EQUIVARIANT MINIMAL SURFACES IN $\mathbb{CH}^2$ AND THEIR HIGGS BUNDLES.

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Abstract. This paper gives a construction for all minimal immersions f of the Poincaré disc into the complex hyperbolic plane $\mathbb{CH}^2$ which are equivariant with respect to an irreducible representation ρ of a hyperbolic surface group into $PU(2,1)$. We exploit the fact that each such immersion is a twisted conformal harmonic map and therefore has a corresponding Higgs bundle. We identify the structure of these Higgs bundles and show how each is determined by properties of the map, including the induced metric and a holomorphic cubic differential on the surface. We show that the moduli space of pairs (ρ, f) is a disjoint union of finitely many complex manifolds, whose structure we fully describe. The holomorphic (or anti-holomorphic) maps provide multiple components of this union, as do the non-holomorphic maps. Each of the latter components has the same dimension as the representation variety for $PU(2,1)$, and is indexed by the number of complex and anti-complex points of the immersion. These numbers determine the Toledo invariant and the Euler number of the normal bundle of the immersion. We show that there is an open set of quasi-Fuchsian representations of Toledo invariant zero for which the minimal surface is unique and Lagrangian.

1. Introduction

In this article we provide a complete classification, together with a parametrisation, for ρ-equivariant minimal immersions $f : \mathcal{D} \to \mathbb{CH}^2$. Here $\mathcal{D}$ is the Poincaré disc and ρ is an irreducible (or more generally, reductive) representation of a hyperbolic surface group (i.e., the fundamental group $\pi_1\Sigma$ of a closed orientable surface Σ of genus at least two) into the group $PU(2,1) = U(2,1)/\text{centre}$. To say f is ρ-equivariant means it intertwines the action of a Fuchsian group on $\mathcal{D}$ with the action of ρ on the complex hyperbolic plane $\mathbb{CH}^2$ by holomorphic isometries. One can also think of f as a section of a $\mathbb{CH}^2$-bundle over Σ or, when ρ is a discrete embedding, as a minimal immersion $f : \Sigma \to \mathbb{CH}^2/\rho$ into the quotient manifold.

We classify these pairs (ρ, f) up to $PU(2,1)$-equivalence, i.e., up to the natural left action of $PU(2,1)$ by conjugation of ρ and the simultaneous ambient isometry of f. In particular, the space of such pairs has a natural "forgetful" map to the moduli space

$$\mathcal{R}(G) = \mathrm{Hom}^+(\pi_1, \Sigma, G)/G,$$

of conjugacy classes of reductive representations into $G = PU(2,1)$. Recall that this is a real analytic variety whose connected components are indexed by the Toledo invariant $\tau(\rho) \in \frac{2}{3}\mathbb{Z}$, $|\tau(\rho)| \le -\chi(\Sigma)$ (see, e.g., [18]). We show that there are families of pairs for every value of τ.

We achieve this classification by exploiting the fact that a minimal immersion of a surface is the same thing as a conformal harmonic map. This allows us to employ the powerful

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machinery developed by Hitchin [28], Donaldson [10], Corlette [9] and others which links equivariant harmonic maps to the Yang-Mills-Higgs equations over a compact Riemann surface. Our starting point is two facts: (i) to each irreducible ρ and marked conformal structure on Σ (i.e., conjugacy class of Fuchsian representations) there is a ρ-equivariant harmonic map f [10, 9]; (ii) when this information is encoded into a G-Higgs bundle (E, Φ) by the Hitchin correspondence [28, 21, 3], f is weakly conformal precisely when $\operatorname{tr}\Phi^2 = 0$.

When $G = PU(2,1)$ the structure of the Higgs bundle is well understood (see, e.g., [21]): the bundle E splits into a sum $V \oplus L$ of a rank two sub-bundle V and a rank one sub-bundle L which are mapped to each other by the Higgs field, i.e., we can write $\Phi = (\Phi_1, \Phi_2)$ where $\Phi_1 : L \to KV$ and $\Phi_2 : V \to KL$ for K the canonical bundle determined by the marked conformal structure. In fact by projective equivalence we may assume that $L = 1$, the trivial bundle. In the Hitchin correspondence Φ corresponds to the differential of f. To be precise, it corresponds to

$$\partial f : T^{1,0}\mathcal{D} \to T^{\mathbb{C}}\mathbb{CH}^2,$$

and the components Φ_1, Φ_2 correspond to the components of ∂f with respect to the type decomposition of $T^{\mathbb{C}}\mathbb{CH}^2$. For a minimal surface there are two possibilities: (i) f is holomorphic ($\Phi_2 = 0$) or anti-holomorphic ($\Phi_1 = 0$), or (ii) f is neither and has isolated complex and anti-complex points which give finite divisors D_2 and D_1 over Σ (where, respectively, Φ_2 and Φ_1 have zeroes). We treat these two possibilities separately, and in fact it is the latter which we treat first, in §2 and §3. An important role is played by a cubic holomorphic differential $\mathcal{Q}$ which can be naturally assigned to any minimal surface in a Kähler manifold of constant holomorphic section curvature [48]. It vanishes identically for holomorphic or anti-holomorphic immersions (but not only for them). When f is neither holomorphic nor anti-holomorphic we show that, with the two bilinear forms $\gamma_j = \frac{1}{2}\operatorname{tr}\Phi_j\Phi_j^\dagger$, which carry the information of the metric γ induced by f, the data $(\gamma_1, \gamma_2, \mathcal{Q})$ completely determines the minimal immersion up to ambient isometries. The principal results of §2 and §3 (Theorems 2.3 and 3.1) can be summarised as follows.

Theorem 1.1. *Let ρ be irreducible and f be minimal and ρ-equivariant. If f is neither holomorphic nor anti-holomorphic then the pair (ρ, f) is faithfully determined, up to G-equivalence, by data $(\Sigma_c, D_1, D_2, \xi)$ where c is a marked conformal structure on Σ, D_1, D_2 are non-negative divisors on Σ whose degrees d_1, d_2 satisfy*

$$2d_1 + d_2 < 6(g-1) \text{ and } d_1 + 2d_2 < 6(g-1),$$

and $\xi \in H^1(\Sigma_c, K^{-2}(D_1 + D_2))$ represents an extension class determining the sub-bundle V of the Higgs bundle $E = V \oplus 1$. Under the Dolbeault isomorphism this extension class corresponds to the cohomology class of $-\bar{\mathcal{Q}}/\gamma_1\gamma_2$ in $H^{0,1}(\Sigma_c, K^{-2}(D_1 + D_2))$, and $\xi = 0$ if and only if $\mathcal{Q} = 0$. In this correspondence, ρ has Toledo invariant $\tau(\rho) = \frac{2}{3}(d_2 - d_1)$, and the Euler number of the normal bundle of f is $\chi(T\Sigma^\perp) = 2(g-1) - d_1 - d_2$.

In particular, the numbers d_1, d_2 determine, and are determined by, the Toledo invariant of ρ and the Euler number of the normal bundle of f.

The correspondence is constructive in the sense that, given data $(\Sigma_c, D_1, D_2, \xi)$ satisfying the conditions above, we give an explicit construction of a stable Higgs bundle (E, Φ) with $\operatorname{tr}\Phi^2 = 0$. Of course, D_1, D_2 are the divisors of anti-complex and complex points of the minimal immersion. For f to be strictly an immersion they must have no common points, but the construction still works to produce branched minimal immersions if they intersect, with branch points at the intersections.

This construction includes and greatly extends the construction of minimal Lagrangian embeddings we gave in [36]. Indeed, up to this time we were not aware of any other examples of non-holomorphic equivariant minimal immersions into $\mathbb{CH}^2$ (save for the Lagrangian examples whose existence is a consequence of the "mountain-pass" solutions to the Gauss equation described in [30]). The theorem above not only gives all examples for reductive representations (when we allow branch points) but allows us to describe the structure of the moduli space of these as a complex manifold (Theorem 1.3 below).

By a theorem of Wolfson [47], f is Lagrangian precisely when it has no complex or anti-complex points, and is therefore parametrised by the pair (Σ_c, ξ). The embeddings constructed in [36] all have the property that the exponential map on the normal bundle provides a diffeomorphism between $T\mathcal{D}^\perp$ and $\mathbb{CH}^2$. It follows that ρ has a finite fundamental domain (given by the normal bundle over a finite fundamental domain for the action of $\pi_1\Sigma$ on $\mathcal{D}$). Here we call such embeddings *almost* $\mathbb{R}$*-Fuchsian*, because they are deformations of the embedding $\mathbb{RH}^2 \to \mathbb{CH}^2$, which is equivariant with respect to every Fuchsian representation into $PO(2,1) \subset PU(2,1)$. In §4 we improve on the results in [36] by showing that whenever f is minimal Lagrangian with $\|\mathcal{Q}\|_\gamma^2 < 2$ it is almost $\mathbb{R}$-Fuchsian and the unique ρ-equivariant minimal immersion. The $\mathbb{R}$-Fuchsian case corresponds to $\mathcal{Q} = 0$, and therefore $\xi = 0$ by the theorem above. The uniqueness of f proved here implies that the data (Σ_c, ξ) also parametrises the almost $\mathbb{R}$-Fuchsian family, although at present we do not understand the appropriate bound on ξ. It is preferable to have the parametrisation in terms of (Σ_c, ξ) since that gets us directly to the Higgs bundle and therefore to ρ. The parametrisation in [36] using $\mathcal{Q}$ requires an additional condition to provide a unique solution to the Gauss equation of the immersion. We describe the subtleties of existence and uniqueness for this equation in §4.1, which also draws on earlier work by Huang, Loftin & Lucia [30].

The minimal Lagrangian case suggests that it is important to understand those minimal immersions for which $\mathcal{Q} = 0$. These are treated in §5. We show that they have a very interesting interpretation in terms of the Higgs bundle, for $\mathcal{Q} = 0$ exactly when the Higgs bundle is a *Hodge bundle* (or *variation of Hodge structure*). These are known to be the critical points of the Morse function $\|\Phi\|_{L^2}^2$ [21] and come in two flavours: length two or length three. The length-two Hodge bundles all correspond to holomorphic or anti-holomorphic maps. The following theorem summarises our results regarding these.

