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Hermitian–Einstein connections on principal bundles over flat affine manifolds

Indranil Biswas

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Bombay 400005, India
E-mail: indranil@math.tifr.res.in

John Loftin

Department of Mathematics and Computer Science
Rutgers University at Newark
Newark, NJ 07102, USA
E-mail: loftin@rutgers.edu

Let M be a compact connected special flat affine manifold without boundary equipped with a Gauduchon metric g and a covariant constant volume form. Let G be either a connected reductive complex linear algebraic group or the real locus of a split real form of a complex reductive group. We prove that a flat principal G–bundle E_G over M admits a Hermitian–Einstein structure if and only if E_G is polystable. A polystable flat principal G–bundle over M admits a unique Hermitian–Einstein connection. We also prove the existence and uniqueness of a Harder–Narasimhan filtration for flat vector bundles over M. We prove a Bogomolov type inequality for semistable vector bundles under the assumption that the Gauduchon metric g is astheno–Kähler.

Keywords: Flat affine manifold; Hermitian–Einstein connection; principal bundle; Harder–Narasimhan filtration.

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1. Introduction

A flat affine manifold M is a C^∞ manifold equipped with a flat torsionfree connection on TM. Equivalently, a flat affine structure on a manifold is provided by an atlas of coordinate charts whose transition functions are all affine maps $x \mapsto Ax + b$. The total space of the tangent bundle TM of a flat affine manifold admits a complex structure, with the transition maps being $z \mapsto Az + b$, for $z = x + \sqrt{-1}y$, with y representing the fiber coordinates. There is a dictionary between holomorphic objects on TM which are invariant in the fiber directions and locally constant objects on M (cf. [9]). This correspondence between affine and complex manifolds has recently become prominent as a part of the mirror conjecture of Strominger–Yau–Zaslow (in this case, each fiber of the tangent bundle $TM \longrightarrow M$ is quotiented by a lattice to form a special Lagrangian torus in a Calabi–Yau manifold). In particular, a flat vector bundle over M naturally extends to a holomorphic vector bundle over TM.

We briefly recall the set–up and the main result of [9]. An affine manifold M is called special if the induced flat connection on the line bundle $\bigwedge^{\rm top} TM$ has trivial holonomy. Let M be a compact connected special flat affine manifold without boundary. Fix a nonzero flat section ν of $\bigwedge^{\rm top} TM$ (under our dictionary, ν corresponds to a holomorphic volume form on the total space of TM). Also fix an affine Gauduchon metric g on M. This allows us to define the degree of a flat vector bundle over M (see Section 2 below); we will consider both real and complex vector bundles. By a flat vector bundle we will always mean a vector bundle equipped with a flat connection (and thus locally constant transition functions). For a flat vector bundle $V \longrightarrow M$, its degree is denoted by $\deg_g(V)$, and its slope $\deg_g(V)/\mathrm{rank}(V)$ is denoted by $\mu_g(V)$.

Once degree is defined, we can define semistable, stable and polystable flat vector bundles over M by imitating the corresponding definitions for holomorphic vector bundles over compact Gauduchon manifolds. Similarly, Hermitian–Einstein metrics on flat vector bundles over M are defined by imitating the definition of Hermitian–Einstein metrics on holomorphic vector bundles over Gauduchon manifolds.

The following theorem is proved in [9], with the main technical part being to use estimates to prove a version of the Donaldson–Uhlenbeck–Yau Theorem on existence of Hermitian–Einstein connections on stable vector bundles.

Theorem 1.1 ([9]). *A flat vector bundle V over M admits a Hermitian–Einstein metric if and only if V is polystable. A polystable flat vector bundle admits a unique Hermitian–Einstein connection.*

Our aim here is to establish a similar result for flat principal bundles over M.

Let G be a connected Lie group such that it is either a complex reductive linear algebraic group or it is the fixed point locus of an anti-holomorphic involution $\sigma_{G_{\mathbb{C}}} : G_{\mathbb{C}} \longrightarrow G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is a complex reductive linear algebraic group; if G is of the second type, then we assume that $\sigma_{G_{\mathbb{C}}}$ is of split type. Extending the definition of a flat polystable vector bundle on M, we define polystable flat principal G–bundles over M. A flat principal GL_r–bundle is polystable if and only if the corresponding vector bundle of rank r is polystable.

Fix a maximal compact subgroup $K \subset G$. A Hermitian structure on a flat principal G–bundle E_G on M is a C^∞ reduction of structure group $E_K \subset E_G$ to the subgroup K. If G is the real points of $G_{\mathbb{C}}$, given a flat principal G–bundle E_G over M, we get a holomorphic principal $G_{\mathbb{C}}$–bundle $E_{G_{\mathbb{C}}}$ over the total space of TM. For any Hermitian structure on E_G, there is a naturally associated connection on the principal $G_{\mathbb{C}}$–bundle $E_{G_{\mathbb{C}}}$.

A Hermitian structure on E_G produces a connection on E_G. Contracting using g the curvature form of the connection, we obtain a section of the adjoint bundle $\mathrm{ad}(E_G)$. A Hermitian structure on E_G is called Hermitian–Einstein if this section of $\mathrm{ad}(E_G)$ is given by some element in the center of $\mathrm{Lie}(G)$. The connection associated to a Hermitian structure satisfying this condition is called a Hermitian–Einstein connection.

We prove the following generalization of Theorem 1.1 (see Theorem 6.1):

Theorem 1.2. *A flat principal G–bundle $E_G \longrightarrow M$ admits a Hermitian–Einstein structure if and only if E_G is polystable. A polystable flat principal G–bundle admits a unique Hermitian–Einstein connection.*

We prove the following analog of the Harder–Narasimhan filtration (see Theorem 2.1):

Theorem 1.3. *For any flat vector bundle $V \longrightarrow M$, there is a unique filtration of flat subbundles*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{\ell-1} \subset F_\ell = V$$

such that for each $i \in [1\,, \ell]$, the flat vector bundle F_i/F_{i-1} is semistable, and

$$\mu_g(F_1) > \mu_g(F_2/F_1) > \cdots > \mu_g(F_\ell/F_{\ell-1})\,.$$

Our proof of Theorem 1.2 crucially uses Theorem 1.3.

One goal of the present work and of [9] is to develop analytic tools to study affine manifolds. In the complex case, Li–Yau–Zheng and also Teleman have used Hermitian–Einstein metrics on vector bundles over non–Kähler surfaces equipped with Gauduchon metrics to partially classify surfaces of Kodaira class VII [8], [12], [13]. There are similar open problems in classifying affine manifolds in low dimension. As suggested by Bill Goldman, one case that may be tractable is that of flat affine symplectic four–manifolds (flat affine four–manifolds admitting a flat nondegenerate closed two–form). Little is known about these manifolds.

In particular, we hope to use the results of this paper to study representations of $\pi_1(M)$ into a reductive Lie group G. It is well known that a flat principal G–bundle over a manifold M is equivalent to a conjugacy class of homomorphisms from $\pi_1(M)$ to G. Our Theorem 1.2 thus can be rephrased as follows: If M is a special affine manifold equipped with a Gauduchon metric g, every representation $\pi_1(M) \longrightarrow G$ admits either a nontrivial destabilizing subrepresentation or a unique Hermitian–Einstein connection.

For a general affine manifold M and representation $\pi_1(M) \longrightarrow G$, we expect there to be few subrepresentations at all, and so the existence of the canonical Hermitian–Einstein connection is to be expected in many cases.

In Section 6.2, we prove the following Bogomolov type inequality as an application of Theorem 1.2 (see Lemma 6.1):

Lemma 1.1. *Assume that the Gauduchon metric g satisfies the condition that $\partial\overline{\partial}(\omega_g^{d-2}) = 0$, where ω is the corresponding $(1\,,1)$-form (it is called an astheno–Kähler metric). Let $V \longrightarrow M$ be a semistable flat vector bundle of rank r. Then*

$$\int_M \frac{c_2(\mathcal{E}nd(V))\omega_g^{d-2}}{\nu} = \int_M \frac{(2r\cdot c_2(V) - (r-1)c_1(V)^2)\omega_g^{d-2}}{\nu} \geq 0\,.$$

A few notes about the proof are in order. As in [9], we are able to follow the existing proofs in the complex case closely, with a few important simplifications. In [9], the main simplification is that we need only consider flat subbundles (as opposed to singular subsheaves) as destabilizing objects in non–stable vector bundles. In the current work, we find another such simplification in the proof of Lemma 2.2. Flatness implies that the space of all flat subbundles of rank k of a given flat vector bundle V is a closed subset of the Grassmannian of the fiber of V at a given point in M, and thus a simple compactness argument guarantees the existence of a flat subbundle with maximal slope. The corresponding argument in the complex Gauduchon case is more complicated [5].

