabla})\,, \tag{3.1}$$

where

$$\partial^{h,\nabla}\,:\,E\,\longrightarrow\,\mathcal{A}^{1,0}(E)\quad\text{and}\quad\overline{\partial}^{\nabla}\,:\,E\,\longrightarrow\,\mathcal{A}^{0,1}(E)$$

are smooth differential operators. Here we write $\mathcal{A}^{p,q}(E) := \mathcal{A}^{p,q}\otimes E$. This pair $(\partial^h\,,\overline{\partial})$ is called the *extended Hermitian connection* of $(E\,,h)$.

If $(E\,,\nabla)$ is a flat real vector bundle over M and h is a real positive–definite metric on E, then the extended Hermitian connection of the complexified vector bundle $E\otimes\mathbb{C}$ over M equipped with the flat connection induced by ∇ and the Hermitian metric induced by h has an interpretation in terms of the dual connection of ∇ with respect to h. Recall that the dual connection ∇^* on E is defined by

$$d(h(s_1\,,s_2))\;=\;h(\nabla s_1\,,s_2)+h(s_1\,,\nabla^* s_2) \tag{3.2}$$

for smooth sections s_1 and s_2 of E (see, e. g., [AN00]). Then we have the following statement from [Lo09].

Proposition 2 ([Lo09, Lemma 1])**.** *If $(E\,,\nabla)$ is a flat real vector bundle over an affine manifold M equipped with a real positive–definite metric h, then the extended Hermitian connection on $E\otimes\mathbb{C}$ equipped with the flat connection induced by ∇ and the Hermitian metric induced by h is given by*

$$(\partial^{h,\nabla}\,,\overline{\partial}^{\nabla})\;=\;(\nabla^*\otimes\tfrac{1}{2}\,,\tfrac{1}{2}\otimes\nabla)\,,$$

where the dual connection ∇^* is defined in (3.2) *and is flat.*

Similarly, there are locally defined *extended connection forms*

$$\theta\;\in\;C^\infty(\mathcal{A}^{1,0}(\operatorname{End}E))\,, \tag{3.3}$$

an *extended curvature form*

$$R\;=\;\overline{\partial}\theta\;\in\;C^\infty(M\,,\mathcal{A}^{1,1}(\operatorname{End}E))\,,$$

an *extended mean curvature*

$$K\;=\;\operatorname{tr}_g R\;\in\;C^\infty(M\,,\operatorname{End}E)\,, \tag{3.4}$$

and an *extended first Chern form*

$$c_1(E\,,h)\;=\;\operatorname{tr}R\;\in\;C^\infty(M\,,\mathcal{A}^{1,1})\,,$$

which are the restrictions of the corresponding objects on $E^{\mathbb{C}}$. Here tr_g denotes contraction of differential forms using the Riemannian metric g, and tr denotes the trace homomorphism on the fibers of $\operatorname{End}E$.

The extended first Chern form is given by

$$c_1(E\,,h)\;=\;-\partial\overline{\partial}(\log\det(h_{\alpha\bar\beta}))\,,$$

where $h_{\alpha\bar\beta}=h(s_\alpha\,,s_\beta)$ in a locally constant frame $\{s_\alpha\}$ of E.

The extended first Chern form and the extended mean curvature are related by

$$(\operatorname{tr}K)\,\omega_g^n\;=\;n\cdot c_1(E\,,h)\wedge\omega_g^{n-1}\,. \tag{3.5}$$

Definition 3. A Hermitian metric h on E is called a *Hermitian–Einstein metric* (with respect to g) if its extended mean curvature K_h is of the form

$$K_h = \gamma \cdot \mathrm{Id}_E$$

for some real constant γ.

The *degree* of (E, ∇) with respect to a Gauduchon metric g on M is defined to be

$$\deg_g(E) := \int_M \frac{c_1(E, h) \wedge \omega_g^{n-1}}{\nu}. \tag{3.6}$$

To see that it is well–defined, observe that for any two Hermitian metrics h and h' on E, we have

$$c_1(E, h') - c_1(E, h) = \partial\overline{\partial}(\log\det(h_{\alpha\bar{\beta}}) - \log\det(h'_{\alpha\bar{\beta}})),$$

which is $\partial\overline{\partial}$ of a function on M. Since by Proposition 1, we can integrate by parts as in the usual case, and g is a Gauduchon metric, it follows that the degree in (3.6) is well–defined. Note that even though E admits a flat connection ∇, there is no reason in general for the degree to be zero in the Gauduchon case. In particular, we can extend ∇ to a flat extended connection on E and then define an extended first Chern form $c_1(E, \nabla)$. But

$$c_1(E, \nabla) - c_1(E, h) = \mathrm{tr}\,\overline{\partial}\theta_\nabla - \overline{\partial}\partial \log\det h_{\alpha\bar{\beta}}$$

is $\overline{\partial}$–exact but not necessarily $\partial\overline{\partial}$–exact. Thus, by integration by parts, the Gauduchon condition is insufficient to force the degree to be zero.

As usual, if $\mathrm{rank}(E) > 0$, the *slope* of E with respect to g is defined to be

$$\mu_g(E) := \frac{\deg_g(E)}{\mathrm{rank}(E)}.$$

Definition 4.

(i) (E, ∇) is called *semistable* (respectively, *stable*) (with respect to g) if for every proper non–zero flat subbundle E' of E, we have

$$\mu_g(E') \leqslant \mu_g(E) \quad (\text{respectively, } \mu_g(E') < \mu_g(E)).$$

(ii) (E, ∇) is called *polystable* (with respect to g) if

$$(E, \nabla) = \bigoplus_{i=1}^{N} (E^i, \nabla^i),$$

where each (E^i, ∇^i) is a stable flat vector bundle with slope $\mu_g(E^i) = \mu_g(E)$.

Note that in our situation of flat vector bundles over affine manifolds it suffices to consider subbundles rather than (singular) subsheaves in the definition of stability.