Theorem 1.2. *Let ρ be irreducible and f be branched holomorphic and ρ-equivariant. Then the pair (ρ, f) is faithfully determined by data (Σ_c, B, L, η) where B is a non-negative divisor of degree b, L is a holomorphic line bundle of degree l satisfying*

$$3(g-1) + \tfrac{1}{2}b < l < 6(g-1) - b, \quad 0 \le b < 2(g-1),$$

and $\eta \in H^1(\Sigma_c, KL^{-1}B)$ is a non-trivial extension class determining V. The Toledo invariant of ρ is $\frac{2}{3}(6g - 6 - b - l) > 0$.

This also accounts for anti-holomorphic immersions, since f is anti-holomorphic and ρ-equivariant if and only if $\bar{f}$ is holomorphic and $\bar{\rho}$-equivariant. The latter has the dual Higgs bundle to ρ.

As with the non-holomorphic case, the extension class η corresponds to the Dolbeault cohomology class of a tensor over Σ_c which has geometrical significance and is related to the second fundamental form of f (see Theorem 5.2). We explain how the limiting value $\eta = 0$ corresponds to reducible representations which are not *maximal*, i.e., do not have $\tau = \pm\chi(\Sigma)$.

By contrast, the length-three Hodge bundles correspond to those pairs (ρ, f) coming from Theorem 1.1 with $\xi = 0$. By using the method of harmonic sequences [4, 7, 14, 15] we show that $\xi = 0$ precisely when the harmonic sequence of f contains a holomorphic ρ-equivariant (and "timelike") map into complex de Sitter 2-space. This is a pseudo-Hermitian symmetric space, the analogue of the real 2-dimensional de Sitter space which complements $\mathbb{RH}^2$.

In the final section, §6, we describe the moduli space

$$\mathcal{V} = \{(\rho, f) : \rho \text{ irreducible, } f \text{ branched minimal}\}/G.$$

By the results stated above it is a union of components $\mathcal{V}(d_1, d_2)$ containing those pairs described by Theorem 1.1 and $\mathcal{W}^+(b, l)$ (respectively, $\mathcal{W}^-(b, l)$) containing the holomorphic (resp., anti-holomorphic) immersions described in Theorem 1.2. Of these we prove:

Theorem 1.3. *Each $\mathcal{V}(d_1, d_2)$ is a complex manifold of dimension $8g-8$, while each $\mathcal{W}^\pm(b, l)$ is a complex manifold of dimension $3(g-1)+l+1$. All of these are diffeomorphic to bundles over the Teichmüller space of Σ, and the fibres are complex analytic submanifolds. For $\mathcal{V}(d_1, d_2)$ each fibre $\mathcal{V}_c(d_1, d_2)$ is a rank $5g-5-d_1-d_2$ bundle over $S^{d_1}\Sigma_c \times S^{d_2}\Sigma_c$, while for $\mathcal{W}^\pm(b, l)$ each fibre is a punctured rank $l+1-b-g$ bundle (i.e., without its zero section) over $S^b\Sigma_c \times \mathrm{Pic}_l(\Sigma_c)$.*

We finish in §6.3 with a brief discussion of the map $\mathcal{V} \to \mathcal{R}(G)$ given by forgetting the immersion. There is much yet to be understood about this map. For example, we do not know if this map is onto for non-maximal representations, or the dimension of its image on components of $\mathcal{V}$. Nevertheless, we can make some salient remarks about its restriction to any fibre over Teichmüller space. We point out that on the image of $\mathcal{V}$ the L^2-norm of the Higgs fields equals the area of the minimal immersion. The critical points of $\|\Phi\|^2_{L^2}$ are all accounted for by the Hodge bundles, and therefore lie in the image of $\mathcal{V}$. A comparison of the structure of $\mathcal{V}_c(d_1, d_2)$ with what is known of the Morse index from [21], together with an area bound established in §3, suggests that the fibres of the vector bundle $\mathcal{V}_c(d_1, d_2)$ map onto the downward Morse flow of $\|\Phi\|^2_{L^2}$.

For us, one of the outstanding challenges is to use this construction to study the *quasi-Fuchsian* representations, where "quasi-Fuchsian" is meant in the sense of Parker & Platis [40], i.e., a convex cocompact, totally loxodromic, discrete embedding. Recent work of Guichard & Wienhard [24, Thm 1.8] has shown that ρ is a convex cocompact embedding precisely when it is an *Anosov embedding.* Since the latter are totally loxodromic, the notions "quasi-Fuchsian", "convex cocompact embedding" and "Anosov embedding" coincide for $PU(2,1)$ (and more generally, any semisimple real Lie group of rank one). One knows from [23] that quasi-Fuchsian representations comprise an open subset of the representation variety $\mathcal{R}(G)$. There are examples for every even value of $\tau(\rho)$ [19], and all almost $\mathbb{R}$-Fuchsian representations are quasi-Fuchsian [36] (it seems certain that this includes the family constructed by different means by Parker & Platis in [39]). But the latter type give the only known open family of quasi-Fuchsian representations outside the maximal representations.

Beyond this, there is very little known; it is not even known whether there are quasi-Fuchsian representations for every value of the Toledo invariant. One compelling reason for looking to minimal immersions to provide more insight is the theorem of Goldman & Wentworth [20], that for convex cocompact representations the harmonic map energy functional on Teichmüller space is a proper function. It therefore has at least one critical point, and it is a well-known result of Sacks & Uhlenbeck [41] that each critical point corresponds to a weakly conformal harmonic (i.e., branched minimal) map. Since our construction includes branched minimal maps, it follows that the map $\mathcal{V} \to \mathcal{R}(G)$ has all quasi-Fuchsian representations in its image.

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2. Minimal surfaces and their Higgs bundles.

We begin by setting up the notation and standard constructions for the minimal surfaces and the Higgs bundles we will be working with.

2.1. Equivariant minimal surfaces in $\mathbb{CH}^2$.

Our model for $\mathbb{CH}^2$ will be the projective model, as follows. Let $\mathbb{C}^{2,1}$ denote the vector space $\mathbb{C}^3$ equipped with the (indefinite) Hermitian metric

$$\langle v, v\rangle = v_1\bar{v}_1 + v_2\bar{v}_2 - v_3\bar{v}_3.$$

Let $\mathbb{C}^{2,1}_- = \{v \in \mathbb{C}^{2,1} : \langle v, v\rangle < 0\}$, so that $\mathbb{CH}^2 \simeq \mathbb{PC}^{2,1}_-$. Thus we consider $\mathbb{CH}^2$ as the orbit of the line $[0, 0, 1] \in \mathbb{PC}^{2,1}_-$ under the standard action of $G = PU(2, 1)$. Consequently $\mathbb{CH}^2 \simeq G/H$, where $H \simeq P(U(2) \times U(1))$ is a maximal compact subgroup of G. We equip $\mathbb{CH}^2$ with its Hermitian metric of constant holomorphic sectional curvature -4; so that its sectional curvature has bounds $-4 \le \kappa \le -1$. We write the Hermitian metric on $\mathbb{CH}^2$ as $h = g - i\omega$, where $\omega(X, Y) = g(JX, Y)$, and recall that $(\mathbb{CH}^2, h)$ is a Kähler-Einstein manifold.

We will always think of a minimal immersion $f : \mathcal{D} \to \mathbb{CH}^2$ as a conformal harmonic immersion, so the induced metric $\gamma = f^*g$ is conformally equivalent to the hyperbolic metric μ, with $\gamma = e^u\mu$ for a smooth function $u : \Sigma \to \mathbb{R}$. To say that f is ρ-equivariant means it intertwines ρ with a Fuchsian representation $\pi_1\Sigma \to \mathrm{Isom}(\mathcal{D})$. The conjugacy class of such a representation is equivalent to a choice of a *marked conformal structure* on Σ, i.e., a point $c \in \mathcal{T}_g$ in the Teichmüller space of Σ. We will write Σ_c to denote the surface with this structure. From now on we will assume f is ρ-equivariant.

To understand the properties of such minimal immersions we need some notation for the type decomposition of the (complexified) differential $df : T^{\mathbb{C}}\mathcal{D} \to T^{\mathbb{C}}\mathbb{CH}^2$. Both the domain and the codomain are complex manifolds, so to distinguish between the type decompositions of their tangent vectors we will write

$$T^{\mathbb{C}}\mathcal{D} = T^{1,0}\mathcal{D} \oplus T^{0,1}\mathcal{D}, \quad T^{\mathbb{C}}\mathbb{CH}^2 = T'\mathbb{CH}^2 + T''\mathbb{CH}^2.$$

The projections will be such that $X = X^{1,0} + \overline{X^{1,0}}$ (or $X' + \overline{X'}$ as appropriate) whenever X is real. Our primary model for $T^{\mathbb{C}}\mathbb{CH}^2$ will be the projective model: viz, at any point $\ell \in \mathbb{PC}^{2,1}_-$ we use the isomorphism

$$\begin{aligned} T'_\ell\mathbb{CH}^2 \oplus T''_\ell\mathbb{CH}^2 &\to \mathrm{Hom}(\ell, \ell^\perp) \oplus \mathrm{Hom}(\ell^\perp, \ell) \subset \mathrm{End}(\mathbb{C}^{2,1}); \\ (Z, W) &\mapsto (\pi_\ell^\perp \circ Z, \pi_\ell \circ W), \end{aligned} \tag{2.1}$$

where $\pi_\ell : \mathbb{C}^{2,1} \to \ell$ is the orthogonal projection and we think of Z, W as operations of differentiation on local sections of $\mathbb{CH}^2 \times \mathbb{C}^3$. In particular, conjugation in $T^{\mathbb{C}}\mathbb{CH}^2$ corresponds to taking the Hermitian transpose in $\mathrm{End}(\mathbb{C}^{2,1})$, i.e., whose fixed subspace is $\mathfrak{u}(2,1)$. The isomorphism can be derived from the symmetric space model for $\mathbb{CH}^2$, which we will occasionally need to use (cf. the related model for $\mathbb{CP}^n$ in, for example, [5]). For that model, let $\mathfrak{g} = \mathfrak{su}(2,1)$ and let $\mathfrak{h} \subset \mathfrak{g}$ denote the Lie subalgebra for H. Then the symmetric space decomposition is $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ where $\mathfrak{m} = \mathfrak{h}^\perp$ with respect to the Killing form, and $T\mathbb{CH}^2 \simeq [\mathfrak{m}] = G \times_H \mathfrak{m}$, where the action of H on $\mathfrak{m}$ is its right adjoint action. It is easy to check that the fibre of $[\mathfrak{m}^\mathbb{C}]$ at ℓ agrees with the codomain of (2.1), and that the metric corresponds to $g(A,B) = \frac{1}{2}\operatorname{tr}(AB)$ whenever $A, B \in \mathfrak{m}$.