2. Harder–Narasimhan filtration of a vector bundle

We recall from [9] some basic definitions. Consider a flat affine manifold M of dimension n as the zero section of its tangent bundle TM. The unifying idea behind all these definitions is to consider flat objects on M to be restrictions of holomorphic objects on TM. We may define operators ∂ and $\bar\partial$ on the affine Dolbeault complex, where (p, q) forms are represented as sections of $\Lambda^p(T^*M) \otimes \Lambda^q(T^*M)$. A Riemannian metric g on M can be extended to a natural Hermitian metric on the total space of TM. Let ω_g be the associated $(1, 1)$ form. The metric g is called *affine Gauduchon* if $\partial\bar\partial(\omega_g^{n-1}) = 0$. Given a flat vector bundle $V \longrightarrow M$ equipped with a Hermitian bundle metric h, we may define its first Chern form $c_1(h)$. Assume there is a covariant constant volume form ν on M. Define the degree of the vector bundle as

$$\deg_g(V) := \int_M \frac{c_1(h) \wedge \omega_g^{n-1}}{\nu} .$$

Then the *slope* of V is defined to be

$$\mu_g(V) := \deg_g(V)/\mathrm{rank}(V) .$$

The vector bundle V is said to be *stable* if every flat subbundle W of V with $0 < \mathrm{rank}(W) < \mathrm{rank}(V)$ satisfies the inequality $\mu_g(W) < \mu_g(V)$. The vector bundle V is called *semistable* if $\mu_g(W) \le \mu_g(V)$ for all such W, and V is said to be *polystable* if it is a direct sum of stable flat bundles of the same slope.

Let

$$V \longrightarrow M$$

be a flat vector bundle; it is allowed to be real or complex. Fix a filtration of flat subbundles

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V \tag{2.1}$$

such that all the successive quotients V_i/V_{i-1}, $1 \le i \le n$, are stable.

Define

$$\delta := \mathrm{Max}\{\mu_g(V_i/V_{i-1})\}_{i=1}^n \in \mathbb{R} . \tag{2.2}$$

Lemma 2.1. *Let $F \subset V$ be any flat subbundle of V of positive rank. Then $\mu_g(F) \leq \delta$.*

Proof. If F is not semistable, then there is a flat subbundle $F' \subset F$ such that

$$0 < \operatorname{rank}(F') < \operatorname{rank}(F) \quad \text{and} \quad \mu_g(F') > \mu_g(F)\,.$$

Furthermore, if F is semistable, then there is a flat subbundle $F'' \subset F$ such that $\operatorname{rank}(F'')$ is smallest among the ranks of all flat subbundles W of F with $\mu_g(W) = \mu_g(F)$. Note that such a smallest rank flat vector bundle F'' is automatically stable. Therefore, it is enough to check the inequality in the lemma under the assumption that F is stable.

Assume that F is stable, and $\mu_g(F) > \delta$.

Let F_1 and F_2 be semistable flat vector bundles over M such that either both of them are real or both are complex. Let

$$\varphi : F_1 \longrightarrow F_2$$

be a nonzero flat homomorphism of vector bundles. Then

$$\mu_g(F_1) \leq \mu_g(\text{Image}(\varphi)) \leq \mu_g(F_2)$$

because F_1 and F_2 are semistable. Therefore, there is no nonzero flat homomorphism of vector bundles from F_1 to F_2 if $\mu_g(F_1) > \mu_g(F_2)$.

From the above observation we conclude that for each $i \in [1\,, n]$, there is no nonzero flat homomorphism of vector bundles from F to V_i/V_{i-1}. This immediately implies that there is no nonzero flat homomorphism of vector bundles from F to V. This contradicts the fact that F is a flat subbundle of V. Therefore, we conclude that $\mu_g(F) \leq \delta$. □

Define

$$\delta_0(V) := \text{Sup}\,\{\mu_g(F) \mid F \text{ is a flat subbundle of } V\} \tag{2.3}$$

which is a finite number due to Lemma 2.1.

Lemma 2.2. *There is a flat subbundle $F \subset V$ such that $\mu_g(F) = \delta_0(V)$, where $\delta_0(V)$ is defined in (2.3).*

Proof. Fix an integer $k \in [1\,, \operatorname{rank}(V)]$ such that

$$\delta_0(V) = \text{Sup}\,\{\mu_g(F) \mid F \subset V \text{ is a flat subbundle of rank } k\}\,;$$

such a k clearly exists because $\operatorname{rank}(V)$ is a finite integer. Fix a point

$$x_0 \in M\,.$$

Let $\text{Gr}(V_{x_0}, k)$ be the Grassmannian parametrizing all linear subspaces of dimension k of the fiber V_{x_0}.

For any flat subbundle $F \subset V$ of rank k, consider the subspace $F_{x_0} \in \mathrm{Gr}(V_{x_0}, k)$. We note that the flat subbundle F is uniquely determined by the point $F_{x_0} \in \mathrm{Gr}(V_{x_0}, k)$, because M is connected. Let

$$S \subset \mathrm{Gr}(V_{x_0}, k) \tag{2.4}$$

be the locus of all subspaces that are fibers of flat subbundles of V of rank k. We will now describe S explicitly.

Let

$$\rho : \pi_1(M, x_0) \longrightarrow \mathrm{GL}(V_{x_0})$$

be the monodromy representation for the flat connection on V. The group $\mathrm{GL}(V_{x_0})$ acts on $\mathrm{Gr}(V_{x_0}, k)$ in a natural way. The subset S in (2.4) is the fixed point locus

$$S = \mathrm{Gr}(V_{x_0}, k)^{\rho(\pi_1(M,x_0))}$$

for the action of $\rho(\pi_1(M, x_0))$ on $\mathrm{Gr}(V_{x_0}, k)$. Note that

$$S = \mathrm{Gr}(V_{x_0}, k)^{\overline{\rho(\pi_1(M,x_0))}}.$$

This implies that S is a closed subset of $\mathrm{Gr}(V_{x_0}, k)$. In particular, S is compact.

For each point $z \in S$, let $F^z \subset V$ be the unique flat subbundle of V such that $(F^z)_{x_0} = z$. We have a continuous function

$$f_V^k : S \longrightarrow \mathbb{R}$$

defined by $z \longmapsto \mu_g(F^z)$. Since S is compact, there is a point $z_0 \in S$ at which the function f_V^k takes the maximum value $\delta_0(V)$. The corresponding flat subbundle $F^{z_0} = F$ satisfies the condition in the lemma. □

Proposition 2.1. *There is a unique maximal flat subbundle $F \subset V$ with $\mu_g(F) = \delta_0(V)$, where $\delta_0(V)$ is defined in* (2.3).

Proof. Let F_1 and F_2 be two flat subbundles of V such that

$$\mu_g(F_1) = \mu_g(F_2) = \delta_0(V). \tag{2.5}$$

Note that F_1 and F_2 are automatically semistable. Let

$$F_1 + F_2 \subset V$$

be the flat subbundle generated by F_1 and F_2. We have a short exact sequence of flat vector bundles

$$0 \longrightarrow F_1 \cap F_2 \longrightarrow F_1 \oplus F_2 \longrightarrow F_1 + F_2 \longrightarrow 0. \tag{2.6}$$

Since F_1 and F_2 are both semistable, from (2.5) it follows immediately that $F_1 \oplus F_2$ is also semistable with

$$\mu_g(F_1 \oplus F_2) = \delta_0(V). \tag{2.7}$$

We will show that $F_1 + F_2$ is semistable with $\mu_g(F_1 + F_2) = \delta_0(V)$.