In [Lo09], the following Donaldson–Uhlenbeck–Yau type correspondence was established.

Theorem 5 ([Lo09, Theorems 1 and 4]). *Let (M, D, ν) be a compact connected special affine manifold equipped with an affine Gauduchon metric g, and let (E, ∇) be a flat complex vector bundle over M. Then E admits a Hermitian–Einstein metric with respect to g if and only if it is polystable.*

The main result of [CY82] provides an important antecedent and special case of Theorem 5. On a closed special affine Kähler manifold M, Cheng–Yau produce an affine Kähler metric g whose induced metric on $M^{\mathbb{C}}$ is Kähler and Ricci–flat. In this case, the Kähler metric, considered as a bundle metric on the tangent bundle $T(M^{\mathbb{C}})$, is Hermitian–Einstein with zero mean curvature. Therefore, g is affine Hermitian–Einstein with respect to its own Kähler form ω_g. Cheng–Yau also show that such metrics are flat, and deduce by Bieberbach's Theorem that such an M is a finite quotient of a flat torus. (Cheng–Yau [CY82] also produce affine Kähler–Einstein metrics with negative extended Ricci curvature on affine manifolds which are quotients of convex pointed cones. The induced metrics on the tangent bundles of these manifolds are not direct examples of the affine Hermitian–Einstein metrics we consider, however, as such affine manifolds do not admit parallel volume forms.)

The proof of Theorem 5 is an adaptation to the affine situation of the methods of Uhlenbeck and Yau, [UY86], [UY89], for compact Kähler manifolds and their modification by Li and Yau, [LY87], for the complex Gauduchon case. As mentioned above, all local calculations are identical to the complex case due to the correspondence between flat vector bundles on the affine manifold M and holomorphic vector bundles on the complex manifold $M^{\mathbb{C}}$ which are constant along the fibers of $M^{\mathbb{C}} = TM \longrightarrow M$, and integration by parts is performed using Proposition 1. A major simplification compared to the complex case is the avoidance of the intricate proof of Uhlenbeck–Yau that a weakly holomorphic subbundle of a holomorphic vector bundle on a complex manifold is a reflexive analytic subsheaf. The corresponding regularity statement on affine manifolds is that a *weakly flat subbundle* of a flat Hermitian vector bundle $(E\,,h_0)$, meaning a subbundle given by an h_0–orthogonal L^2_1 projection π which satisfies $(\mathrm{Id}_E - \pi)\circ \overline{\partial}\pi = 0$ in L^1, is in fact a flat subbundle; this is much easier to prove (see [Lo09, Proposition 27]).

We give an analogue of the above theorem for flat real vector bundles.

Definition 6. Let $(E\,,\nabla)$ be a flat real vector bundle over M.

(i) $(E\,,\nabla)$ is called $\mathbb{R}$–*semistable* (respectively, $\mathbb{R}$–*stable*) (with respect to g) if for every proper non–zero flat real subbundle E' of E, we have

$$\mu_g(E') \leqslant \mu_g(E) \quad (\text{respectively, } \mu_g(E') < \mu_g(E))\,.$$

(ii) $(E\,,\nabla)$ is called $\mathbb{R}$–*polystable* (with respect to g) if

$$(E\,,\nabla) = \bigoplus_{i=1}^{N} (E^i\,,\nabla^i)\,,$$

where each $(E^i\,,\nabla^i)$ is an $\mathbb{R}$–stable flat real vector bundle with slope $\mu_g(E^i) = \mu_g(E)$.

Corollary 7 ([Lo09, Corollary 33 and Theorem 4])**.** *Let $(M\,,D\,,\nu)$ be a compact connected special affine manifold equipped with an affine Gauduchon metric g, and let $(E\,,\nabla)$ be a flat real vector bundle over M. Then E admits a Hermitian–Einstein metric with respect to g if and only if it is $\mathbb{R}$–polystable.*

4. Hermitian–Einstein connections on flat principal bundles

Let (M, D, ν) be a compact connected special affine manifold equipped with an affine Gauduchon metric g. In this section, we will explain how to generalize the results of the previous section to flat principal bundles over M.

4.1. **Preliminaries.** Let H be a Lie group. A principal H–bundle over M is a triple (E_H, p, ψ), where E_H is a smooth real manifold, $p : E_H \longrightarrow M$ is a smooth surjective submersion, and

$$\psi : E_H \times H \longrightarrow E_H$$

is a smooth action of H on E_H, such that

(1) $p \circ \psi = p \circ p_1$, where p_1 is the natural projection of $E_H \times H$ to E_H, and
(2) for each point $x \in M$, there is an open neighborhood $U \subset M$ of x and a smooth diffeomorphism

$$\phi : p^{-1}(U) \longrightarrow U \times H,$$

such that ϕ commutes with the actions of H (the group H acts on $U \times H$ through right translations of H), and $q_1 \circ \phi = p$, where q_1 is the natural projection of $U \times H$ to U.

Let $dp : TE_H \longrightarrow p^*TM$ be the differential of the projection p. A *flat connection* on E_H is a smooth homomorphism

$$\nabla : p^*TM \longrightarrow TE_H,$$

such that

(1) $dp \circ \nabla = \mathrm{Id}_{p^*TM}$,
(2) the distribution $\nabla(p^*TM) \subset TE_H$ is integrable, and
(3) $\nabla(p^*TM)$ is invariant under the action of H on TE_H given by the action of H on E_H.

A pair (E_H, ∇) consisting of a principal H–bundle E_H over M and a flat connection ∇ on E_H will be called a *flat principal H–bundle*. We also write E_H for (E_H, ∇) if ∇ is clear from the context.