By extending ℓ to mean the tautological sub-bundle; the Hermitian metric h on $T'\mathbb{CH}^2$ is then equivalent to the inner product

$$h(Z_1, Z_2) = \langle \pi_\ell^\perp Z_1 \sigma_0, \pi_\ell^\perp Z_2 \sigma_0 \rangle, \quad \sigma_0 \in \Gamma(\ell),\ \langle \sigma_0, \sigma_0 \rangle = -1.$$

Then type decomposition induces an isometry $T\mathbb{CH}^2 \to T'\mathbb{CH}^2$. These type decompositions give four complex linear parts of df:

$$\partial f' : T^{1,0}\mathcal{D} \to T'\mathbb{CH}^2, \quad \partial f'' : T^{1,0}\mathcal{D} \to T''\mathbb{CH}^2, \tag{2.2}$$

$$\bar\partial f' : T^{0,1}\mathcal{D} \to T'\mathbb{CH}^2, \quad \bar\partial f'' : T^{0,1}\mathcal{D} \to T''\mathbb{CH}^2, \tag{2.3}$$

which are related by $\bar\partial f'' = \overline{\partial f'}$ and $\bar\partial f' = \overline{\partial f''}$ using simultaneously the conjugation in $T^\mathbb{C}\mathcal{D}$ and $T^\mathbb{C}\mathbb{CH}^2$. Since f is ρ-equivariant, so is df, i.e., $df d\delta = d\rho(\delta) df$ whenever $\delta \in \pi_1\Sigma \subset \mathrm{Isom}(\mathcal{D})$. Therefore we can think of df as a section of the bundle $T^\mathbb{C}\Sigma^* \otimes (f^{-1}T^\mathbb{C}\mathbb{CH}^2/\rho)$ over Σ_c. In particular,

$$\partial f = \partial f' + \partial f'',$$

is a smooth section of the vector bundle $K \otimes (f^{-1}T^\mathbb{C}\mathbb{CH}^2/\rho)$ over Σ_c, where K is the canonical bundle of Σ_c.

One says that f has a *complex point* at $p = f(z)$ when $\partial f''$ vanishes at p (i.e., when $df(T^{1,0}\mathcal{D}) \subset T'\mathbb{CH}^2$), and an *anti-complex point* when $\partial f'$ vanishes at p. The Levi-Civita connexion induces a holomorphic structure on $f^{-1}T^\mathbb{C}\mathbb{CH}^2/\rho$ which preserves type decomposition, and f is harmonic when $\nabla^{\mathbb{CH}^2}_{\bar Z} \partial f(Z) = 0$ for local holomorphic sections Z of $T^{1,0}\mathcal{D}$, i.e., when

$$\nabla^{\mathbb{CH}^2}_{\bar Z} \partial f'(Z) = 0 \text{ and } \nabla^{\mathbb{CH}^2}_{\bar Z} \partial f''(Z) = 0.$$

Thus $\partial f'$ and $\partial f''$ are holomorphic sections of their respective bundles; so a harmonic immersion which is not holomorphic or anti-holomorphic must have isolated anti-complex and complex points. We will denote the divisors of zeroes of $\partial f'$ and $\partial f''$ on Σ by D_1 and D_2 respectively .

Through these identifications there is a sesqui-linear form $h(df', df')$ on $T^\mathbb{C}\Sigma$ which gives the induced metric as $\gamma = \mathrm{Re}\, h(df', df') = g(df', df')$. The map f is *weakly conformal* when

$$h(\partial f', \bar\partial f') = 0,$$

and conformal when additionally df' does not vanish. Therefore, for conformal maps the induced metric and the pull-back of the Kähler form are expressed, with respect to a local complex coordinate z, as

$$\gamma = f^*g = (u_1^2 + u_2^2)|dz|^2, \quad f^*\omega = \frac{i}{2}(u_1^2 - u_2^2) dz \wedge d\bar z. \tag{2.4}$$

where

$$u_1 = \|\partial f'(Z)\|,\ u_2 = \|\partial f''(Z)\|, \quad Z = \partial/\partial z. \tag{2.5}$$

The functions u_1, u_2 are locally real analytic and vanish precisely at the anti-complex and complex points, respectively. They correspond to Hermitian metrics

$$\gamma_1 = h(\partial f', \partial f') = u_1^2 dz\overline{dz}, \quad \gamma_2 = h(\partial f'', \partial f'') = u_2^2 dz\overline{dz} \tag{2.6}$$

on $K^{-1}(D_1)$ and $K^{-1}(D_2)$ respectively. Note that, to apply γ to all elements of $T^{\mathbb{C}}\Sigma$ consistently, the meaning of "$|dz|^2$" above is

$$|dz|^2 = \tfrac{1}{2}(dz\overline{dz} + d\bar{z}\overline{d\bar{z}}),$$

in terms of the local complex linear forms $dz, d\bar{z}$. For this reason we do not write $\gamma = \gamma_1 + \gamma_2$.

Because the forms $\gamma, f^*\omega$ live on Σ we can use some of the arguments which apply to compact minimal surfaces in Kähler-Einstein manifolds [45, 47, 6] to relate numerical invariants of a minimal immersion.

Theorem 2.1. *Let $f : \mathcal{D} \to \mathbb{CH}^2$ be a ρ-equivariant minimal immersion which is neither holomorphic nor anti-holomorphic. Let d_1, d_2 be the degrees of the divisors D_1 and D_2 of anti-complex and complex points. Then*

$$\begin{aligned} c_1(\rho) &= d_1 - d_2, & (2.7)\\ \chi(\Sigma) + \chi(T\Sigma^\perp) &= -d_1 - d_2, & (2.8) \end{aligned}$$

where $c_1(\rho)$ is the first Chern class of the bundle $f^{-1}T\mathbb{CH}^2/\rho$ over Σ and $T\Sigma^\perp \subset f^{-1}T\mathbb{CH}^2/\rho$ is the normal sub-bundle.

Wolfson also showed that in the absence of complex or anti-complex points a minimal immersion into a Kähler-Einstein surface of negative scalar curvature must be Lagrangian [47, Thm 2.1]. His argument extends to ρ-equivariant maps, so that in the setting of the previous theorem f will be Lagrangian if and only if $d_1 = 0 = d_2$.

Now we recall that the Toledo invariant $\tau(\rho)$ is the integer

$$\tau(\rho) = \frac{2}{\pi}\int_\Sigma f^*\omega. \tag{2.9}$$

This is the normalisation which fits with the metric of holomorphic sectional curvature -4. It is known that for any representation ρ into $PU(2,1)$, $|\tau(\rho)| \le -\chi(\Sigma)$ [11] and $\tau(\rho) \in \frac{2}{3}\mathbb{Z}$ [19]. Since $\mathbb{CH}^2$ has Einstein constant -6, equation (2.7) tells us that

$$\tau(\rho) = -\tfrac{2}{3}c_1(\rho) = \tfrac{2}{3}(d_2 - d_1). \tag{2.10}$$

Finally, as well as the degenerate metrics γ_j above, there is a third important invariant of minimal equivariant immersions [48, Cor 2.7], the cubic holomorphic differential $\mathcal{Q} \in H^0(\Sigma, K^3)$ defined by

$$\mathcal{Q}(Z,Z,Z) = h(\nabla_Z^{\mathbb{CH}^2}\partial f'(Z), \bar\partial f'(\bar Z)) = -h(\partial f'(Z), \nabla_{\bar Z}^{\mathbb{CH}^2}\bar\partial f'(\bar Z)), \tag{2.11}$$

for $Z \in T^{1,0}\mathcal{D}$. It follows at once from this that $\mathcal{Q}$ vanishes identically for holomorphic or anti-holomorphic immersions (but not only for them, as we shall see). When f is neither holomorphic nor anti-holomorphic we will see later that the quantities $\gamma_1, \gamma_2, \mathcal{Q}$ uniquely determine f up to ambient isometries.

2.2. **G-Higgs bundles and representations.** As well as fixing our notation for Higgs bundles, we also need to summarise their correspondence with (projectively) flat connexions, and hence representations and harmonic maps, since we will be making explicit use of this correspondence for most of this article.

Suppose Σ has been given a fixed conformal structure. With the notation of the previous section, a G-Higgs bundle for $G = PU(2,1)$ (cf. [21, 49]) is a pair (E,Φ) consisting of a holomorphic rank three vector bundle E over Σ equipped with a splitting $E = V \oplus L$ into a rank two sub-bundle V and a line sub-bundle L (both holomorphic) together with a holomorphic section

$$\Phi \in H^0(K\,\mathrm{Hom}(L,V) \oplus K\,\mathrm{Hom}(V,L)), \tag{2.12}$$

called the *Higgs field*. We will write $\Phi = (\Phi_1, \Phi_2)$ to denote the two summands implied by the direct sum. It is also convenient to write the holomorphic structure on E as a $\bar\partial$-operator on smooth sections, $\bar\partial_E : \mathcal{E}^0(E) \to \mathcal{E}^{0,1}(E)$. A Higgs bundle is *stable* if for any proper (non-zero) Φ-invariant sub-bundle $W \subset E$ the slope condition

$$\frac{\deg(W)}{\mathrm{rk}(W)} < \tfrac{1}{3}\deg(E), \tag{2.13}$$

is satisfied. It is *polystable* when it is either stable or the direct sum of stable proper Higgs sub-bundles all having the same slope (these latter type are called *strictly polystable*). Polystable Higgs bundles correspond to reductive representations $\rho : \pi_1\Sigma \to G$ via the Hitchin correspondence (cf. [28, 16]), and the stable Higgs bundles pick out the irreducible representations (hence strictly polystable Higgs bundles correspond to reducible reductive representations). We shall state this correspondence in one direction as follows.