To prove this, first note that if $F_1 \cap F_2 = 0$, then $F_1 \oplus F_2 = F_1 + F_2$, hence it is equivalent to the above observation. So assume that $\text{rank}(F_1 \cap F_2) > 0$.

From (2.6),

$$\deg_g(F_1 \oplus F_2) = \deg_g(F_1 \cap F_2) + \deg_g(F_1 + F_2)\,.$$

Hence

$$\mu_g(F_1 \oplus F_2) = \frac{\mu_g(F_1 \cap F_2) \cdot \text{rank}(F_1 \cap F_2) + \mu_g(F_1 + F_2) \cdot \text{rank}(F_1 + F_2)}{\text{rank}(F_1 \cap F_2) + \text{rank}(F_1 + F_2)}\,. \tag{2.8}$$

Since $F_1 \cap F_2$ and $F_1 + F_2$ are flat subbundles of V, we have

$$\mu_g(F_1 \cap F_2)\,, \mu_g(F_1 + F_2) \le \delta_0(V)\,. \tag{2.9}$$

Using (2.7) and (2.8) and (2.9) we conclude that

$$\mu_g(F_1 + F_2) = \mu_g(F_1 \cap F_2) = \delta_0(V)\,.$$

Therefore, we have proved that $F_1 + F_2$ is semistable with $\mu_g(F_1 + F_2) = \delta_0(V)$.

Consider the flat subbundle $F \subset V$ generated by all flat subbundles W with $\mu_g(W) = \delta_0(V)$. Since $F_1 + F_2$ is semistable with $\mu_g(F_1 + F_2) = \delta_0(V)$ whenever F_1 and F_2 are semistable with slope $\delta_0(V)$, if follows immediately that the flat subbundle F satisfies the condition in the proposition. □

The unique maximal flat semistable subbundle $F \subset V$ with $\mu_g(F) = \delta_0(V)$ in Proposition 2.1 will be called the *maximal semistable subbundle* of V.

Proposition 2.1 has the following corollary:

Corollary 2.1. *There is a unique filtration of flat subbundles*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{\ell-1} \subset F_\ell = V$$

such that for each $i \in [1\,, \ell]$, *the flat subbundle* $F_i/F_{i-1} \subset V/F_{i-1}$ *is the unique maximal semistable subbundle.*

The filtration in Corollary 2.1 can be reformulated as follows:

Theorem 2.1. *Let* V *be a flat vector bundle over* M. *Then there is a unique filtration of* V *by flat subbundles*

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_j \subsetneq F_{j+1} \subsetneq \cdots \subsetneq F_{\ell-1} \subsetneq F_\ell = V$$

such that for each $i \in [1\,, \ell]$, *the flat vector bundle* F_i/F_{i-1} *is semistable, and*

$$\mu_g(F_1) > \mu_g(F_2/F_1) > \cdots > \mu_g(F_{j+1}/F_j) > \cdots > \mu_g(F_\ell/F_{\ell-1})\,.$$

Proof. The filtration in Corollary 2.1 clearly has the property that for each $i \in [1\,, \ell]$, the flat subbundle F_i/F_{i-1} is semistable, and

$$\mu_g(F_1) > \mu_g(F_2/F_1) > \cdots > \mu_g(F_\ell/F_{\ell-1})\,.$$

Now, let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = V \tag{2.10}$$

be another filtration of flat subbundles of V such that for each $i \in [1\,,n]$, the flat subbundle E_i/E_{i-1} is semistable, and

$$\mu_g(E_1) > \mu_g(E_2/E_1) > \cdots > \mu_g(E_n/E_{n-1})\,.$$

To show that the filtration in Corollary 2.1 coincides with the filtration in (2.10), it suffices to prove that $E_1 = F_1$, because we may replace V by V/F_i and use induction on i.

If $n = 1$, then V is semistable. Hence $F_1 = V$, and the theorem is evident.

Hence assume that $n \geq 2$.

We have

$$\mu_g(E_n/E_{n-1}) < \mu_g(E_1) \leq \mu_g(F_1)$$

because F_1 is the maximal semistable subbundle of V. Therefore, case there is no nonzero flat homomorphism from F_1 to E_n/E_{n-1} (see the proof of Lemma 2.1). Now, by induction, there is no nonzero flat homomorphism from F_1 to E_i/E_{i-1} for all $i \geq 2$. Hence there is no nonzero flat homomorphism from F_1 to V/E_1. Consequently, F_1 is a subbundle of E_1.

We have $\mu_g(E_1) \geq \mu_g(F_1)$ because E_1 is semistable and F_1 is a subbundle of E_1. On the other hand, we have $\mu_g(F_1) \geq \mu_g(E_1)$ because F_1 is the maximal semistable subbundle of V. Therefore, $\mu_g(E_1) = \mu_g(F_1)$. Again from the fact that F_1 is the maximal semistable subbundle of V we conclude that the subbundle $F_1 \subset E_1$ must coincide with E_1. □

3. Semistability of tensor product

A flat vector bundle $(V\,,D)$ over M will be called *polystable* if

$$(V\,,D) = (\bigoplus_{i=1}^m W_i\,,\bigoplus_{i=1}^m D_i)\,,$$

where $(W_i\,,D_i)$, $1 \leq i \leq m$, are flat stable vector bundles, and

$$\mu_g(W_1) = \cdots = \mu_g(W_m)\,.$$

Let V_1 and V_2 be flat vector bundles over M such that either both are real or both are complex.

Lemma 3.1. *If V_1 and V_2 are stable, then the flat vector bundle $V_1 \otimes V_2$ is polystable.*

Proof. Assume that V_1 and V_2 are stable. Then each one them admits an affine Hermitian–Einstein metric (see Theorem 1 in page 102 of [9] for the complex case and Corollary 33 in page 129 of [9] for the real case). Let h_1 and h_2 be affine Hermitian–Einstein metrics on V_1 and V_2 respectively. The Hermitian metric on

$V_1 \otimes V_2$ induced by h_1 and h_2 is clearly an affine Hermitian–Einstein one. Therefore, $V_1 \otimes V_2$ is polystable (Theorem 4 in page 110 of [9]). □

Corollary 3.1. *If V_1 and V_2 are polystable, then the flat vector bundle $V_1 \otimes V_2$ is also polystable.*

Proof. This follows from Lemma 3.1 after writing the flat polystable vector bundles V_1 and V_2 as direct sums of flat stable vector bundles. □

Proposition 3.1. *If V_1 and V_2 are semistable, then the flat vector bundle $V_1 \otimes V_2$ is semistable.*

Proof. First assume that V_2 is stable. Let

$$0 = W_0 \subset W_1 \subset \cdots \subset W_{n-1} \subset W_n = V_1 \tag{3.1}$$

be a filtration of flat subbundles such that each successive quotient W_i/W_{i-1}, $1 \leq i \leq n$, is stable with $\mu_g(W_i/W_{i-1}) = \mu_g(V_1)$. Consider the filtration of flat subbundles

$$0 = W_0 \otimes V_2 \subset W_1 \otimes V_2 \subset \cdots \subset W_{n-1} \otimes V_2 \subset W_n \otimes V_2 = V_1 \otimes V_2 \tag{3.2}$$

obtained by tensoring the filtration in (3.1) by V_2. Each successive quotient in this filtration is polystable by Lemma 3.1; also,

$$\mu_g((W_i/W_{i-1}) \otimes V_2) = \mu_g(W_i/W_{i-1}) + \mu_g(V_2) = \mu_g(V_1) + \mu_g(V_2)\,.$$

In view of these properties of the successive quotients for the filtration in (3.2) we conclude that $V_1 \otimes V_2$ is semistable.