Let $H' \subset H$ be a closed subgroup. A *reduction of structure group* of a principal H–bundle E_H to H' is a principal H'–bundle $E_{H'} \subset E_H$; the action of H' on $E_{H'}$ is the restriction of the action of H on E_H. A reduction of structure group of E_H to H' is given by a smooth section of the fiber bundle $E_H/H' \longrightarrow M$. We note that if a reduction $E_{H'} \subset E_H$ corresponds to a section σ, then $E_{H'}$ is the inverse image of $\sigma(M)$ under the quotient map $E_H \longrightarrow E_H/H'$.

Let ∇ be a flat connection on E_H. A reduction of structure group $E_{H'} \subset E_H$ to H' is said to be *compatible* with ∇ if for each point $z \in E_{H'}$, the subspace $\nabla(T_{p(z)}M) \subset T_zE_H$ is contained in the subspace $T_zE_{H'} \subset T_zE_H$. Note that this condition ensures that ∇ produces a flat connection on $E_{H'}$.

Consider the adjoint action of H on its Lie algebra $\text{Lie}(H)$. Let

$$\text{ad}(E_H) := E_H \times^H \text{Lie}(H) \longrightarrow M \tag{4.1}$$

be the vector bundle over M associated to the principal H–bundle E_H for this action; it is known as the *adjoint vector bundle* of E_H. Since the adjoint action of H on $\text{Lie}(H)$ preserves the Lie algebra structure, the fibers of $\text{ad}(E_H)$ are Lie algebras isomorphic to $\text{Lie}(H)$. The connection ∇ on E_H induces a connection on every fiber bundle associated to E_H. In particular, ∇ induces a connection on the vector bundle $\text{ad}(E_H)$; this induced connection on $\text{ad}(E_H)$ will be denoted by ∇^{ad}. The connection ∇^{ad} is compatible with the Lie algebra structure of the fibers of $\text{ad}(E_H)$, meaning

$$\nabla^{\text{ad}}([s\,,t]) = [\nabla^{\text{ad}}(s)\,,t] + [s\,,\nabla^{\text{ad}}(t)]$$

for all locally defined smooth sections s and t of $\text{ad}(E_H)$.

4.2. **Stable and semistable principal bundles.** Let $G_{\mathbb{C}}$ be a complex reductive linear algebraic group. A *real form* on $G_{\mathbb{C}}$ is an anti–holomorphic involution

$$\sigma_{G_{\mathbb{C}}} : G_{\mathbb{C}} \longrightarrow G_{\mathbb{C}}\,.$$

The real form $\sigma_{G_{\mathbb{C}}}$ is said to be of *split type* if there is a maximal torus $T \subset G_{\mathbb{C}}$ such that $\sigma_{G_{\mathbb{C}}}(T) = T$ and the fixed point locus of the involution $\sigma_{G_{\mathbb{C}}}|_T$ of T is a product of copies of $\mathbb{R}^*$ (the group of non–zero real numbers).

Let G be a connected Lie group such that either it is a complex reductive linear algebraic group, or it is the fixed point locus of a split real form $\sigma_{G_{\mathbb{C}}} \in \text{Aut}(G_{\mathbb{C}})$, where $G_{\mathbb{C}}$ and $\sigma_{G_{\mathbb{C}}}$ are as above.

If G is a complex reductive group, a connected closed algebraic subgroup $P \subset G$ is called a *parabolic subgroup* if the quotient variety G/P is complete. So, in particular, G itself is a parabolic subgroup. Let P be a parabolic subgroup of G. A character χ of P is called *strictly anti–dominant* if the following two conditions hold:

(1) the line bundle over G/P associated to the principal P–bundle $G \longrightarrow G/P$ for χ is ample, and
(2) the character χ is trivial on the connected component of the center of P containing the identity element.

Let $R_u(P) \subset P$ be the unipotent radical. The group $P/R_u(P)$ is called the *Levi quotient* of P. A *Levi subgroup* of P is a connected reductive subgroup $L(P) \subset P$ such that the composition

$$L(P) \hookrightarrow P \longrightarrow P/R_u(P)$$

is an isomorphism. A Levi subgroup always exists (see [Bo91, p. 158, §11.22] and [Hu81, p. 184, §30.2]).

If G is the fixed point locus of a real form $(G_{\mathbb{C}}\,,\sigma_{G_{\mathbb{C}}})$, by a *parabolic subgroup* of G we will mean a subgroup $P \subset G$ such that there is a parabolic subgroup $P' \subset G_{\mathbb{C}}$ satisfying the conditions that $\sigma_{G_{\mathbb{C}}}(P') = P'$ and $P' \cap G = P$. By a *Levi subgroup* of a parabolic subgroup P we will mean a subgroup $L(P) \subset P$ such that there is a Levi subgroup $L(P') \subset P'$ satisfying the conditions that $\sigma_{G_{\mathbb{C}}}(L(P')) = L(P')$ and $L(P') \cap G = L(P)$.

Definition 8. Let $(E_G\,,\nabla)$ be a flat principal G–bundle over M.

$(E_G\,,\nabla)$ is called *semistable* (respectively, *stable*) if for every triple $(Q\,,E_Q\,,\lambda)$, where $Q\subset G$ is a proper parabolic subgroup, $E_Q\subset E_G$ is a reduction of structure group of E_G to Q compatible with ∇, and λ is a strictly anti–dominant character of Q, the inequality

$$\deg_g(E_Q(\lambda))\,\geqslant\,0\quad(\text{respectively, }\deg_g(E_Q(\lambda))\,>\,0)$$

holds, where $E_Q(\lambda)$ is the flat line bundle over M associated to the flat principal Q–bundle E_Q for the character λ of Q, and the degree is defined in (3.6).