Theorem 2.2. *For each polystable stable Higgs bundle (E,Φ) there is a $\mathbb{C}^{2,1}$ metric on E for which $L = V^\perp$ and the corresponding Chern connexion ∇_E and Hermitian adjoint $\Phi^\dagger$ yield a projectively flat connexion $\nabla = \nabla_E + \Phi + \Phi^\dagger$, i.e., with curvature*

$$R^\nabla = R^{\nabla_E} + [\Phi \wedge \Phi^\dagger] = \omega I,$$

for a scalar 2-form ω. The holonomy of ∇ yields a reductive representation $\rho : \pi_1\Sigma \to PU(2,1)$ for which $\mathbb{P}E \simeq \mathcal{D} \times_\rho \mathbb{PC}^{2,1}$, and ρ is irreducible precisely when (E,Φ) is stable.

Remark 2.1. From the Higgs bundle perspective, the Toledo invariant is sometimes defined to be $\frac{2}{3}\deg(VL^{-1})$ (see, for example, [49]). This differs by a sign from our convention, since

$$\tfrac{2}{3}\deg(VL^{-1}) = \tfrac{2}{3}\deg\mathrm{Hom}(L,V) = \tfrac{2}{3}c_1(\rho) = -\tau(\rho). \tag{2.14}$$

The $\mathbb{C}^{2,1}$ metric is negative definite on L, and therefore L determines a smooth section of the $\mathbb{CH}^2$ bundle $\mathbb{P}E_-$ (where E_- denotes the bundle of negative length vectors in E). Since $\mathbb{P}E_- \simeq \mathcal{D} \times_\rho \mathbb{CH}^2$ this section is equivalent to a ρ-equivariant map $f : \mathcal{D} \to \mathbb{CH}^2$ in such a way that $\Phi = \partial f$. Moreover, ∇_E induces a metric connexion on $f^{-1}T\mathbb{CH}^2$ which agrees with the pull-back of the Levi-Civita connexion, so that the equation $\bar\partial_E \Phi = 0$ is the harmonic map condition for f.

In the reverse direction, a representation $\rho : \pi_1\Sigma \to G$ determines the projective bundle $\mathbb{P}E$ uniquely, but not E itself. Nevertheless, it does determine a class of projectively equivalent $\mathbb{C}^{2,1}$ bundles, each with a projectively flat connexion. By Corlette's results [9], there is a ρ-equivariant harmonic map $f : \mathcal{D} \to \mathbb{CH}^2$ precisely when ρ is reductive, and this map is unique. The map corresponds to a line sub-bundle $L \subset E$ and therefore a splitting $E = L^\perp \oplus L$.

The splitting determines a bundle automorphism $\sigma \in \mathrm{End}(E,E)$ for which $\sigma|L^\perp = 1$ and $\sigma|L = -1$, and therefore a decomposition $\nabla = \nabla_E + \Psi$, where

$$\nabla_E = \tfrac{1}{2}(\nabla + \sigma\nabla\sigma), \quad \Psi = \tfrac{1}{2}(\nabla - \sigma\nabla\sigma).$$

The Higgs field is $\Phi = \Psi^{1,0}$. The harmonic map equation, when paired with the projective flatness of ∇, asserts that $\bar\partial_E \Psi^{1,0} = 0$, and thus the Higgs field satisfies (2.12) when we take $V = L^\perp$.

Two such bundles, (E,∇) and (E',∇'), are projectively equivalent when there is a line bundle $\mathcal{L}$ equipped with a connexion $\nabla_{\mathcal{L}}$ for which $E' \simeq E \otimes \mathcal{L}$ and $\nabla' \simeq \nabla \otimes \nabla_{\mathcal{L}}$ (the induced connexion on the tensor product). In particular, by taking $\mathcal{L} = L^{-1}$ equipped with the connexion obtained from the restriction of ∇ to L, we may assume without loss of generality that $E = V \oplus 1$, where 1 denotes the trivial bundle, and the restriction of ∇ to 1 is the canonical flat connexion. In that case the Toledo invariant of ρ is $-\frac{2}{3}\deg(V)$.

Remark 2.2. The alternative normalisation, used by Xia [49], is to note that since $\deg(E\otimes\mathcal{L}) = \deg(E) + 3\deg(\mathcal{L})$ one can normalise by degree, i.e., insist that $0 \le \deg(E) < 3$. In particular, the topological type of $\mathbb{P}E$ is determined by $\deg(E)$ mod 3, and the representation ρ only lifts to $SU(2,1)$ when there exists an $\mathcal{L}$ for which $E \otimes \mathcal{L} \simeq \mathcal{D} \times_{\hat\rho} \mathbb{C}^{2,1}$ for some representation $\hat\rho : \pi_1\Sigma \to U(2,1)$. This happens if and only if $\deg(E) \equiv 0$ mod 3, i.e., when $\tau \in 2\mathbb{Z}$.

2.3. **Minimal surfaces and their Higgs bundles.** We are now in a position to classify, in terms of Higgs bundle data, the minimal ρ-equivariant surfaces which are neither holomorphic nor anti-holomorphic, when ρ is irreducible.

Theorem 2.3. *An irreducible representation $\rho \in \mathrm{Hom}(\pi_1\Sigma, G)$ admits a ρ-equivariant minimal immersion $f : \mathcal{D} \to \mathbb{CH}^2$ which is neither holomorphic nor anti-holomorphic if and only if it corresponds to a Higgs bundle (E,Φ) for which $E = V \oplus 1$ where V is a rank 2 holomorphic extension*

$$0 \to K^{-1}(D_1) \overset{\Phi_1}{\to} V \overset{\Phi_2}{\to} K(-D_2) \to 0. \tag{2.15}$$

Here D_1, D_2 are non-negative divisors with no common points, whose degrees d_1, d_2 satisfy the stability inequalities

$$2d_1 + d_2 < 6(g-1) \text{ and } d_1 + 2d_2 < 6(g-1), \tag{2.16}$$

where g is the genus of Σ. The Higgs field is given by $\Phi = (\Phi_1, \Phi_2)$, after making the canonical identifications

$$\begin{aligned} \mathrm{Hom}(K^{-1}(D_1), V) &\simeq K\,\mathrm{Hom}(1,V) \otimes \mathcal{O}(-D_1), \\ \mathrm{Hom}(V, K(-D_2)) &\simeq K\,\mathrm{Hom}(V,1) \otimes \mathcal{O}(-D_2). \end{aligned} \tag{2.17}$$

These divisors are, respectively, the divisors of anti-complex and complex points of the minimal immersion f. The representation ρ has $\tau(\rho) = \frac{2}{3}(d_2 - d_1)$.

Proof. Given ρ we choose a projectively flat $\mathbb{C}^{2,1}$ bundle of the form $E = V \oplus 1$, and then f provides holomorphic sections

$$\begin{aligned} \Phi_1 &= \partial f' \in K\,\mathrm{Hom}(1,V) \simeq \mathrm{Hom}(K^{-1}, V), \\ \Phi_2 &= \partial f'' \in K\,\mathrm{Hom}(V,1) \simeq \mathrm{Hom}(V, K), \end{aligned}$$

whose sum is $\Phi = \partial f$. Since f is conformal we have

$$0 = g(\partial f, \partial f) = \tfrac{1}{2}\,\mathrm{tr}(\Phi^2).$$

From this we can show that $\Phi_2 \circ \Phi_1 = 0$, by the following local frame argument. Neither of Φ_1, Φ_2 are identically zero since f is not $\pm$-holomorphic. So, away from its zero locus, the image of Φ_1 is a rank one sub-bundle $V_1 \subset V$. We can locally frame E by sections σ_1, σ_2 of V and σ_3 of 1 such that σ_1 generates V_1, and we may assume this is a $U(2,1)$ frame with respect to the metric on E. It follows that there are locally holomorphic sections a, b, c of K for which

$$\Phi_2(\sigma_1) = a\sigma_3, \quad \Phi_2(\sigma_2) = b\sigma_3, \quad \Phi_1(\sigma_3) = c\sigma_1.$$

Since f is not holomorphic $c \neq 0$. Now

$$\begin{aligned}\mathrm{tr}(\Phi^2) &= \mathrm{tr}(\Phi_1 \circ \Phi_2 + \Phi_2 \circ \Phi_1)\\ &= \langle \Phi_1 \circ \Phi_2(\sigma_1), \sigma_1\rangle + \langle \Phi_1 \circ \Phi_2(\sigma_2), \sigma_2\rangle - \langle \Phi_2 \circ \Phi_1(\sigma_3), \sigma_3\rangle\\ &= 2ac.\end{aligned}$$

Therefore $\mathrm{tr}(\Phi^2) = 0$ implies $a = 0$, i.e., $\Phi_2 \circ \Phi_1 = 0$. Thus

$$K^{-1} \stackrel{\Phi_1}{\to} V \stackrel{\Phi_2}{\to} K,$$

has the image of Φ_1 in the kernel of Φ_2. Now Φ_1 vanishes precisely on anti-complex points, while Φ_2 vanishes precisely on complex points, so we have

$$0 \to K^{-1}(D_1) \stackrel{\Phi_1}{\to} V \stackrel{\Phi_2}{\to} K(-D_2) \to 0.$$

This must be exact at the middle since V has rank 2.