If V_2 is not stable, then fix a filtration

$$0 = W'_0 \subset W'_1 \subset \cdots \subset W'_{m-1} \subset W'_m = V_2$$

such that each successive quotient W'_i/W'_{i-1}, $1 \leq i \leq m$, is stable, and $\mu_g(W'_i/W'_{i-1}) = \mu_g(V_2)$. Consider the filtration of $V_1 \otimes V_2$

$$0 = V_1 \otimes W'_0 \subset V_1 \otimes W'_1 \subset \cdots \subset V_1 \otimes W'_{m-1} \subset V_1 \otimes W'_m = V_1 \otimes V_2 \tag{3.3}$$

obtained by tensoring the above filtration with V_1. For each $1 \leq i \leq m$, the quotient

$$(V_1 \otimes W'_i)/(V_1 \otimes W'_{i-1}) = V_1 \otimes (W'_i/W'_{i-1})$$

in (3.3) is semistable by the earlier observation, and furthermore,

$$\mu_g(V_1 \otimes (W'_i/W'_{i-1})) = \mu_g(V_1) + \mu_g(W'_i/W'_{i-1}) = \mu_g(V_1) + \mu_g(V_2)\,.$$

Hence $V_1 \otimes V_2$ is semistable. □

Corollary 3.2. *Let V be a flat vector bundle over M. Take any integer $j \in [1\,, \mathrm{rank}(V)]$. If V is polystable, then the exterior power $\bigwedge^j V$ equipped with the* *induced flat connection is polystable. If V is semistable, then $\bigwedge^j V$ equipped with the induced flat connection is semistable.*

Proof. If V is polystable, then from Corollary 3.1 it follows that $V^{\otimes j}$ equipped with the induced flat connection is polystable. Since the flat vector bundle $\bigwedge^j V$ is a direct summand of the flat vector bundle $V^{\otimes j}$, we conclude that $\bigwedge^j V$ is polystable if $V^{\otimes j}$ is so.

If V is semistable, then from Proposition 3.1 it follows that $V^{\otimes j}$ equipped with the induced flat connection is semistable. Therefore, the direct summand $\bigwedge^j V \subset V^{\otimes j}$ is semistable. □

4. Principal bundles on flat affine manifolds

4.1. *Preliminaries*

Let H be a Lie group. A principal H–bundle on M is a triple of the form $(E_H\, , p\, , \psi)$, where E_H is a C^∞ manifold, $p\, :\, E_H\, \longrightarrow\, M$ is a C^∞ surjective submersion, and

$$\psi\, :\, E_H \times H\, \longrightarrow\, E_H$$

is a smooth action of H on E_H, such that

(1) $p\circ\psi\, =\, p\circ p_1$, where p_1 is the natural projection of $E_H\times H$ to E_H, and
(2) for each point $x\,\in\, M$, there is an open neighborhood $U\,\subset\, M$ of x, and a smooth diffeomorphism

$$\phi\, :\, p^{-1}(U)\, \longrightarrow\, U\times H\, ,$$

such that ϕ commutes with the actions of H (the group H acts on $U\times H$ through right translations of H), and $q_1\circ\phi\, =\, p$, where q_1 is the natural projection of $U\times H$ to U.

Let $dp\, :\, TE_H\, \longrightarrow\, p^*TM$ be the differential of the projection p. A *flat connection* on E_H is a C^∞ homomorphism

$$D\, :\, p^*TM\, \longrightarrow\, TE_H$$

such that

- $dp\circ D\, =\, \mathrm{Id}_{p^*TM}$,
- the distribution $D(p^*TM)\,\subset\, TE_H$ is integrable, and
- $D(p^*TM)$ is invariant under the action of H on TE_H given by the action of H on E_H.

Let $H'\,\subset\, H$ be a closed subgroup. A *reduction* of structure group of a principal H–bundle E_H to H' is a principal H'–bundle $E_{H'}\,\subset\, E_H$; the action of H' on $E_{H'}$ is the restriction of the action of H on E_H. A reduction of structure group of E_H to H' is given by a smooth section of the fiber bundle $E_H/H'\, \longrightarrow\, M$. We note that if a reduction $E_{H'} \subset E_H$ corresponds to a section σ, then E'_H is the inverse image of $\sigma(M)$ for the quotient map $E_H \longrightarrow E_H/H'$.

Let D be a flat connection on E_H. A reduction of structure group $E_{H'} \subset E_H$ to H' is said to be *compatible* with D if for each point $z \in E_{H'}$, the subspace $D(T_{p(z)}M) \subset T_z E_H$ is contained in the subspace $T_z E_{H'} \subset T_z E_H$. Note that this condition ensures that D produces a flat connection on $E_{H'}$.

Consider the adjoint action of H on its Lie algebra $\mathrm{Lie}(H)$. Let

$$\mathrm{ad}(E_H) := E_H \times^H \mathrm{Lie}(H) \longrightarrow M \tag{4.1}$$

be the vector bundle over M associated to the principal H–bundle E_H for this action; it is known as the *adjoint vector bundle* for E_H. Since the adjoint action of H on $\mathrm{Lie}(H)$ preserves the Lie algebra structure, the fibers of $\mathrm{ad}(E_H)$ are Lie algebras isomorphic to $\mathrm{Lie}(H)$. The connection D on E_H induces a connection on every fiber bundle associated to E_H. In particular, D induces a connection on the vector bundle $\mathrm{ad}(E_H)$; this induced connection on $\mathrm{ad}(E_H)$ will be denoted by D^{ad}. The connection D^{ad} is compatible with the Lie algebra structure of the fibers of $\mathrm{ad}(E_H)$, meaning

$$D^{\mathrm{ad}}([s\,,t]) = [D^{\mathrm{ad}}(s)\,,t] + [s\,,D^{\mathrm{ad}}(t)]$$

for all locally defined smooth sections s and t of $\mathrm{ad}(E_H)$.

4.2. *Stable and semistable principal bundles*

Let $G_{\mathbb{C}}$ be a complex reductive linear algebraic group. A real form on $G_{\mathbb{C}}$ is an anti-holomorphic involution

$$\sigma_{G_{\mathbb{C}}} : G_{\mathbb{C}} \longrightarrow G_{\mathbb{C}}\,.$$

The real form $\sigma_{G_{\mathbb{C}}}$ is said to be of *split type* if there is a maximal torus $T \subset G_{\mathbb{C}}$ such that $\sigma_{G_{\mathbb{C}}}(T) = T$ and the fixed point locus of the involution $\sigma_{G_{\mathbb{C}}}|_T$ of T is a product of copies of $\mathbb{R}^*$ (the group of nonzero real numbers).

Let G be a connected Lie group such that either it is a complex reductive linear algebraic group or it is the fixed point locus of a split real form $\sigma_{G_{\mathbb{C}}} \in \mathrm{Aut}(G_{\mathbb{C}})$, where $G_{\mathbb{C}}$ and $\sigma_{G_{\mathbb{C}}}$ are as above.

If G is a complex reductive group, a connected closed algebraic subgroup $P \subset G$ is called a *parabolic* subgroup if the quotient variety G/P is complete. So, in particular, G itself is a parabolic subgroup. Let P be a parabolic subgroup of G. A character χ of P is called *strictly anti–dominant* if the following two conditions hold:

- the line bundle over G/P associated to the principal P–bundle $G \longrightarrow G/P$ for χ is ample, and
- the character χ is trivial on the connected component of the center of P containing the identity element.

Let $R_u(P) \subset P$ be the unipotent radical. The group $P/R_u(P)$ is called the *Levi quotient* of P. A *Levi subgroup* of P is a connected reductive subgroup $L(P) \subset P$ such that the composition

$$L(P) \longrightarrow P \longrightarrow P/R_u(P)$$

is an isomorphism. A Levi subgroup always exists (see § 11.22 in page 158 of [4] and § 30.2 in page 184 of [6]).

If G is the fixed point locus of a real form $(G_{\mathbb C}\,,\sigma_{G_{\mathbb C}})$, by a parabolic subgroup of G we will mean a subgroup $P\,\subset\, G$ such that there is a parabolic subgroup $P'\,\subset\, G_{\mathbb C}$ satisfying the conditions that $\sigma_{G_{\mathbb C}}(P')\,=\,P'$ and $P'\bigcap G\,=\,P$. By a Levi subgroup of the parabolic subgroup P we will mean a subgroup $L(P)\,\subset\, P$ such that there is a Levi subgroup $L(P')\,\subset\, P'$ satisfying the conditions that $\sigma_{G_{\mathbb C}}(L(P'))\,=\,L(P')$ and $L(P')\bigcap G\,=\,L(P)$.