In order to decide whether $(E_G\,,\nabla)$ is semistable (respectively, stable), it suffices to verify the above inequality (respectively, strict inequality) only for those Q which are proper maximal parabolic subgroups of G. More precisely, E_G is semistable (respectively, stable) if and only if for every pair $(Q\,,\sigma)$, where $Q\subset G$ is a proper maximal parabolic subgroup, and $\sigma:M\longrightarrow E_G/Q$ is a reduction of structure group of E_G to Q compatible with ∇, the inequality

$$\deg_g(\sigma^*T_{\rm rel})\,\geqslant\,0\quad(\text{respectively, }\deg_g(\sigma^*T_{\rm rel})\,>\,0)$$

holds, where $T_{\rm rel}$ is the relative tangent bundle over E_G/Q for the projection $E_G/Q\longrightarrow M$. (See [Ra75, p. 129, Definition 1.1] and [Ra75, p. 131, Lemma 2.1].) It should be mentioned that the flat connection ∇ on E_G induces a flat connection on the associated fiber bundle $E_G/Q\longrightarrow M$. Since the section σ is flat with respect to this induced connection (it is flat because the reduction E_Q is compatible with ∇), the pullback $\sigma^*T_{\rm rel}$ gets a flat connection.

Let $(E_G\,,\nabla)$ be a flat principal G–bundle over M. A reduction of structure group

$$E_Q\,\subset\,E_G$$

of E_G to a parabolic subgroup $Q\,\subset\,G$ compatible with ∇ is called *admissible* if for each character λ of Q which is trivial on the center of G, the associated flat line bundle $E_Q(\lambda)\longrightarrow M$ satisfies the following condition:

$$\deg_g(E_Q(\lambda))\,=\,0\,.$$

Definition 9. A flat principal G–bundle $(E_G\,,\nabla)$ over M is called *polystable* if either it is stable, or there is a proper parabolic subgroup $Q\,\subset\,G$ and a reduction of structure group $E_{L(Q)}\subset E_G$ of E_G to a Levi subgroup $L(Q)$ of Q compatible with ∇ such that the following conditions are satisfied:

(1) the flat principal $L(Q)$–bundle $E_{L(Q)}$ is stable, and
(2) the reduction of structure group of E_G to Q, obtained by extending the structure group of $E_{L(Q)}$ using the inclusion of $L(Q)$ in Q, is admissible.

We note that a polystable flat principal G–bundle over M is semistable.

4.3. **Hermitian–Einstein connections.** Fix a maximal compact subgroup

$$K\,\subset\,G$$

of the reductive group G. Let E_G be a flat principal G–bundle over M. A *Hermitian structure* on E_G is a smooth reduction of structure group

$$E_K \subset E_G .$$

Recall that G is either the fixed point locus of a split real form on a complex reductive group $G_{\mathbb{C}}$, or G is complex reductive. In the second case, by $G_{\mathbb{C}}$ we will denote G itself for notational convenience.

By analogous arguments to those given in Section 3 for flat vector bundles, the flat principal G–bundle E_G over M extends to a holomorphic principal $G_{\mathbb{C}}$–bundle $E_{G_{\mathbb{C}}}$ over the complex manifold $M^{\mathbb{C}}$. Also, given a Hermitian structure on E_G, there is a naturally associated connection on the principal $G_{\mathbb{C}}$–bundle $E_{G_{\mathbb{C}}}$ over $M^{\mathbb{C}}$.

Any element z of the center of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ defines a flat section of $\mathrm{ad}(E_G)$, because z is fixed by the adjoint action of G on $\mathfrak{g}$; this section of $\mathrm{ad}(E_G)$ given by z will be denoted by $\underline{z}$.

Let $E_K \subset E_G$ be a Hermitian structure on E_G, and let d^{E_K} be the corresponding connection on $E_{G_{\mathbb{C}}}$. The curvature of d^{E_K} will be denoted by R_{E_K}. So R_{E_K} is a smooth $(1\,,1)$–form on $M^{\mathbb{C}}$ with values in the adjoint vector bundle $\mathrm{ad}(E_{G_{\mathbb{C}}})$. Contracting it using the metric g, we get a smooth section $\mathrm{tr}_g\, R_{E_K}$ of $\mathrm{ad}(E_G)$.

Definition 10. The Hermitian structure E_K is called *Hermitian–Einstein* if there is an element z in the center of the Lie algebra $\mathfrak{g}$ such that

$$\mathrm{tr}_g\, R_{E_K} = \underline{z} .$$

If E_K is a Hermitian–Einstein structure, then the corresponding connection d^{E_K} is called a *Hermitian–Einstein connection.*

Theorem 11 ([BL12, Theorem 1.2])**.** *A flat principal G–bundle E_G over M admits a Hermitian–Einstein structure if and only if it is polystable. A polystable flat principal G–bundle admits a unique Hermitian–Einstein connection.*

An important ingredient in the proof of Theorem 11 is the following analogue of the Harder–Narasimhan filtration for flat vector bundles.

Theorem 12 ([BL12, Theorem 1.3])**.** *Let V be a flat vector bundle over M. Then there is a unique filtration of V by flat subbundles*

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_j \subsetneq F_{j+1} \subsetneq \cdots \subsetneq F_{\ell-1} \subsetneq F_\ell = V$$

such that for each $i \in [1\,,\ell]$, the flat vector bundle F_i/F_{i-1} is semistable, and

$$\mu_g(F_1) > \mu_g(F_2/F_1) > \cdots > \mu_g(F_{j+1}/F_j) > \cdots > \mu_g(F_\ell/F_{\ell-1}) .$$

The proof of Theorem 12 given in [BL12] makes use of a major simplification compared to the complex case. Namely, flatness implies that the space of all flat subbundles of some rank k of a given flat vector bundle V is a closed subset of the Grassmannian of k–dimensional subspaces of the fiber of V at a fixed point of M, and thus a simple compactness argument guarantees the existence of a flat subbundle with maximal slope. The corresponding argument in the complex Gauduchon case is more complicated; see [Br01].

4.4. **A Bogomolov type inequality.** We explain an application of Theorem 11 given in [BL12]. As before, let (M, D, ν) be a compact connected special affine manifold of dimension n equipped with an affine Gauduchon metric g. Recall that the Gauduchon condition says that

$$\partial\overline{\partial}(\omega_g^{n-1}) = 0,$$

where ω_g is the $(1,1)$–form associated to g. The Gauduchon metric g is called *astheno–Kähler* if

$$\partial\overline{\partial}(\omega_g^{n-2}) = 0 \tag{4.2}$$

(see [JY93, p. 246]). Note that if $n = 2$, then g is automatically astheno–Kähler. If g is Kähler, then it is also astheno–Kähler.