For stability we need to identify the Φ-invariant sub-bundles. With respect to the local frame $\sigma_1, \sigma_2, \sigma_3$ above Φ is represented by the matrix

$$\begin{pmatrix} 0 & 0 & c\\ 0 & 0 & 0\\ 0 & b & 0\end{pmatrix}.$$

It follows that the only Φ-invariant proper sub-bundles of E are the image V_1 of Φ_1 and $V_1 \oplus 1$. So stability requires

$$\deg(V_1) < \tfrac{1}{3}\deg(E), \quad \tfrac{1}{2}\deg(V_1) < \tfrac{1}{3}\deg(E).$$

Since $\deg(V_1) = d_1 - 2(g-1)$ and $\deg(E) = d_1 - d_2$ these inequalities are equivalent to the inequalities (2.16).

Reversing the argument is straightforward: when the Higgs bundle has this form we have $\Phi_2 \circ \Phi_1 = 0$, hence $\mathrm{tr}(\Phi^2) = 0$. So when the Higgs bundle is stable the sub-bundle $1 \subset E$ provides a conformal harmonic ρ-equivariant map f for some irreducible ρ. □

The proof works perfectly well in the case where D_1, D_2 have common points, in which case the map f is a branched minimal immersion with branch points on $D_1 \cap D_2$. We will follow common usage and say D_1, D_2 are *co-prime* when they are disjoint.

The Higgs bundle data which appears in the previous theorem can be written as a quadruple $(\Sigma_c, D_1, D_2, \xi)$, where $\xi \in H^1(\Sigma_c, K^{-2}(D_1 + D_2))$ is the extension class of (2.15). The theorem shows that this data determines the pair (ρ, f) up to G-equivalance, i.e., up to the simultaneous action of G by conjugacy of ρ and ambient isometry of f. The exact sequence (2.15), which gives us V and Φ_1, Φ_2, is completely and uniquely determined by its extension class ξ [25, Ch. 5]. Moreover, since the isomorphisms (2.17) require D_1, D_2, not just their linear equivalence classes, the assignment from $(\Sigma_c, D_1, D_2, \xi)$ to the G-equivalance class of (ρ, f) is bijective. The moduli space parametrised by this data will be described in §6.

Remark 2.3. The proof above shows that when (E,Φ) has $\mathrm{tr}(\Phi^2)=0$ it cannot split into a direct sum of Φ-invariant sub-bundles unless one of Φ_1 or Φ_2 is identically zero. It follows that a reducible ρ can only admit ρ-equivariant minimal surfaces which are holomorphic or anti-holomorphic. We describe all these reducible representations in Remark 5.1 below. However, there are also irreducible representations which admit holomorphic or anti-holomorphic surfaces: we give a complete classification in §5.

3. The Higgs bundle data in terms of minimal surface data.

In this section we will give the explicit correspondence between the minimal surface data $(\gamma_1,\gamma_2,\mathcal{Q})$ and the Higgs bundle data (Σ_c,D_1,D_2,ξ) of the previous section. This is achieved by exploiting the harmonic sequence for a minimal surface in $\mathbb{CH}^2$. It provides us with a preferred system of local $U(2,1)$ frames for the bundle E in Theorem 2.3, and which we will call *Toda frames*. These frames are explictly determined by the minimal surface data, and through them we calculate the extension class ξ of the bundle V in Theorem 2.3. The correspondence between the Higgs data and the minimal surface data then comes through the Dolbeault isomorphism

$$H^1(\Sigma_c,K^{-2}(D_1+D_2))\simeq H^{0,1}(\Sigma_c,K^{-2}(D_1+D_2)).$$

Theorem 3.1. *Let the pair (ρ,f) correspond to the Higgs data (Σ_c,D_1,D_2,ξ) as in the previous section. Let $\gamma_1,\gamma_2,\mathcal{Q}$ be the minimal surface data determined by f through* (2.6) *and* (2.11). *Then the extension class ξ corresponds, under the Dolbeault isomorphism, to the cohomology class*

$$-\left[\frac{\bar{\mathcal{Q}}}{\gamma_1\gamma_2}\right]\in H^{0,1}(\Sigma_c,K^{-2}(D_1+D_2)).$$

Moreover $\xi=0$ if and only if $\mathcal{Q}=0$.

In particular, this means that $(\gamma_1,\gamma_2,\mathcal{Q})$ determines (Σ_c,D_1,D_2,ξ), since we get the conformal structure of the induced metric and the divisors D_1,D_2 from γ_1,γ_2. Therefore Theorems 2.3 and 3.1 together have the following corollary.

Corollary 3.2. *If two ρ-equivariant minimal surfaces have the same data $(\gamma_1,\gamma_2,\mathcal{Q})$ then they are identical up to ambient isometry.*

Before we begin the proof of Theorem 3.1 we describe the local Toda frames which link the minimal surface data to the local geometry of the Higgs bundle. Fix a pair (ρ,f) and let $E=V\oplus 1$ be the Higgs bundle over Σ_c corresponding to them as above, equipped with its Higgs field Φ, its $\mathbb{C}^{2,1}$-metric and the Chern connexion ∇_E.

Lemma 3.3. *Let (U,z) be holomorphic chart on Σ_c for which U contains no complex or anti-complex points of f. Then over U there is a local trivialisation $\varphi:E|U\to U\times\mathbb{C}^3$ for which*

$$\varphi\circ\bar\partial_E\circ\varphi^{-1}=d\bar z\left[\frac{\partial}{\partial\bar z}+\begin{pmatrix}-\bar Z\log u_1 & -\bar{Q}/u_1u_2 & 0\\ 0 & \bar Z\log u_2 & 0\\ 0 & 0 & 0\end{pmatrix}\right] \tag{3.1}$$

and

$$\varphi\circ\Phi\circ\varphi^{-1}=\begin{pmatrix}0 & 0 & u_1\\ 0 & 0 & 0\\ 0 & u_2 & 0\end{pmatrix}dz, \tag{3.2}$$

where $\bar Z=\partial/\partial\bar z$ and u_1,u_2,Q are given by (2.5) *and* (2.11).

Proof. For notational convenience, let us set $\ell_0 = 1 \subset E$. Then $V = \ell_0^\perp$ with respect to the $\mathbb{C}^{2,1}$-metric, and ℓ_0 is the section of $\mathbb{P}E_-$ which represents f. Thus $\partial f', \bar\partial f' \in \Omega^1_\Sigma(\mathrm{Hom}(\ell_0, \ell_0^\perp))$ and they determine line sub-bundles $\ell_1, \ell_2 \subset \ell_0^\perp$ via their images, i.e.,

$$\ell_2 \otimes \bar K \overset{\bar\partial f'}{\leftarrow} \ell_0 \overset{\partial f'}{\rightarrow} \ell_1 \otimes K. \tag{3.3}$$

These two sub-bundles are orthogonal since f is conformal. We give each of these line bundles the holomorphic structure it inherits from $\bar\partial_E$, i.e., a local section σ_j of ℓ_j is holomorphic when $\pi_j \bar\partial_E \sigma_j = 0$, where $\pi_j : E \to \ell_j$ is the orthogonal projection. Note that since $\Phi \in \mathrm{Hom}(\ell_0, \ell_0^\perp)$ this holomorphic structure on each ℓ_j agrees with the one induced by the projectively flat connexion $\nabla = \nabla_E + \Phi + \Phi^\dagger$. Then $\partial f' = \Phi_1$ is a holomorphic map, while $\bar\partial f' = \Phi_2^\dagger$ is anti-holomorphic, since f is harmonic. Since ∇ induces the canonical flat connexion on 1 we can choose a globally flat section f_0 of 1, i.e., $\langle \nabla f_0, f_0\rangle = 0$ with $\langle f_0, f_0\rangle = -1$.

Now fix a chart (U, z) and for any section σ of $E|U$ write $X\sigma$ to mean $\nabla_X \sigma$ with respect to the projectively flat connexion ∇. Then the maps above can be written locally as

$$\ell_2 \overset{\pi_0^\perp \bar Z}{\leftarrow} \ell_0 \overset{\pi_0^\perp Z}{\rightarrow} \ell_1.$$

Define local sections $\sigma_j \in \Gamma(U, \ell_j)$, $j = 1, 2$ by

$$\sigma_1 = \partial f'(Z) f_0 = \pi_0^\perp Z f_0, \quad \sigma_2 = \bar\partial f'(\bar Z) f_0 = \pi_0^\perp \bar Z f_0.$$

We assume that U does not contain any complex or anti-complex points, and therefore the functions u_1, u_2 in (2.5) are non-vanishing. Clearly $|\sigma_j| = u_j$, and we set $f_j = \sigma_j / u_j$. Then f_1, f_2, f_0 is a $U(2,1)$ frame for E. We claim that these satisfy the equations

$$Z f_1 = (Z \log u_1) f_1 + (Q/u_1 u_2) f_2, \tag{3.4}$$
$$Z f_2 = -(Z \log u_2) f_2 + u_2 f_0, \tag{3.5}$$
$$Z f_0 = u_1 f_1. \tag{3.6}$$

The last of these is obvious, since $Z f_0 = \pi_0^\perp Z f_0$.