Let $(E_G\,,D)$ be a flat principal G–bundle over M.

It is called *semistable* (respectively, *stable*) if for every triple of the form $(Q\,,E_Q\,,\lambda)$, where $Q\,\subset\, G$ is a proper parabolic subgroup, and $E_Q\,\subset\, E_G$ is a reduction of structure group of E_G to Q compatible with D, and λ is a strictly anti–dominant character of Q, the inequality

$$\deg_g(E_Q(\lambda))\,\geq\, 0 \tag{4.2}$$

(respectively, $\deg_g(E_Q(\lambda))\,>\,0$) holds, where $E_Q(\lambda)$ is the flat line bundle over M associated to the flat principal Q–bundle E_Q for the character λ of Q.

In order to decide whether $(E_G\,,D)$ is semistable (respectively, stable), it suffices to verify the above inequality (respectively, strict inequality) only for those Q which are proper maximal parabolic subgroups of G. More precisely, E_G is semistable (respectively, stable) if and only if for every pair $(Q\,,\sigma)$, where $Q\,\subset\, G$ is a proper maximal parabolic subgroup, and $\sigma\,:\,M\,\longrightarrow\, E_G/Q$ is a reduction of structure group of E_G to Q compatible with D, the inequality

$$\deg_g(\sigma^* T_{\rm rel})\,\geq\, 0 \tag{4.3}$$

(respectively, $\deg_g(\sigma^* T_{\rm rel})\,>\,0$) holds, where $T_{\rm rel}$ is the relative tangent bundle over E_G/Q for the projection $E_G/Q\,\longrightarrow\, M$. (See Definition 1.1 in page 129 of [11] and Lemma 2.1 in page 131 of [11].) It should be mentioned that the connection D on E_G induces a flat connection on the associated fiber bundle $E_G/Q\,\longrightarrow\, M$. Since the section σ is flat with respect to this induced connection (it is flat because the reduction E_Q is compatible with D), the pullback $\sigma^* T_{\rm rel}$ gets a flat connection.

Let $(E_G\,,D)$ be a flat principal G–bundle over M. A reduction of structure group

$$E_Q\,\subset\, E_G$$

to a parabolic subgroup $Q\,\subset\, G$ compatible with D is called *admissible* if for each character λ of Q trivial on the center of G, the associated flat line bundle $E_Q(\lambda)\,\longrightarrow\, M$ satisfies the following condition:

$$\deg_g(E_Q(\lambda))\,=\,0\,. \tag{4.4}$$

We will call $(E_G\,, D)$ to be *polystable* if either E_G is stable, or there is a proper parabolic subgroup Q and a reduction of structure group $E_{L(Q)} \subset E_H$ to a Levi subgroup $L(Q)$ of Q compatible with D such that the flat principal $L(Q)$–bundle $E_{L(Q)}$ is stable, and the reduction of structure group of E_G to Q, obtained by extending the structure group of $E_{L(Q)}$ using the inclusion of $L(Q)$ in Q, is admissible.

We note that a flat polystable principal G–bundle on M is semistable.

For notational convenience, we will omit the symbol of connection for a flat principal bundle. When we will say "E_G be a flat principal G–bundle" it will mean that E_G is equipped with a flat connection.

4.3. *Harder–Narasimhan reduction of principal bundles*

Let G be as before. Let E_G be a flat principal G–bundle over M.

A *Harder–Narasimhan reduction* of E_G is a pair of the form $(P\,, E_P)$, where $P \subset G$ is a parabolic subgroup, and $E_P \subset E_G$ is a reduction of structure group of E_G to P compatible with the connection such that the following two conditions hold:

(1) The principal $P/R_u(P)$–bundle $E_P/R_u(P)$ equipped with the induced flat connection is semistable, where $R_u(P) \subset P$ is the unipotent radical.
(2) For any nontrivial character χ of P which can be expressed as a nonnegative integral combination of simple roots, the flat line bundle over M associated to E_P for χ is of positive degree.

Proposition 4.1. *A flat principal G–bundle E_G admits a Harder–Narasimhan reduction. If $(P\,, E_P)$ and $(P\,, E_P)$ are two Harder–Narasimhan reductions of E_G, then there is an element $g \in G$ such that $Q = g^{-1}Pg$ and $E_Q = E_Pg$.*

Proof. Let $\mathrm{ad}(E_G) \longrightarrow M$ be the adjoint vector bundle of E_G (defined in (4.1)). As mentioned earlier, the flat connection on E_G induces a flat connection on $\mathrm{ad}(E_G)$. Let

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell = \mathrm{ad}(E_G) \tag{4.5}$$

be the Harder–Narasimhan filtration of the flat vector bundle $\mathrm{ad}(E_G)$ constructed in Theorem 2.1. Using Proposition 3.1 it can be deduced that ℓ in (4.5) is an odd integer; its proof is identical to the proof of (3) in page 215 of [1]. The flat subbundle

$$V_{(\ell+1)/2} \subset \mathrm{ad}(E_G) \tag{4.6}$$

in (4.5) is the adjoint vector bundle of a reduction of structure group of E_G to a parabolic subgroup; its proof is identical to the proof of Lemma 4 in page 699 of [2]. After we fix a parabolic subgroup P of G in the conjugacy class of parabolic subgroups of G defined by the subalgebra $(E_{(\ell+1)/2})_x \subset \mathrm{ad}(E_G)_x$, where $x \in M$, we get a reduction of structure group of E_G to P compatible with the connection. This reduction satisfies all the conditions in the proposition. The details of the argument are in [2]. □

The second condition in the Harder–Narasimhan reduction can be reformulated in other equivalent ways; see [2].

Remark 4.1. Proposition 4.1 can also be proved by imitating the proof of Proposition 3.1 in [3].

Proposition 4.1 has the following corollary:

Corollary 4.1. *A flat principal G–bundle E_G over M is semistable if and only if the flat vector bundle $\mathrm{ad}(E_G)$ is semistable.*

Proof. Assume that $\mathrm{ad}(E_G)$ is not semistable. Then $E_{(\ell+1)/2}$ in (4.5) is a proper subbundle of $\mathrm{ad}(E_G)$. Hence E_G has a nontrivial Harder–Narasimhan reduction $(P\,, E_P)$. Let $\mathfrak{g}$ and $\mathfrak{p}$ be the Lie algebras of G and P respectively. The group P has the adjoint action on $\mathfrak{g}/\mathfrak{p}$. The vector bundle over M associated to the principal P–bundle E_P for the P–module $\mathfrak{g}/\mathfrak{p}$ is identified with the vector bundle $\mathrm{ad}(E_G)/E_{(\ell+1)/2}$. Consequently, the reduction $E_P \subset E_G$ and the strictly anti–dominant character of P defined by the P–module $\bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{p})$ violate the inequality in (4.2). Hence the flat principal G–bundle E_G is not semistable.

To prove the converse, assume that the flat vector bundle $\mathrm{ad}(E_G)$ is semistable. Then $E_{(\ell+1)/2} = \mathrm{ad}(E_G)$ (see (4.6)). Hence the Harder–Narasimhan reduction of E_G is $(G\,, E_G)$ itself. Since the Levi quotient of G is G itself, from the first condition in the definition of a Harder–Narasimhan reduction we conclude that E_G is semistable. □

Corollary 4.2. *Assume that G is the fixed point locus of a split real form on $G_{\mathbb{C}}$. Let E_G be a flat principal G–bundle over M. Let $E_{G_{\mathbb{C}}}$ be the flat principal $G_{\mathbb{C}}$–bundle over M obtained by extending the structure group of E_G using the inclusion of G in $G_{\mathbb{C}}$. The principal G–bundle E_G is semistable if and only if the principal $G_{\mathbb{C}}$–bundle $E_{G_{\mathbb{C}}}$ is so.*

Proof. Let V be a flat real vector bundle over M. Let $V_{\mathbb{C}} := V \bigotimes_{\mathbb{R}} \mathbb{C}$ be the flat complex vector bundle. We will show that V is semistable if and only if $V_{\mathbb{C}}$ is so.