For the rest of this subsection, we will assume that g is astheno–Kähler.

If V is a flat vector bundle over M and h is a Hermitian metric on V, then the astheno–Kähler condition (4.2) implies that

$$\int_M \frac{c_1(E, h)^2 \wedge \omega_g^{n-2}}{\nu} \in \mathbb{R} \quad \text{and} \quad \int_M \frac{c_2(E, h) \wedge \omega_g^{n-2}}{\nu} \in \mathbb{R}$$

are independent of the choice of h.

The following Bogomolov type inequality for flat vector bundles was given in [BL12].

Proposition 13 ([BL12, Lemma 1.1])**.** *Let V be a semistable flat vector bundle over M of rank r. Then*

$$\int_M \frac{c_2(\operatorname{End} V) \wedge \omega_g^{n-2}}{\nu} = \int_M \frac{(2r \cdot c_2(V) - (r-1) \cdot c_1(V)^2) \wedge \omega_g^{n-2}}{\nu} \geqslant 0.$$

As before, let G be a connected Lie group which is either a complex reductive linear algebraic group, or the fixed point locus of a split real form on a complex reductive linear algebraic group. Then Proposition 13 has the following corollary.

Corollary 14 ([BL12, Proposition 6.1])**.** *Let E_G be a semistable flat principal G–bundle over M. Then*

$$\int_M \frac{c_2(\operatorname{ad}(E_G)) \wedge \omega_g^{n-2}}{\nu} \geqslant 0.$$

5. Yang–Mills–Higgs metrics

Higgs bundles were first investigated by Hitchin in [Hi87] (see also [Si92]). Hitchin and Donaldson extended the correspondence between polystable bundles and Hermitian–Einstein (or Yang–Mills) connections to Higgs bundles on Riemann surfaces; see [Hi87] and [Do87b]. Simpson extended it to Higgs bundles on compact Kähler manifolds (and also some non–compact cases) using Donaldson's heat flow technique (see [Si88], [Do85], [Do87a]). Recently, this has been adapted for the compact Gauduchon case by Jacob [Ja11].

As before, let (M, D, ν) be a compact connected special affine manifold equipped with an affine Gauduchon metric g. In this section, we will introduce Higgs fields on flat vector bundles over M and give a correspondence between polystable flat Higgs bundles and Yang–Mills–Higgs connections.

The flat connection D on TM induces a flat connection on T^*M; this induced connection on T^*M will be denoted by D^*. Let $(E\,,\nabla)$ be a flat complex vector bundle over M.

Definition 15. A *flat Higgs field* on $(E\,,\nabla)$ is a smooth section φ of $T^*M \otimes \operatorname{End} E$ such that

(1) φ is covariant constant, meaning the connection operator

$$\widetilde{\nabla} : T^*M \otimes \operatorname{End} E \longrightarrow T^*M \otimes T^*M \otimes \operatorname{End} E$$

defined by the connections ∇ on E and D^* on T^*M respectively, annihilates φ, and

(2) $\varphi \wedge \varphi = 0$.

If φ is a flat Higgs field on $(E\,,\nabla)$, then $(E\,,\nabla\,,\varphi)$ (or $(E\,,\varphi)$ if ∇ is understood from the context) is called a *flat Higgs bundle.*

The space of all connections on E is an affine space for the vector space of smooth sections of $T^*M \otimes \operatorname{End} E$; a family of connections $\{\nabla_t\}_{t\in\mathbb{R}}$ is called *affine* if there is a smooth section α of $T^*M \otimes \operatorname{End} E$ such that $\nabla_t = \nabla_0 + t\cdot\alpha$. The following equivalent definition of flat Higgs bundles was given in [BLS12a].

Lemma 16 ([BLS12a, Lemma 2.2])**.** *Giving a flat Higgs bundle $(E\,,\nabla\,,\varphi)$ is equivalent to giving a smooth vector bundle E together with a 1–dimensional affine family*

$$\{\nabla_t \;:=\; \nabla_0 + t\cdot\alpha\}_{t\in\mathbb{R}}$$

of flat connections on E such that the $\operatorname{End} E$*–valued 1–form α is flat with respect to the connection on $T^*M \otimes \operatorname{End} E$ defined by ∇_0 and D^*.*

A Higgs field will always be understood as a section of $\mathcal{A}^{1,0}(\operatorname{End} E)$, meaning it is expressed in local affine coordinates as

$$\varphi \;=\; \sum_{i=1}^{n} \varphi_i \otimes dz^i\,,$$

where φ_i are locally defined flat sections of $\operatorname{End} E$; note that $dz^i = dx^i$ on M. Given a Hermitian metric h on E, the adjoint φ^* of φ with respect to h will be regarded as an element of $\mathcal{A}^{0,1}(\operatorname{End} E)$. In local affine coordinates, this means that

$$\varphi^* \;=\; \sum_{j=1}^{n} (\varphi_j)^* \otimes d\bar{z}^j\,.$$

In particular, the Lie bracket $[\varphi\,,\varphi^*]$ is an element of $\mathcal{A}^{1,1}(\operatorname{End} E)$. Locally,

$$[\varphi\,,\varphi^*] \;=\; \varphi\wedge\varphi^* + \varphi^*\wedge\varphi \;=\; \sum_{i,j=1}^{n} \big(\varphi_i \circ (\varphi_j)^* - (\varphi_j)^* \circ \varphi_i\big) \otimes dz^i \otimes d\bar{z}^j\,.$$

Let E be a flat vector bundle over M equipped with a flat Higgs field φ as well as a Hermitian metric h. As before, we write d^h for the corresponding Chern connection on