Next consider

$$Z f_1 = \langle Z f_1, f_1\rangle + \langle Z f_1, f_2\rangle.$$

Since f_0 is holomorphic, so is σ_1, and therefore $Z\langle \sigma_1, \sigma_1\rangle = \langle Z\sigma_1, \sigma_1\rangle$. Thus

$$\begin{aligned}\langle Z f_1, f_1\rangle &= u_1^{-1} \langle Z(u_1^{-1}\sigma_1), \sigma_1\rangle \\ &= -Z \log(u_1) + u_1^{-2} Z \log\langle \sigma_1, \sigma_1\rangle \\ &= Z \log(u_1).\end{aligned}$$

Now

$$\begin{aligned}\langle Z f_1, f_2\rangle &= \langle Z(\sigma_1/u_1), \sigma_2/u_2\rangle \\ &= \langle Z\sigma_1, \sigma_2\rangle / u_1 u_2 \\ &= \langle \pi_0^\perp Z \pi_0^\perp Z f_0, \pi_0^\perp \bar Z f_0\rangle / u_1 u_2.\end{aligned}$$

On the other hand, using (2.11) and the fact that $\nabla^{\mathbb{CH}^2}$ is the connexion on $\mathrm{Hom}(\ell_0, \ell_0^\perp)$ induced by the connexions on each bundle $\ell_0, \ell_0^\perp$, we have

$$\begin{aligned} Q &= \langle [\nabla_Z^{\mathbb{CH}^2} \partial f'(Z)] f_0, \bar\partial f'(\bar Z)) f_0\rangle \\ &= \langle \pi_0^\perp Z[\partial f'(Z) f_0] - \partial f'(Z)[\pi_0 Z f_0], \bar\partial f'(\bar Z)) f_0\rangle \\ &= \langle \pi_0^\perp Z \pi_0^\perp Z f_0, \pi_0^\perp \bar Z f_0\rangle - \langle \pi_0^\perp Z \pi_0 Z f_0, \pi_0^\perp \bar Z f_0\rangle. \end{aligned} \tag{3.7}$$

The second term in the last line vanishes since ℓ_{-1} is orthogonal to ℓ_1. Thus $\langle Z f_1, f_2\rangle = Q/u_1u_2$.

Finally, consider

$$Z f_2 = \langle Z f_2, f_2\rangle f_2 - \langle Z f_2, f_0\rangle f_0.$$

For the first term we note that σ_2 is anti-holomorphic since f_0 is, so

$$0 = \langle Z\sigma_2, f_2\rangle = Z u_2 + u_2\langle Z f_2, f_2\rangle.$$

Since $\langle f_2, f_0\rangle = 0$ the second term yields

$$-\langle Z f_2, f_0\rangle f_0 = \langle f_2, \bar Z f_0\rangle = \langle f_2, \sigma_2\rangle = u_2.$$

Now that we have established the equations for the frame f_1, f_2, f_0, we take φ to be the corresponding trivialisation. In this frame the equations (3.4)-(3.6) show us that

$$\varphi \circ \nabla^{1,0} \circ \varphi^{-1} = dz \left(\frac{\partial}{\partial z} + \begin{pmatrix} Z\log u_1 & 0 & u_1 \\ Q/u_1u_2 & -Z\log u_2 & 0 \\ 0 & u_2 & 0 \end{pmatrix} \right) \tag{3.8}$$

Thus (3.1) and (3.2) follow. □

At complex or anti-complex points we require a slight adjustment of the frame above. We may choose the chart (U, z) so that U contains precisely one complex or anti-complex point, at $z = 0$. Thus one of σ_1, σ_2 vanishes at $z = 0$. To treat all cases simultaneously, let p, q be the non-negative integers for which $z^{-q}\sigma_1$ and $\bar z^{-p}\sigma_2$ are holomorphic and do not vanish at $z = 0$: at most one of p, q is non-zero. It follows that the functions $u_1/|z|^q = |z^{-q}\sigma_1|$ and $u_2/|z|^p = |\bar z^{-p}\sigma_2|$ do not vanish. By setting

$$\tilde f_1 = \frac{z^{-q}\sigma_1}{|z^{-q}\sigma_1|} = (|z|/z)^q f_1, \quad \tilde f_2 = (|z|/\bar z)^p f_2 = (z/|z|)^p f_2,$$

we obtain a $U(2)$ frame $\tilde f_1, \tilde f_2$ for V throughout U, with corresponding trivialisation $\tilde\varphi$. This is easiest to work with by writing it as

$$\tilde\varphi = S(\varphi|V), \quad S = \begin{pmatrix} (z/|z|)^q & 0 \\ 0 & (|z|/z)^p \end{pmatrix}. \tag{3.9}$$

The next step towards Theorem 3.1 is to represent the extension class ξ of V using a 1-cocycle with values in $K^{-2}(D_1 + D_2)$. For this computation, we choose an atlas $\mathcal{U} = \{(U_j, z_j)\}$ for Σ_c of contractible charts for which each complex or anti-complex point lies only in one chart, at $z_j = 0$, and for simplicity assume charts containing distinct complex or anti-complex points are disjoint.

Lemma 3.4. *There is a holomorphic atlas $\mathcal{U} = \{(U_j, z_j)\}$ for Σ_c in which ξ is represented by the 1-cocycle $\{(\xi_{jk}, U_j, U_k)\} \in H^1(\mathcal{U}, K^{-2}(D_1 + D_2))$ for which*

$$\xi_{jk} = a_j z^{-(p_j+q_j)} dz_j^{-2} - a_k dz_k^{-2} \text{ on } U_j \cap U_k, \tag{3.10}$$

for smooth functions a_j on U_j satisfying

$$\partial a_j/\partial \bar{z}_j = -\frac{\bar{Q}_j z_j^{p_j+q_j}}{u_{1j}^2 u_{2j}^2},$$

(assuming U_k contains no complex or anti-complex points).

Proof. For simplicity, set $w = z/|z|$. Using the previous lemma and the transformation (3.9) we have, on V over a single chart (U, z),

$$\tilde{\varphi} \circ \bar{\partial}_E \circ \tilde{\varphi}^{-1} = d\bar{z} \left[\frac{\partial}{\partial \bar{z}} + \begin{pmatrix} -\bar{Z}\log(u_1/|z|^q) & -\bar{Q}w^{p+q}/u_1u_2 \\ 0 & \bar{Z}\log(u_2/|z|^p) \end{pmatrix} \right].$$

Now suppose $U_j \cap U_k \neq \emptyset$, and assume without loss of generality that U_k contains no complex or anti-complex points, so that $p_k = 0 = q_k$ and $\tilde{\varphi}_k = \varphi_k|V$. Then we have transition relations $\tilde{\varphi}_j = c_{jk}\tilde{\varphi}_k$ where

$$c_{jk} = \begin{pmatrix} w_j^{q_j} \frac{dz_j/dz_k}{|dz_j/dz_k|} & 0 \\ 0 & w_j^{-p_j} \frac{|dz_j/dz_k|}{dz_j/dz_k} \end{pmatrix},$$

where we have used that fact that

$$\frac{d\bar{z}_j/d\bar{z}_k}{|d\bar{z}_j/d\bar{z}_k|} = \overline{\frac{dz_j/dz_k}{|dz_j/dz_k|}} = \frac{|dz_j/dz_k|}{dz_j/dz_k}.$$

It follows that, as a smooth bundle, $V \simeq K^{-1}(D_1) \oplus K(-D_2)$.

To elucidate its holomorphic structure we find local trivialisations χ_j in which

$$\chi_j \circ \bar{\partial}_E \circ \chi_j^{-1} = d\bar{z}_j \frac{\partial}{\partial \bar{z}_j},$$

i.e., we seek local gauge transformations $\chi_j = R_j\tilde{\varphi}_j$ for which

$$R_j[\tilde{\varphi} \circ \bar{\partial}_E \circ \tilde{\varphi}^{-1}]R_j^{-1} = d\bar{z}_j \frac{\partial}{\partial \bar{z}_j}.$$

A straightforward calculation shows that this is achieved by taking

$$R_j = \begin{pmatrix} 1 & a_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (u_{1j}/|z_j|^{q_j})^{-1} & 0 \\ 0 & u_{2j}/|z_j|^{p_j} \end{pmatrix},$$

where

$$\frac{\partial a_j}{\partial \bar{z}_j} = -\frac{\bar{Q}_j z^{p_j+q_j}}{u_{1j}^2 u_{2j}^2}, \tag{3.11}$$

throughout U_j. Such a function a_j exists, by the $\bar{\partial}$-Poincaré Lemma, because the right hand side is smooth throughout U_j since Q_j vanishes to at least order $p_j + q_j$ at $z_j = 0$ by (2.11). Thus the transition between two such charts $\chi_j = b_{jk}\chi_k$ is given by

$$b_{jk} = R_j c_{jk} R_k^{-1} = \begin{pmatrix} z_j^{q_j} dz_j/dz_k & \lambda_{jk} \\ 0 & z_j^{-p_j} dz_k/dz_j \end{pmatrix},$$

where

$$\lambda_{jk} = a_j z_j^{-p_j} \frac{dz_k}{dz_j} - a_k z_j^{q_j} \frac{dz_j}{dz_k},$$

and this is holomorphic on $U_j \cap U_k$ (this follows easily from (3.11)). Now using the same convention as [25, p74] this determines the 1-cocycle with values in $K^{-2}(D_1 + D_2)$ given by

$$\begin{aligned}\xi_{jk} &= (z_j^{q_j} \frac{dz_j}{dz_k})^{-1} \lambda_{jk} dz_k^{-2} \\ &= a_j z_j^{-(p_j+q_j)} dz_j^{-2} - a_k dz_k^{-2}.\end{aligned}$$

□

Proof of Theorem 3.1. For notational simplicity, set $\mathcal{L} = K^{-2}(D_1 + D_2)$. In U_j the quantity

$$z_j^{-(p_j+q_j)} dz_j^{-2}$$

is a local holomorphic section of $\mathcal{L}$. Therefore we have local smooth sections

$$\eta_j = a_j z_j^{-(p_j+q_j)} dz_j^{-2} \in \Gamma(U_j, \mathcal{E}^0(\mathcal{L})),$$

which provide a 0-cochain η for the sheaf $\mathcal{E}^0(\mathcal{L})$ of locally smooth sections of $\mathcal{L}$. By (3.10) the 1-cocyle $\{(\xi_{jk}, U_j, U_k)\}$ is, as a smooth cocycle, the coboundary of η. Now recall that the Dolbeault isomorphism $H^1(\mathcal{U}, \mathcal{L}) \to H^{0,1}(\mathcal{U}, \mathcal{L})$ is derived from the short exact sequence

$$0 \to \mathcal{O}(\mathcal{L}) \to \mathcal{E}^0(\mathcal{L}) \stackrel{\bar\partial}{\to} \mathcal{E}^{0,1}(\mathcal{L}) \stackrel{\bar\partial}{\to} 0,$$

by taking the cohomology class of $\bar\partial\eta$. But

$$\bar\partial\eta_j = -\frac{\bar{Q}_j}{u_{1j}^2 u_{2j}^2} dz_j^{-2} d\bar{z}_j.$$