If a flat subbundle $W \subset V$ violates the semistability condition for V, then the flat subbundle $W \bigotimes_{\mathbb{R}} \mathbb{C} \subset V_{\mathbb{C}}$ violates the semistability condition for $V_{\mathbb{C}}$. Therefore, V is semistable if $V_{\mathbb{C}}$ is so.

To prove the converse, assume that $V_{\mathbb{C}}$ is not semistable. Let $F \subset V_{\mathbb{C}}$ be the maximal semistable subbundle of $V_{\mathbb{C}}$, which is a proper subbundle because $V_{\mathbb{C}}$ is not semistable. From the uniqueness of F it follows immediately that the $\mathbb{R}$–linear conjugation automorphism of $V_{\mathbb{C}} = V \bigotimes_{\mathbb{R}} \mathbb{C}$ defined by $v \otimes \lambda \longmapsto v \otimes \overline{\lambda}$, where $v \in V$ and $\lambda \in \mathbb{C}$, preserves the subbundle F. Hence F is the complexification of a flat subbundle F' of V. This subbundle F' violates the semistability condition for V. Therefore, $V_{\mathbb{C}}$ is semistable if and only if V is so.

We apply the above observation to $V = \mathrm{ad}(E_G)$. Note that

$$\mathrm{ad}(E_{G_{\mathbb{C}}}) = \mathrm{ad}(E_G) \otimes_{\mathbb{R}} \mathbb{C}\,. \tag{4.7}$$

In view of Corollary 4.1, the proof is complete. □

5. The socle reduction

Let $V \longrightarrow M$ be a flat semistable vector bundle; it is allowed to be real or complex. Let F_1 and F_2 be two flat subbundles of V such that both F_1 and F_2 are polystable, and

$$\mu_g(F_1) = \mu_g(F_2) = \mu_g(V). \tag{5.1}$$

Let

$$F_1 + F_2 \subset V$$

be the flat subbundle of V generated by F_1 and F_2.

Proposition 5.1. *The flat vector bundle $F_1 + F_2$ is polystable, and $\mu_g(F_1 + F_2) = \mu_g(V)$.*

Proof. From (5.1),

$$\mu_g(F_1 \oplus F_2) = \mu_g(V). \tag{5.2}$$

Consider the short exact sequence of flat vector bundles

$$0 \longrightarrow F_1 \cap F_2 \longrightarrow F_1 \oplus F_2 \longrightarrow F_1 + F_2 \longrightarrow 0. \tag{5.3}$$

If $F_1 \cap F_2 = 0$, then $F_1 + F_2 = F_1 \oplus F_2$, hence in this case $F_1 + F_2$ is polystable, and $\mu_g(F_1 + F_2) = \mu_g(V)$ from (5.2). Therefore, the proposition is evident if $F_1 \cap F_2 = 0$.

So assume that $\text{rank}(F_1 \cap F_2) > 0$.

Since V is semistable, and both $F_1 \cap F_2$ and $F_1 + F_2$ are flat subbundles of V, we have

$$\mu_g(F_1 \cap F_2), \mu_g(F_1 + F_2) \leq \mu_g(V). \tag{5.4}$$

From (5.3),

$$\mu_g(F_1 \oplus F_2) = \frac{\mu_g(F_1 \cap F_2) \cdot \text{rank}(F_1 \cap F_2) + \mu_g(F_1 + F_2) \cdot \text{rank}(F_1 + F_2)}{\text{rank}(F_1 \cap F_2) + \text{rank}(F_1 + F_2)}.$$

Combining this with (5.2) and (5.4),

$$\mu_g(F_1 + F_2) = \mu_g(F_1 \cap F_2) = \mu_g(V). \tag{5.5}$$

Let r be the rank of the flat vector bundle $F_1 \cap F_2$; recall that it is positive. Consider the vector bundle

$$\mathcal{W} := \mathcal{H}om(\bigwedge\nolimits^r (F_1 \cap F_2), \bigwedge\nolimits^r F_1) = \bigwedge\nolimits^r (F_1 \cap F_2)^* \otimes \bigwedge\nolimits^r F_1. \tag{5.6}$$

Note that the inclusion homomorphism $F_1 \cap F_2 \hookrightarrow F_1$ defines a nonzero flat section

$$\eta \in H^0(M, \mathcal{W}). \tag{5.7}$$

We have

$$\mu_g(\mathcal{W}) \,=\, \mu_g(\bigwedge^r F_1) - \mu_g(\bigwedge^r (F_1 \cap F_2)) \,=\, r \cdot \mu_g(F_1) - r \cdot \mu_g(F_1 \cap F_2)\,.$$

Hence $\mu_g(\mathcal{W}) \,=\, 0$ by (5.1) and (5.5). Hence, $\deg_g(\mathcal{W}) \,=\, 0$; also, from Corollary 3.2 we know that $\mathcal{W}$ is polystable (note that $\bigwedge^r (F_1 \cap F_2)^*$ is a line bundle). We recall from [9] that given any flat vector bundle V on M of degree zero equipped with a Hermitian–Einstein connection ∇_V, any flat section of V is flat with respect to ∇_V (see Theorem 3 in page 110 of [9]). Also, Theorem 1 (in page 102) of [9] and Corollary 33 (in page 129) of [9] say that any polystable vector bundle on M admits a Hermitian–Einstein connection. Hence the vector bundle $\mathcal{W}$ in (5.6) admits a Hermitian–Einstein connection, and the section η in (5.7) is flat with respect to the Hermitian–Einstein connection on $\mathcal{W}$.

Since η is flat with respect to the Hermitian–Einstein connection on $\mathcal{W}$, it follows that the Hermitian–Einstein connection on F_1 preserves the subbundle $F_1 \cap F_2 \,\subset\, F_1$. Consequently, $F_1 \cap F_2$ is polystable (Theorem 4 in page 110 of [9]). This also implies that the orthogonal complement of $F_1 \cap F_2$ with respect to a Hermitian–Einstein metric on F_1

$$F' \,:=\, (F_1 \cap F_2)^\perp \,\subset\, F_1$$

is preserved by the Hermitian–Einstein connection. Hence F' is polystable if $F' \,\neq\, 0$; note that $\mu_g(F') \,=\, \mu_g(F_1)$ if $F' \,\neq\, 0$.

Since $F_1 + F_2 \,=\, F' \oplus F_2$, we now conclude that $F_1 + F_2$ is polystable, and $\mu_g(F_1 + F_2) \,=\, \mu_g(V)$. □

Corollary 5.1. *Let $V \longrightarrow M$ be a flat semistable vector bundle. Then there is a unique maximal polystable flat subbundle $F \,\subset\, V$ such that $\mu_g(F) \,=\, \mu_g(V)$.*

Proof. In view of Proposition 5.1, the flat subbundle $F \,\subset\, V$ generated by all flat polystable subbundles $E \,\subset\, V$ with $\mu_g(E) \,=\, \mu_g(V)$ satisfies the conditions in the corollary. □

The flat polystable subbundle $F \,\subset\, V$ in Corollary 5.1 is called the *socle* of V.

If F is properly contained in V, then we note that V/F is semistable, and $\mu_g(V/F) \,=\, \mu_g(V)$. Therefore, Corollary 5.1 gives the following:

Corollary 5.2. *Let $V \longrightarrow M$ be a flat semistable vector bundle. Then there is a filtration of flat subbundles*

$$0 \,=\, F_0 \,\subset\, F_1 \,\subset\, \cdots \,\subset\, F_{n-1} \,\subset\, F_n \,=\, V$$

such that for each $i \,\in\, [1\,, n]$, the flat subbundle $F_i/F_{i-1} \,\subset\, V/F_{i-1}$ is the socle of the flat semistable vector bundle V/F_{i-1}.

5.1. *Socle reduction of a principal bundle*

Let G be as before. Let $E_G \longrightarrow M$ be a semistable principal G–bundle.

A *socle reduction* of E_G is a pair $(Q_0\,, E_{Q_0})$, where

- $Q_0 \subset H$ is maximal among all the parabolic subgroups Q of G such that E_G admits an admissible reduction of structure group
$$E_Q \subset E_G$$
for which the corresponding principal $Q/R_u(Q)$–bundle $E_Q/R_u(Q) \longrightarrow M$ is polystable, where $R_u(Q)$ is the unipotent radical of Q, and
- $E_{Q_0} \subset E_G$ is an admissible reduction of structure group of E_G to Q_0 such that the associated principal $Q_0/R_u(Q_0)$–bundle $E_{Q_0}/R_u(Q_0)$ is polystable.