$E^{\mathbb{C}}$ over $M^{\mathbb{C}}$ and $(\partial^h, \overline{\partial})$ for the extended Hermitian connection of (E, h) (see (3.1)). The *extended connection form* θ^φ of the Hermitian flat Higgs bundle (E, φ, h) is defined to be

$$\theta^\varphi := (\theta + \varphi, \varphi^*) \in \mathcal{A}^{1,0}(\operatorname{End} E) \oplus \mathcal{A}^{0,1}(\operatorname{End} E),$$

where θ is the extended connection form of (E, h) (see (3.3)) and φ^* denotes the adjoint of φ with respect to h. This extended connection form corresponds to the connection form of $d^h + \varphi + \varphi^*$ on $E^{\mathbb{C}} \longrightarrow M^{\mathbb{C}}$. Analogously, the *extended curvature form* R^φ of (E, φ, h) is defined to be

$$R^\varphi := \left(\partial^h \varphi, \overline{\partial}\theta + [\varphi, \varphi^*], \overline{\partial}(\varphi^*)\right) \in \mathcal{A}^{2,0}(\operatorname{End} E) \oplus \mathcal{A}^{1,1}(\operatorname{End} E) \oplus \mathcal{A}^{0,2}(\operatorname{End} E);$$

it corresponds to the curvature form of $d^h + \varphi + \varphi^*$ on $E^{\mathbb{C}}$. As in the usual case, the *extended mean curvature* K^φ of (E, φ, h) is obtained by contracting the $(1,1)$–part of the extended curvature R^φ using the Riemannian metric g, so

$$K^\varphi := \operatorname{tr}_g \left(\overline{\partial}\theta + [\varphi, \varphi^*]\right) \in \mathcal{A}^{0,0}(\operatorname{End} E).$$

Since $\operatorname{tr}[\varphi, \varphi^*] = 0$, we have $\operatorname{tr} K^\varphi = \operatorname{tr} K$, and so by (3.5), the extended mean curvature K^φ of (E, φ, h) is related to the first Chern form $c_1(E, h)$ by

$$(\operatorname{tr} K^\varphi)\, \omega_g^n = n \cdot c_1(E, h) \wedge \omega_g^{n-1}.$$

Definition 17. A *Yang–Mills–Higgs metric* on (E, φ) is a Hermitian metric h on E such that the extended mean curvature K^φ of (E, φ, h) is of the form

$$K^\varphi = \gamma \cdot \operatorname{Id}_E$$

for some real constant γ. If h is a Yang–Mills–Higgs metric, then the corresponding connection d^h is called a *Yang–Mills–Higgs connection.*

Definition 18. Let (E, φ) be a flat Higgs bundle over M.

(i) (E, φ) is called *semistable* (respectively, *stable*) (with respect to g) if for every proper non–zero flat subbundle E' of E which is preserved by φ, meaning $\varphi(E') \subset T^*M \otimes E'$, we have

$$\mu_g(E') \leqslant \mu_g(E) \quad \text{(respectively, } \mu_g(E') < \mu_g(E)\text{)}. \tag{5.1}$$

(ii) (E, φ) is called *polystable* (with respect to g) if

$$(E, \varphi) = \bigoplus_{i=1}^{N} (E^i, \varphi^i),$$

where each (E^i, φ^i) is a stable flat Higgs bundle with slope $\mu_g(E^i) = \mu_g(E)$.

Remark 19. *If* $\{\nabla_t\}_{t\in\mathbb{R}}$ *is the family of flat connections on* E *satisfying the condition in Lemma 16 and corresponding to the flat Higgs bundle* (E, φ)*, then Definition 18 (i) is equivalent to the condition that* (5.1) *holds for every proper non–zero smooth subbundle* E' *of* E *which is preserved by* ∇_t *for all* t*.*

The following Donaldson–Uhlenbeck–Yau type correspondence was proved in [BLS12a].

Theorem 20 ([BLS12a, Theorem 1.1 and Corollary 4.1])**.** *Let $(M\,,D\,,\nu)$ be a compact connected special affine manifold equipped with an affine Gauduchon metric g, and let $(E\,,\nabla\,,\varphi)$ be a flat complex Higgs bundle over M. Then E admits a Yang–Mills–Higgs metric with respect to g if and only if it is polystable. A polystable flat complex Higgs bundle admits a unique Yang–Mills–Higgs connection.*

As in the usual case, there is an analogue of this theorem for flat real Higgs bundles.

Definition 21. Let $(E\,,\varphi)$ be a flat real Higgs bundle over M.

(i) $(E\,,\varphi)$ is called $\mathbb{R}$*–semistable* (respectively, $\mathbb{R}$*–stable*) (with respect to g) if for every proper non–zero flat real subbundle E' of E which is preserved by φ, we have
$$\mu_g(E') \leqslant \mu_g(E) \quad \text{(respectively, } \mu_g(E') < \mu_g(E)) .$$
(ii) $(E\,,\varphi)$ is called $\mathbb{R}$*–polystable* if
$$(E\,,\varphi) = \bigoplus_{i=1}^{N} (E^i\,,\varphi^i)\,,$$
where each $(E^i\,,\varphi^i)$ is an $\mathbb{R}$–stable flat real Higgs bundle with slope $\mu_g(E^i) = \mu_g(E)$.

Corollary 22 ([BLS12a, Corollary 4.3])**.** *Let $(M\,,D\,,\nu)$ be a compact connected special affine manifold equipped with an affine Gauduchon metric g, and let $(E\,,\nabla\,,\varphi)$ be a flat real Higgs bundle over M. Then E admits a Yang–Mills–Higgs metric with respect to g if and only if it is $\mathbb{R}$–polystable. An $\mathbb{R}$–polystable flat real Higgs bundle admits a unique Yang–Mills–Higgs connection.*

5.1. **Flat Higgs principal bundles.** As before, let $(M\,,D\,,\nu)$ be a compact connected special affine manifold equipped with an affine Gauduchon metric g. Let G be a reductive algebraic group, and let $(E_G\,,\nabla)$ be a flat principal G–bundle over M (see Subsection 4.1). Denote by $\mathrm{ad}(E_G)$ the adjoint vector bundle of E_G (see (4.1)), and recall that the flat connection ∇ on E_G induces a flat connection ∇^{ad} on $\mathrm{ad}(E_G)$.