So $\bar\partial\eta$ is represented by the cohomology class of the smooth form

$$-\bar{\mathcal{Q}}/\gamma_1\gamma_2 \in \Gamma(\Sigma_c, \mathcal{E}^{0,1}(\mathcal{L})).$$

Finally, we show that this vanishes when it is $\bar\partial$-exact, by showing that it is harmonic with respect to Hermitian metric $B = \gamma_1\gamma_2$ on $\mathcal{L}$. With respect to the Hodge inner product on $\mathcal{E}^{*,*}(\mathcal{L})$ determined by B on $\mathcal{L}$ and the induced metric γ on Σ_c, the adjoint of $\bar\partial : \mathcal{E}^0(\mathcal{L}) \to \mathcal{E}^{0,1}(\mathcal{L})$ is given by

$$\mathcal{E}^{0,1}(\mathcal{L}) \stackrel{\bar\partial^*}{\to} \mathcal{E}^0(\mathcal{L}); \quad \bar\partial^* = -\bar\star\bar\partial\bar\star,$$

where $\bar\star$ is the conjugate linear Hodge star operator for our choice of metrics (see, for example, [46, p168]). It therefore suffices for us to show that

$$\bar\partial\bar\star(\bar{\mathcal{Q}}/\gamma_1\gamma_2) = 0.$$

Let (U, z) be a chart in which the divisor $D_1 + D_2$ has at most one point, of degree d at $z = 0$. Then $\tau = z^{-d} dz^{-2}$ is a local holomorphic trivialising section of $\mathcal{L}$ over U and

$$\mathcal{E}^{0,1}(U, \mathcal{L}) \stackrel{\bar\star}{\to} \mathcal{E}^{1,0}(U, \mathcal{L}^*); \quad a\tau d\bar{z} \mapsto -i\bar{a}B(\cdot, \tau)dz,$$

for any locally smooth complex function a over U. Now in U we write

$$\bar{\mathcal{Q}}/\gamma_1\gamma_2 = \frac{\bar{Q}z^d}{u_1^2 u_2^2} \tau d\bar{z},$$

and therefore

$$\bar\partial\bar\star(\bar{\mathcal{Q}}/\gamma_1\gamma_2) = -i\frac{\partial}{\partial\bar{z}}\left(\frac{Q\bar{z}^d\|\tau\|^2}{u_1^2 u_2^2}\right)(z^d dz^2) d\bar{z} \wedge dz.$$

But

$$\|\tau\|^2 = B(\tau,\tau) = |z|^{-2d}u_1^2u_2^2,$$

and therefore $\bar\partial\bar\star(\bar{\mathcal{Q}}/\gamma_1\gamma_2) = 0$ throughout U since Q is holomorphic. □

We finish this section by giving a global expression which relates the curvatures of the immersion f to the norms of $\mathcal{Q}$ and $\mathcal{Q}/\gamma_1\gamma_2$ with respect to the induced metric. This in turn provides an area bound for such immersions.

First we note that a straightfoward calculation using (3.8) shows that the projective flatness of the connexion ∇ is equivalent to the local equations

$$Z\bar{Z}\log u_1^2 = 2u_1^2 + |Q|^2/u_1^2u_2^2 - u_2^2, \tag{3.12}$$

$$Z\bar{Z}\log u_2^2 = 2u_2^2 + |Q|^2/u_1^2u_2^2 - u_1^2, \tag{3.13}$$

in a chart (U,z) containing no complex or anti-complex points. These are the appropriate version of the Toda equations for this geometry (cf. [4] for the $\mathbb{CP}^n$ version). They are also related to the two equations Wolfson derived for the *Kähler angle* in [47]. Recall that the Kähler angle is a continuous function $\theta : \Sigma_c \to \mathbb{R}$ for which $f^*\omega = \cos(\theta)v_\gamma$, where v_γ is the area form for the induced metric. Wolfson showed that, except at complex or anti-complex points where θ is not differentiable,

$$i\partial\bar\partial\log\tan^2(\theta/2) = f^*\,\mathrm{Ric}, \tag{3.14}$$

$$i\partial\bar\partial\log\sin^2(\theta) = (\kappa_\gamma + \kappa^\perp)v_\gamma, \tag{3.15}$$

where Ric is the Ricci form and $\kappa_\gamma, \kappa^\perp$ are the Gaussian and normal curvatures of the immersion respectively. In our case we have, from (2.4), $\cos(\theta) = (u_1^1 - u_2^2)/(u_1^2+u_2^2)$. Therefore

$$\tan^2(\theta/2) = \frac{u_2^2}{u_1^2}, \quad \sin^2(\theta) = \frac{4u_1^2u_2^2}{(u_1^2+u_2^2)^2},$$

and (3.14) is just the difference of (3.12) and (3.13) since $f^*\,\mathrm{Ric} = -6f^*\omega$. Now

$$\begin{aligned} Z\bar{Z}\log\sin^2\theta &= Z\bar{Z}u_1^2 + Z\bar{Z}u_2^2 - 2Z\bar{Z}\log(u_1^2+u_2^2) \\ &= u_1^2 + \frac{2|Q|^2}{u_1^2u_2^2} + u_2^2 - \frac{2}{u_1^2+u_2^2}[Z\bar{Z}\log(u_1^2+u_2^2)](u_1^2+u_2^2), \end{aligned}$$

using (3.12) and (3.13). The second term on the right contains the local expression for κ_γ, so substituting this equation into (3.15) reveals that

$$\kappa^\perp - \kappa_\gamma = 2\left(1 + \frac{2|Q|^2}{u_1^2u_2^2(u_1^2+u_2^2)}\right).$$

The right hand side can be written in terms of the globally defined quantities

$$\|\mathcal{Q}\|_\gamma = \frac{2\sqrt{2}|Q|}{(u_1^2+u_2^2)^{3/2}}, \quad \|\frac{\mathcal{Q}}{\gamma_1\gamma_2}\|_\gamma = \frac{|Q|\sqrt{u_1^2+u_2^2}}{\sqrt{2}u_1^2u_2^2},$$

to obtain

$$\kappa^\perp - \kappa_\gamma = 2(1 + \|\frac{\mathcal{Q}}{\gamma_1\gamma_2}\|_\gamma\|\mathcal{Q}\|_\gamma). \tag{3.16}$$

This must hold globally, since all the terms are smooth on Σ. By integration over Σ, and using (2.8), we arrive at the following conclusion.

Proposition 3.5. *Let f be a ρ-equivariant minimal immersion which is neither holomorphic nor anti-holomorphic, with induced metric γ, cubic holomorphic differential $\mathcal{Q}$, d_1 anti-complex points and d_2 complex points. Then*

$$(4(g-1)-d_1-d_2)\pi \geq \mathrm{Area}_\gamma(\Sigma) + \int_\Sigma \|\mathcal{Q}\|_\gamma^2 v_\gamma, \tag{3.17}$$

with equality if and only if either $\mathcal{Q}=0$ or when f is Lagrangian.

Note that the stability inequalities (2.16) confirm that the left hand side is positive. The last statement follows because if $\mathcal{Q}\neq 0$ equality requires $\|\mathcal{Q}/\gamma_1\gamma_2\|_\gamma = \|\mathcal{Q}\|_\gamma$, which in turn requires $u_1 = u_2$, whence $\cos(\theta)=0$.

Remark 3.1. The local equations (3.12), (3.13) are clearly invariant under any unimodular scaling $Q\mapsto e^{i\alpha}Q$. Globally this corresponds to the symmetry $\mathcal{Q}\mapsto e^{i\alpha}\mathcal{Q}$, and by Theorem 3.1 this corresponds in turn to $\xi\mapsto e^{-i\alpha}\xi$. In fact this is equivalent to the well-known symmetry of the Higgs bundle equations $\Phi\mapsto e^{i\psi}\Phi$ (taking $\alpha=2\psi$) which Hitchin showed is a Hamiltonian circle action on the moduli space of $SL(2,\mathbb{C})$-Higgs bundles [28]. To see this equivalence it is enough to perform the following gauge transformation for $\partial_E + e^{i\psi}\Phi$ using the local gauge (3.8):

$$\begin{aligned}&\mathrm{Ad}\begin{pmatrix} e^{-2i\psi}0 & 0 & \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\psi}\end{pmatrix}\cdot\left(\frac{\partial}{\partial z} + \begin{pmatrix} Z\log u_1 & 0 & e^{i\psi}u_1 \\ Q/u_1u_2 & -Z\log u_2 & 0 \\ 0 & e^{i\psi}u_2 & 0\end{pmatrix}\right)\\ = \;&\frac{\partial}{\partial z} + \begin{pmatrix} Z\log u_1 & 0 & u_1 \\ e^{2i\psi}Q/u_1u_2 & -Z\log u_2 & 0 \\ 0 & u_2 & 0\end{pmatrix}\end{aligned} \tag{3.18}$$

Note in particular that, unlike the $SL(2,\mathbb{C})$ case, the map $(E,\Phi)\mapsto(E,-\Phi)$ fixes every $PU(2,1)$-Higgs bundle since $\partial_E+\Phi$ and $\partial_E-\Phi$ are gauge equivalent (by the symmetric space involution).

4. Minimal Lagrangian immersions.

By Wolfson's theorem [47, Thm 2.1] Theorem 2.3 yields minimal Lagrangian immersions when both divisors D_1, D_1 are zero. Therefore Theorem 2.3 implies the following characterisation of all equivariant minimal Lagrangian immersions.

Corollary 4.1. *Given a closed oriented surface Σ of hyperbolic type, minimal Lagrangian immersions $f:\mathcal{D}\to\mathbb{CH}^2$ which are equivariant with respect to an irreducible representation of $\pi_1\Sigma$ into $PU(2,1)$ are in one-to-one correspondence with the Higgs data (Σ_c,ξ), where c is a point in Teichmüller space and $\xi\in H^{0,1}(\Sigma_c,K^{-2})$.*

It is not strictly necessary to say that ρ is irreducible in this statement: this is implied by a combination of Corlette's result that twisted harmonic maps correspond to reductive representations [9] and Remark 2.3.