(Admissible reductions were defined in (4.4).)

Proposition 5.2. *Let $E_G \longrightarrow M$ be a semistable principal G–bundle. Then E_G admits a socle reduction. If $(Q_1\,, E_{Q_1})$ and $(Q_2\,, E_{Q_2})$ are two socle reductions of E_G, then there is an element $g \in G$ such that $Q_2 = g^{-1}Q_1 g$ and $E_{Q_2} = E_{Q_1}g$.*

Proof. From Corollary 4.1 we know that the flat adjoint bundle $\mathrm{ad}(E_G)$ is semistable. Let

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = \mathrm{ad}(E_G) \tag{5.8}$$

be the filtration constructed in Corollary 5.2. Using Corollary 3.1 it can be shown that n is an odd integer; see page 218 of [1] for the details. The flat subbundle

$$F_{(n+1)/2} \subset \mathrm{ad}(E_G) \tag{5.9}$$

in (5.8) is the adjoint vector bundle of a reduction of structure group of E_G to a parabolic subgroup. After we fix a parabolic subgroup Q_0 of G in the conjugacy class of parabolic subgroups of G defined by the subalgebra $(E_{(n+1)/2})_x \subset \mathrm{ad}(E_G)_x$, where $x \in M$, we get a reduction of structure group $E_{Q_0} \subset E_G$ to Q_0 compatible with the connection on E_G. It can be shown that this pair $(Q_0\,, E_{Q_0})$ is a socle reduction of E_G [1]. The uniqueness statement is also proved in [1]. □

From Proposition 5.2 and its proof we have the following corollary:

Corollary 5.3. *Let E_G be a flat principal G–bundle over M. Then E_G is polystable if and only if the flat vector bundle* $\mathrm{ad}(E_G)$ *is polystable.*

Proof. First we assume that E_G is polystable. Then the flat vector bundle $\mathrm{ad}(E_G)$ is semistable by Corollary 4.1. If $\mathrm{ad}(E_G)$ is not polystable, and $(Q_0\,, E_{Q_0})$ is a socle reduction of E_G, then Q_0 is a proper parabolic subgroup of G. Therefore, E_G is not polystable, which contradicts the assumption. Hence $\mathrm{ad}(E_G)$ is polystable.

To prove the converse, assume that $\mathrm{ad}(E_G)$ is polystable. Then the principal G–bundle E_G is semistable (see Corollary 4.1). Consider the socle of E_G. Since $\mathrm{ad}(E_G)$

is polystable, we have $E_{(n+1)/2} = \text{ad}(E_G)$ (see (5.9)). Hence $(G\,,E_G)$ is the socle of E_G. Therefore, from the definition of a socle we conclude that E_G is polystable. □

Corollary 5.4. *Assume that G is the fixed point locus of a split real form on $G_{\mathbb{C}}$. Let E_G be a flat principal G–bundle over M. Let $E_{G_{\mathbb{C}}}$ be the flat principal $G_{\mathbb{C}}$–bundle over M obtained by extending the structure group of E_G using the inclusion of G in $G_{\mathbb{C}}$. The principal G–bundle E_G is polystable if and only if the principal $G_{\mathbb{C}}$–bundle $E_{G_{\mathbb{C}}}$ is so.*

Proof. We note that E_G is polystable if and only if $\text{ad}(E_G)$ is polystable by Corollary 5.3. Since

$$\text{ad}(E_{G_{\mathbb{C}}}) = \text{ad}(E_G) \otimes_{\mathbb{R}} \mathbb{C} = \text{ad}(E_G) \oplus \sqrt{-1} \cdot \text{ad}(E_G)\,,$$

it follows that $\text{ad}(E_G)$ is polystable if and only if $\text{ad}(E_{G_{\mathbb{C}}})$ is polystable. From Corollary 5.3, the adjoint vector bundle $\text{ad}(E_{G_{\mathbb{C}}})$ is polystable if and only if $E_{G_{\mathbb{C}}}$ is polystable. □

6. Hermitian–Einstein connection on stable bundles and Bogomolov inequality

6.1. *Hermitian–Einstein connection and stable principal bundles*

Fix a maximal compact subgroup

$$K \subset G$$

of the reductive group G. Let E_G be a flat principal G–bundle over M. A *Hermitian structure* on E_G is a C^∞ reduction of structure group

$$E_K \subset E_G\,.$$

Recall that G is either the fixed point locus of a split real form on a complex reductive group $G_{\mathbb{C}}$ or G is complex reductive. In the second case, by $G_{\mathbb{C}}$ we will denote G itself; this is for notational convenience.

Given a flat principal G–bundle E_G over M, we get a holomorphic principal $G_{\mathbb{C}}$–bundle $E_{G_{\mathbb{C}}}$ over $M_{\mathbb{C}}$ (the total space of TM); see page 102 of [9].

Given a Hermitian structure on E_G, there is a naturally associated connection on the principal $G_{\mathbb{C}}$–bundle $E_{G_{\mathbb{C}}}$ over $M_{\mathbb{C}}$ (Lemma 1 in page 106 of [9]); although this Lemma 1 of [9] is only for vector bundles, the proof for principal bundles is identical.

Any element z of the center of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ defines a flat section of $\text{ad}(E_G)$, because z is fixed by the adjoint action of G on $\mathfrak{g}$; this section of $\text{ad}(E_G)$ given by z will be denoted by $\underline{z}$.

Let $E_K \subset E_G$ be a Hermitian structure on E_G. Let ∇ be the corresponding connection on $E_{G_{\mathbb{C}}}$. The curvature of ∇ will be denoted by $K(\nabla)$. So $K(\nabla)$ is a smooth $(1\,,1)$–form on $M_{\mathbb{C}}$ with values in the adjoint vector bundle $\text{ad}(E_{G_{\mathbb{C}}})$.

Contracting it using the metric g, we get a smooth section $\Lambda_g K(\nabla)$ of $\mathrm{ad}(E_G)$. The Hermitian structure E_K is called *Hermitian–Einstein* if there is an element z in the center of the Lie algebra $\mathfrak{g}$ such that

$$\Lambda_g K(\nabla) = \underline{z}\,.$$

If E_K is a Hermitian–Einstein structure, then the corresponding connection ∇ is called a *Hermitian–Einstein connection.*

Theorem 6.1. *A flat principal G–bundle $E_G \longrightarrow M$ admits a Hermitian–Einstein structure if and only if E_G is polystable. A polystable flat principal G–bundle admits a unique Hermitian–Einstein connection.*

Proof. First we assume that E_G admits a Hermitian–Einstein structure. A Hermitian–Einstein structure on E_G induces a Hermitian–Einstein metric on the adjoint vector bundle $\mathrm{ad}(E_G)$. Hence $\mathrm{ad}(E_G)$ is polystable (Theorem 4 in page 110 of [9]). Hence E_G is polystable by Corollary 5.3.

To prove the converse, assume E_G is polystable. We will first reduce to the case that G is complex reductive.

If G is the fixed point locus of a split real form on $G_{\mathbb{C}}$, then Corollary 5.4 says that the corresponding principal $G_{\mathbb{C}}$–bundle $E_{G_{\mathbb{C}}}$ is also polystable. In the following paragraphs, we produce a unique Hermitian–Einstein connection on $E_{G_{\mathbb{C}}}$. Uniqueness implies that it is invariant under a natural complex conjugation, as in the proof of Corollary 4.2 above, and so the Hermitian–Einstein connection on $E_{G_{\mathbb{C}}}$ reduces to a connection on E_G.

We assume that G is complex reductive. Let $Z_G \subset G$ be the center of G. The adjoint action of G/Z_G on $\mathfrak{g}$ is faithful. Since G is reductive, the quotient $G/[G\,,G]$ is a product of copies of $\mathbb{C}^*$.