Let
$$\widetilde{\nabla}^{\mathrm{ad}} : T^*M \otimes \mathrm{ad}(E_G) \longrightarrow T^*M \otimes T^*M \otimes \mathrm{ad}(E_G)$$
be the flat connection on $T^*M \otimes \mathrm{ad}(E_G)$ defined by ∇^{ad} and the connection D^* on T^*M.

If φ is a smooth section of $T^*M \otimes \mathrm{ad}(E_G)$, then using the Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$ and the obvious projection $T^*M \otimes T^*M \longrightarrow \bigwedge^2 T^*M$, we get a smooth section of $\left(\bigwedge^2 T^*M\right) \otimes \mathrm{ad}(E_G)$, which will be denoted by $[\varphi\,,\varphi]$.

Definition 23. A *flat Higgs field* on the flat principal G–bundle $(E_G\,,\nabla)$ is a smooth section φ of $T^*M \otimes \mathrm{ad}(E_G)$ such that

(1) the section φ is flat with respect to the connection $\widetilde{\nabla}^{\mathrm{ad}}$ on $T^*M \otimes \mathrm{ad}(E_G)$, and
(2) $[\varphi\,,\varphi] = 0$.

If φ is a flat Higgs field on $(E_G\,,\nabla)$, then $(E_G\,,\nabla\,,\varphi)$ is called a *flat Higgs G–bundle.* (See [Si92] for Higgs G–bundles on complex manifolds.)

Let $(E_G\,,\nabla\,,\varphi)$ be a flat Higgs G–bundle over M. Fix a maximal compact subgroup $K \subset G$. Given a smooth reduction of structure group $E_K \subset E_G$ to K, we have a natural connection d^{E_K} on the principal K–bundle E_K constructed using ∇; the connection on E_G induced by d^{E_K} will also be denoted by d^{E_K}. We may define as before the $(1\,,1)$–part of the extended curvature

$$\overline{\partial}\theta + [\varphi\,,\varphi^*]\,,$$

which is a $(1\,,1)$–form with values in $\mathrm{ad}(E_G)$; as before, θ is a $(1\,,0)$–form with values in $\mathrm{ad}(E_G)$.

The reduction E_K is called a *Yang–Mills–Higgs reduction* of $(E_G\,,\nabla\,,\varphi)$ if there is an element z in the center of $\mathrm{Lie}(G)$ such that the section

$$\mathrm{tr}_g(\overline{\partial}\theta + [\varphi\,,\varphi^*])$$

of $\mathrm{ad}(E_G)$ coincides with the one given by z. If E_K is a Yang–Mills–Higgs reduction, then the connection d^{E_K} on E_G is called a *Yang–Mills–Higgs connection.*

The proofs of Theorems 11 and 20 also give the following:

Corollary 24 ([BLS12a, Corollary 4.5])**.** *Let $(M\,,D\,,\nu)$ be a compact connected special affine manifold equipped with an affine Gauduchon metric g. Let G be either a reductive affine algebraic group over $\mathbb{C}$ or a reductive affine algebraic group over $\mathbb{R}$ of split type. Then a flat Higgs G–bundle $(E_G\,,\nabla\,,\varphi)$ over M admits a Yang–Mills–Higgs connection if and only if it is polystable. Further, the Yang–Mills–Higgs connection on a polystable flat Higgs G–bundle is unique.*

6. The vortex equation

A holomorphic pair on a compact Kähler manifold X is a pair $(E\,,\phi)$ consisting of a holomorphic vector bundle E over X and a holomorphic section ϕ of E which is not identically equal to zero. These objects were introduced by Bradlow in [Bd90] and [Bd91] (see also [GP93], [GP94a]). He defined the notion of τ–stability, where τ is a real number, and established a Donaldson–Uhlenbeck–Yau type correspondence for holomorphic pairs. This correspondence relates τ–stability to the existence of a Hermitian metric solving the τ–vortex equation, which is similar to the Hermitian–Einstein equation but additionally involves the section ϕ. In [GP94b], García–Prada showed that the vortex equation is a dimensional reduction of the Hermitian–Einstein equation for an $\mathrm{SU}(2)$–equivariant holomorphic vector bundle over $X \times \mathbb{P}^1_{\mathbb{C}}$, where $\mathrm{SU}(2)$ acts trivially on X and in the standard way on $\mathbb{P}^1_{\mathbb{C}}$.

Let $(M\,,D\,,\nu)$ be a compact connected special affine manifold equipped with an affine Gauduchon metric g. In this section, we will introduce the vortex equation for a pair $(E\,,\phi)$ consisting of a flat vector bundle E over M and a flat non–zero section ϕ of E, and give a correspondence between polystability of such pairs and the existence of a solution to the vortex equation.

Definition 25. A *flat pair* on M is a pair $((E\,,\nabla)\,,\phi)$ (or $(E\,,\phi)$ if ∇ is understood from the context) consisting of a flat complex vector bundle $(E\,,\nabla)$ over M, and a non–zero flat section ϕ of E.

Definition 26. Let $(E\,,\phi)$ be a flat pair on M, and let τ be a real number.

(i) $(E\,,\phi)$ is called *τ–stable* (with respect to g) if the following two conditions are satisfied:
  (1) $\mu_g(E') < \tau$ for every flat subbundle E' of E with $\operatorname{rank}(E') > 0$.
  (2) $\mu_g(E/E') > \tau$ for every flat subbundle E' of E with $0 < \operatorname{rank}(E') < \operatorname{rank}(E)$ containing the image of the section ϕ.

(ii) $(E\,,\phi)$ is called *τ–polystable* (with respect to g) if either it is τ–stable, or E decomposes as a direct sum of flat subbundles

$$E \;=\; E' \oplus E''$$

such that ϕ is a section of E', the flat pair $(E'\,,\phi)$ is τ–stable, and the flat vector bundle E'' is polystable with slope $\mu_g(E'') = \frac{\tau}{n}$, where $n = \dim M$.

Definition 27. Given a flat pair $(E\,,\phi)$ on M and a real number τ, a smooth Hermitian metric h on E is said to satisfy the *τ–vortex equation* if

$$K_h + \frac{1}{2}\,\phi\circ\phi^* - \frac{\tau}{2}\,\mathrm{Id}_E \;=\; 0\,,$$

where K_h is the extended mean curvature of $(E\,,h)$ (see (3.4)), ϕ is regarded as a homomorphism from the trivial Hermitian line bundle on M to E, and ϕ^* denotes its adjoint with respect to h.

The following correspondence was established in [BLS12b].

Theorem 28 ([BLS12b, Theorem 1])**.** *Let $(M\,,D\,,\nu)$ be an n–dimensional compact connected special affine manifold equipped with an affine Gauduchon metric g with associated $(1\,,1)$–form ω_g, and let $(E\,,\phi)$ be a flat pair on M. Let τ be a real number, and let*

$$\widehat{\tau} \;=\; \frac{\tau}{2}\int_M \frac{\omega_g^n}{\nu}\,.$$

Then E admits a smooth Hermitian metric satisfying the τ–vortex equation if and only if it is $\widehat{\tau}$–polystable.

We give a brief review of the proof of Theorem 28. Let $(M\,,D\,,\nu)$ be a compact connected special affine manifold of dimension n. Denote by $\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}}$ the complex projective line. Consider the product manifold

$$X \;:=\; M\times\mathbb{P}^1\,,$$

which is a smooth real manifold of dimension $n+2$. Let

$$p:\, M\times\mathbb{P}^1 \;\longrightarrow\; M \quad\text{and}\quad q:\, M\times\mathbb{P}^1 \;\longrightarrow\; \mathbb{P}^1$$

be the natural projections.

The complexified tangent bundle $T_{\mathbb{C}}X := T_{\mathbb{R}}X\otimes\mathbb{C}$ of X can be decomposed as

$$T_{\mathbb{C}}X \;=\; p^*T_{\mathbb{C}}M\oplus q^*T_{\mathbb{C}}\mathbb{P}^1 \;=\; p^*T_{\mathbb{C}}M\oplus q^*T^{1,0}\mathbb{P}^1\oplus q^*T^{0,1}\mathbb{P}^1\,.$$

Here $T^{1,0}\mathbb{P}^1$ and $T^{0,1}\mathbb{P}^1$ are respectively the holomorphic and anti–holomorphic tangent bundles of $\mathbb{P}^1$.

In [BLS12b], the theory of Hermitian–Einstein metrics on a flat vector bundle over the affine manifold M was adapted to a smooth complex vector bundle over the product manifold $X = M \times \mathbb{P}^1$ equipped with a flat partial connection in the direction of

$$S^{0,1} := p^* T_{\mathbb{C}} M \oplus q^* T^{0,1}\mathbb{P}^1 \subset T_{\mathbb{C}} X$$

(see [BLS12b, Subsection 2.2] for partial connections); such a bundle is called an $S^{0,1}$–*partially flat vector bundle*.

A smooth complex vector bundle E over X admits a flat partial connection in the direction of $S^{0,1}$ if and only if it admits local trivializations with transition functions which are locally constant in the direction of M and holomorphic in the direction of $\mathbb{P}^1$. Denote by $E^{\mathbb{C}}$ the pullback of E to $M^{\mathbb{C}} \times \mathbb{P}^1$ by the natural projection

$$M^{\mathbb{C}} \times \mathbb{P}^1 = TM \times \mathbb{P}^1 \longrightarrow M \times \mathbb{P}^1 = X\,.$$

The transition functions for $E^{\mathbb{C}}$ are obtained by extending the transition functions of E in a constant way along the fibers of TM. Consequently, $E^{\mathbb{C}}$ is a holomorphic vector bundle if and only if E is an $S^{0,1}$–partially flat vector bundle. Therefore, the map $E \longmapsto E^{\mathbb{C}}$ gives a bijective correspondence between $S^{0,1}$–partially flat vector bundles over X and holomorphic vector bundles over $M^{\mathbb{C}} \times \mathbb{P}^1$ that are constant along the fibers of TM.

Let g_M be an affine Gauduchon metric on M, and let $g_{\mathbb{P}^1}$ be the Fubini–Study metric on $\mathbb{P}^1$ with Kähler form $\omega_{\mathbb{P}^1}$, normalized so that

$$\int_{\mathbb{P}^1} \omega_{\mathbb{P}^1} = 1\,.$$

Define a Riemannian metric g on X by

$$g := p^* g_M \oplus q^* g_{\mathbb{P}^1}\,. \tag{6.1}$$

The following analogue of Theorem 5 was given in [BLS12b].

Theorem 29 ([BLS12b, Theorem 8])**.** *Let $(M\,, D\,, \nu)$ be a compact connected special affine manifold equipped with an affine Gauduchon metric g_M, and consider $X := M \times \mathbb{P}^1$ together with the Riemannian metric g defined in* (6.1)*. Let $(E\,, \nabla)$ be an $S^{0,1}$–partially flat vector bundle over X. Then E admits a Hermitian–Einstein metric with respect to g if and only if it is polystable.*

Using this, Theorem 28 was proved in [BLS12b] by showing that the vortex equation for a flat pair on M is a dimensional reduction of the Hermitian–Einstein equation for an SU(2)–equivariant $S^{0,1}$–partially flat vector bundle over $M \times \mathbb{P}^1$.

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