Let $E_G \longrightarrow M$ be a flat polystable principal G–bundle. Then the vector bundle $\mathrm{ad}(E_G)$ is polystable by Corollary 5.3. Let $\nabla(\mathrm{ad})$ be the Hermitian–Einstein connection on $\mathrm{ad}(E_G)$.

We will show that the connection $\nabla(\mathrm{ad})$ preserves the Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$.

Let

$$\theta_0 \,:\, \mathrm{ad}(E_G)\otimes \mathrm{ad}(E_G) \longrightarrow \mathrm{ad}(E_G)$$

be the homomorphism defined by the Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$. Define the flat vector bundle

$$\mathcal{W} := \mathcal{H}om(\mathrm{ad}(E_G)\otimes \mathrm{ad}(E_G)\,, \mathrm{ad}(E_G)) = (\mathrm{ad}(E_G)\otimes \mathrm{ad}(E_G))^* \otimes \mathrm{ad}(E_G)\,. \quad (6.1)$$

Let

$$\theta \in C^\infty(M,\, \mathcal{W}) \quad (6.2)$$

be the smooth section defined by the above homomorphism θ_0. We note that θ is flat with respect to the flat connection on $\mathcal{W}$ induced by the flat connection on E_G.

Fix a nondegenerate G–invariant symmetric bilinear form B on $\mathfrak{g}$; such a form exists because G is either complex reductive or the fixed point locus of a real form of a complex reductive group.Since B is G–invariant, it produces a symmetric bilinear form $\widetilde{B}$ on $\text{ad}(E_G)$ which is fiberwise nondegenerate and is preserved by the flat connection on $\text{ad}(E_G)$. Therefore, $\widetilde{B}$ produces an isomorphism of the flat vector bundle $\text{ad}(E_G)$ with its dual $\text{ad}(E_G)^*$. Hence $\deg_g(\text{ad}(E_G)) = -\deg_g(\text{ad}(E_G)^*) = -\deg_g(\text{ad}(E_G))$, implying that

$$\deg_g(\text{ad}(E_G)) = 0 .$$

Hence

$$\deg_g(\mathcal{W}) = 0 , \tag{6.3}$$

where $\mathcal{W}$ is defined in (6.1).

The Hermitian–Einstein connection $\nabla(\text{ad})$ on $\text{ad}(E_G)$ induces a connection on the vector bundle $\mathcal{W}$; this induced connection will be denoted by $\widetilde{\nabla}$. Since the connection $\nabla(\text{ad})$ is Hermitian–Einstein, the connection $\widetilde{\nabla}$ is also Hermitian–Einstein. Therefore, from (6.3) it follows that any flat section of $\mathcal{W}$ is also flat with respect to the Hermitian–Einstein connection $\widetilde{\nabla}$ (Theorem 3 in page 110 of [9]). In particular, the section θ in (6.2) is flat with respect to $\widetilde{\nabla}$. This immediately implies that the connection $\nabla(\text{ad})$ preserves the Lie algebra structure of the fibers of $\text{ad}(E_G)$.

Hence the connection $\nabla(\text{ad})$ produces a connection on the flat principal G/Z_G–bundle E_G/Z_G. This connection on E_G/Z_G will be denoted by ∇'. The connection ∇' is Hermitian–Einstein because $\nabla(\text{ad})$ is so.

Since $G/[G\,,G] = (\mathbb{C}^*)^d$, where d is the dimension of Z_G, the flat principal $G/[G\,,G]$–bundle $E_G/[G\,,G]$ admits a unique Hermitian–Einstein connection; this connection will be denoted by ∇''.

The quotient map $G \longrightarrow (G/Z_G) \times (G/[G\,,G])$ has the property that the corresponding homomorphism of Lie algebras is an isomorphism. Therefore, there is a natural bijection between the connections on a principal G–bundle F_G and the connections on the principal $G/Z_G) \times (G/[G\,,G])$–bundle obtained by extending the structure group of F_G using the above homomorphism $G \longrightarrow (G/Z_G) \times (G/[G\,,G])$. The connections ∇' and ∇'' on E_G/Z_G and $E_G/[G\,,G]$ together define a connection on the principal $G/Z_G) \times (G/[G\,,G])$–bundle $(E_G/Z_G) \times_M (E_G/[G\,,G])$ over M. By the above remark on bijection of connections, this connection on $(E_G/Z_G) \times_M (E_G/[G\,,G])$ produces a connection on E_G. The connection on E_G obtained this way is Hermitian–Einstein because both ∇' and ∇'' are so.

The uniqueness of a Hermitian–Einstein connection on E_G follows from the uniqueness of the Hermitian–Einstein connections on the vector bundle $\text{ad}(E_G)$ and the principal $G/[G\,,G]$–bundle $E_G/[G\,,G]$. This completes the proof of the theorem. □

6.2. *A Bogomolov type inequality*

As before, M is a compact connected special flat affine manifold, g is a Gauduchon metric on M, and ν is a nonzero covariant constant volume form on M. Let d be the dimension of M. The $(1\,,1)$–form given by g will be denoted by ω_g. We recall that the Gauduchon condition says that

$$\partial\overline{\partial}(\omega_g^{d-1}) \,=\, 0$$

(see page 109 of [9]).

The Gauduchon metric g is called *astheno–Kähler* if

$$\partial\overline{\partial}(\omega_g^{d-2}) \,=\, 0 \tag{6.4}$$

(see page 246 of [7]).

We note that if $d \,=\, 2$, then g is astheno–Kähler. If g is Kähler, then g is also astheno–Kähler.

We assume that the Gauduchon metric g is astheno–Kähler.

Let E be a flat vector bundle on M. Take a Hermitian structure h on E. Using (6.4) it follows that

$$\int_M \frac{c_1(E,h)^2\omega_g^{d-2}}{\nu} \in \mathbb{R} \quad \text{and} \quad \int_M \frac{c_2(E,h)\omega_g^{d-2}}{\nu} \in \mathbb{R}$$

are independent of the choice of h.

Lemma 6.1. *Let $V \longrightarrow M$ be a semistable flat vector bundle of rank r. Then*

$$\int_M \frac{c_2(\mathcal{E}nd(V))\omega_g^{d-2}}{\nu} \,=\, \int_M \frac{(2r\cdot c_2(V)-(r-1)c_1(V)^2)\omega_g^{d-2}}{\nu} \,\geq\, 0\,.$$

Proof. First assume that V is polystable. Therefore, V admits a Hermitian–Einstein connection (see Theorem 1 in page 102 of [9] for the complex case and Corollary 33 in page 129 of [9] for the real case). Let h be a Hermitian–Einstein metric on V. Then the d–form

$$\frac{(2r\cdot c_2(V,h)-(r-1)c_1(V,h)^2)\omega_g^{d-2}}{\nu}$$

on M is pointwise nonnegative (see [10] and page 107 of [8] for the computation). Therefore, the lemma is proved for polystable vector bundles.

If V is semistable, then there is a filtration of flat subbundles

$$0 \,=\, V_0 \,\subset\, V_1 \,\subset\, \cdots \,\subset\, V_{n-1} \,\subset\, V_\ell \,=\, V$$

such that V_i/V_{i-1} is polystable for all $i \,\in\, [1\,,n]$, and also $\mu_g(V_i/V_{i-1}) \,=\, \mu_g(V)$. Since the inequality in the statement of the lemma holds for all V_i/V_{i-1}, it also holds for V. □

Let G be a connected Lie group such that it is either a complex reductive linear algebraic group or it is the fixed point locus of a split real form on a complex reductive linear algebraic group.

Proposition 6.1. *Let $E_G \longrightarrow M$ be a flat semistable principal G–bundle. Then*

$$\int_M \frac{c_2(\mathrm{ad}(E_G))\omega_g^{d-2}}{\nu} \geq 0\,.$$

Proof. The vector bundle $\mathrm{ad}(E_G)$ is semistable because E_G is semistable (see Corollary 4.1). Since $\mathrm{ad}(E_G) = \mathrm{ad}(E_G)^*$, we have

$$\int_M \frac{c_1(\mathrm{ad}(E_G))^2\omega_g^{d-2}}{\nu} = 0\,.$$

Hence the proof is completed by Lemma 6.1. □

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