For a Lagrangian immersion the Kähler angle satisfies $\sin\theta = 1$, so that (3.15) implies $\kappa^\perp = -\kappa_\gamma$. Moreover, $\|\mathcal{Q}\|_\gamma^2 = 2\|A_f\|^2$, where A_f is the shape operator of f [37, Lemma 2.8] and

$$\|A_f\|^2 = \sup\{\tfrac{1}{2}(\mathrm{tr}_\gamma A_f(\nu)^2) : \nu\in T_p\mathcal{D}^\perp\}.$$

Since $\mathbb{CH}^2$ has constant Lagrangian sectional curvature -1, (3.16) reduces to the Gauss (and Ricci) equation for minimal Lagrangian immersions:

$$-1 = \kappa_\gamma + 2\|A_f\|^2. \tag{4.1}$$

In [36] we wrote this as an equation for the conformal factor $\gamma = e^u\mu$ of the induced metric with respect to the hyperbolic metric μ:

$$\Delta_\mu u - 2\|\mathcal{Q}\|_\mu^2 e^{-2u} - 2e^u + 2 = 0. \tag{4.2}$$

We gave existence results for this in terms of pairs $(\Sigma_c, \mathcal{Q})$, and showed that there is a constant k, independent of $c \in \mathcal{T}_g$, for which $\|\mathcal{Q}\|_\mu \le k$ yields a minimal ρ-equivariant embedding for which the normal bundle exponential map $\exp^\perp : T\mathcal{D}^\perp \to \mathbb{CH}^2$ is a diffeomorphism. We then showed that ρ is quasi-Fuchsian, since the image under $\exp^\perp$ of a fundamental domain for $\pi_1\Sigma$ gives a globally finite fundamental domain for ρ. Taking $\mathcal{Q} = 0$ gives the $\mathbb{R}$-Fuchsian representations, i.e., those which factor through the canonical inclusion $PO(2,1) \to PU(2,1)$. It is therefore convenient to adopt here the following terminology[1].

Definition 4.2. *A representation $\rho \in \mathrm{Hom}(\pi_1, PU(2,1))$ will be called* almost $\mathbb{R}$-Fuchsian *if it admits a ρ-equivariant minimal Lagrangian embedding $f : \mathcal{D} \to \mathbb{CH}^2$ whose normal bundle exponential map $\exp^\perp$ is a diffeomorphism. We will call f an* almost $\mathbb{R}$-Fuchsian embedding.

A necessary and sufficient condition for $\exp^\perp$ to be an immersion is that $\|\mathcal{Q}\|_\gamma^2 \le 2$ [36, Thm 7.1]. The following theorem improves substantially on the existence results in [36] by showing that almost $\mathbb{R}$-Fuchsian immersions exist, and are the unique minimal immersion, up to this optimal bound on $\mathcal{Q}$: this is the direct analogue of Uhlenbeck's result [43, Thm 3.3] for almost Fuchsian minimal surfaces in $\mathbb{RH}^3$.

Theorem 4.3. *Let f be a ρ-equivariant minimal Lagrangian immersion for which $\|\mathcal{Q}\|_\gamma^2 < 2$. Then f is an almost $\mathbb{R}$-Fuchsian embedding (and ρ is almost $\mathbb{R}$-Fuchsian). Moreover, f is the unique ρ-equivariant minimal immersion.*

Remark 4.1. This theorem should also be compared with the parametrisation of hyperbolic affine sphere immersions $\mathcal{D} \to \mathbb{R}^3$ equivariant with respect to an irreducible representation into $PSL(3,\mathbb{R})$, or equally, to the parametrisation of all convex real projective structures on Σ [44, 35, 34]. The data is a pair $(\Sigma_c, \mathcal{Q})$ where $\mathcal{Q}$ is a cubic holomorphic differential. By a theorem of Choi & Goldman [8] this data parametrises an entire component, the Hitchin component, of the representation space $\mathcal{R}(SL(3,\mathbb{R}))$. Labourie [34] directly related the hyperbolic affine sphere data to the Higgs bundles identified by Hitchin in [29]. In that case the Hitchin component is parametrized by Higgs bundles over Σ_c with bundle $K^{-1} \oplus 1 \oplus K$ and Higgs field

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mathcal{Q} & 0 & 0 \end{pmatrix}.$$

Theorem 4.3 shows that Corollary 4.1 provides a similar parametrisation of the almost Fuchsian locus inside the $\tau = 0$ component $\mathcal{R}(PU(2,1))$. It is unlikely, however, to parametrise the entire component. In the analogous case of equivariant minimal surfaces in $\mathbb{RH}^3$, there are examples of quasi-Fuchisan hyperbolic 3-manifolds which admit more than one minimal surface [1, 31].

[1] Similar terminology is used in the study of minimal surfaces in $\mathbb{RH}^3$, where it was introduced in [33].

We will break the proof of Theorem 4.3 into two parts, starting with the proof that f is an embedding. For this we need to recall from [36, §7] an explicit expression for $\exp^\perp : T\mathcal{D}^\perp \to \mathbb{CH}^2$ when f is minimal Lagrangian. In this case the frame f_1, f_2, f_0 used in the proof of Theorem 3.1 has the simple form

$$f_1 = \frac{1}{s}(f_0)_z, \quad f_2 = \frac{1}{s}(f_0)_{\bar z}, \quad s = u_1 = u_2.$$

Let $S_- \subset \mathbb{C}^{2,1}_-$ be the pseudo-sphere into which f_0 maps, and $\pi : S_- \to \mathbb{CH}^2$ be the projection. When f is Lagrangian $T\mathcal{D}^\perp = Jf_*T\mathcal{D}$ is a Lagrangian 2-plane in $T\mathbb{CH}^2$. This implies that $\exp^\perp(T_z\mathcal{D}^\perp)$ is a totally geodesic Lagrangian disc normal to $f(\mathcal{D})$ at $f(z)$. This Lagrangian disc has the form

$$\{\pi[(ia-b)f_1(z) + (ia+b)f_2(z) + f_0(z)] : a^2 + b^2 < \tfrac{1}{2}\}.$$

Let $\Delta \subset \mathbb{C}$ denote the open disc of radius 1/2. Then we have an identification between $T\mathcal{D}^\perp$ and $\mathcal{D} \times \Delta$ for which $\exp^\perp$ is represented by the map

$$\Theta : \mathcal{D} \times \Delta \to \mathbb{CH}^2; \quad \Theta(z,w) = \pi(-\bar w f_1(z) + w f_2(z) + f_0(z)).$$

The following result improves [36, Thm 8.1].

Lemma 4.4. *When $\|\mathcal{Q}\|^2_\gamma < 2$ the pullback metric Θ^*g is complete, and therefore Θ is a proper map and hence a diffeomorphism. In that case f is an embedding and ρ is almost $\mathbb{R}$-Fuchsian.*

Proof. Fix a point $p \in \mathcal{D}$. We can normalise the frame f_1, f_2, f_0 so that these are the standard basis vectors e_1, e_2, e_3 at p, and choose a conformal normal coordinate z centred at p so that $\gamma(p) = |dz|^2$ (i.e., $s(0) = 1/\sqrt{2}$) with $s_z = 0 = s_{\bar z}$ at this point. We may also rotate z so that $Q_0 = Q(p)$ is real and non-negative. Since $\mathcal{Q} = Q_0 dz^3$ and $\|dz\| = \sqrt{2}$ we have $0 \le Q_0 < \frac{1}{2}$. With such choices, in [36, §7], we computed the differential of Θ (in affine coordinates) at p to be given by

$$\begin{pmatrix} d\Theta_1 \\ d\bar\Theta_1 \\ d\Theta_2 \\ d\bar\Theta_2 \end{pmatrix} = \begin{pmatrix} l & k & 0 & -1 \\ \bar k & l & -1 & 0 \\ \bar k & l & 1 & 0 \\ l & k & 0 & 1 \end{pmatrix} \begin{pmatrix} dz \\ d\bar z \\ dw \\ d\bar w \end{pmatrix},$$

where

$$l = \tfrac{1}{\sqrt{2}}(1 + |w|^2), \quad k = -2Q_0 w - \tfrac{1}{\sqrt{2}}\bar w^2.$$

Now set $\phi = ldz + kd\bar{z}$ and notice that in affine coordinates $\Theta(p) = (-\bar{w}, w)$. Then at p we compute the pull-back of g to be

$$\begin{aligned}
(\Theta^* g)_p =& \sum_{i,j=1}^{2} \frac{1}{1-\|\Theta\|^2}\left(\delta_{ij} + \frac{\bar{\Theta}_i\Theta_j}{1-\|\Theta\|^2}\right) d\Theta_i d\bar{\Theta}_j \\
=& \frac{1}{1-2|w|^2}\left[\left(1+\frac{(-w)(-\bar{w})}{1-2|w|^2}\right)(\phi - d\bar{w})(\bar{\phi} - dw)\right. \\
&+ \left(0+\frac{(-w)w}{1-2|w|^2}\right)(\phi - d\bar{w})(\bar{\phi} + d\bar{w}) \\
&+ \left(0+\frac{\bar{w}(-\bar{w})}{1-2|w|^2}\right)(\bar{\phi} + dw)(\bar{\phi} - dw) \\
&+ \left.\left(1+\frac{\bar{w}w)}{1-2|w|^2}\right)(\bar{\phi} + dw)(\phi + d\bar{w})\right] \\
=& \frac{1}{(1-2|w|^2)^2}\left([2|\phi|^2 - (w\phi + \bar{w}\bar{\phi})^2] + [2|dw|^2 + (wd\bar{w} - \bar{w}dw)^2]\right).
\end{aligned} \tag{4.3}$$

Now consider the two terms in this expression:

$$\theta_1 = \frac{1}{(1-2|w|^2)^2}\left(2|\phi|^2 - (w\phi + \bar{w}\bar{\phi})^2\right), \quad \theta_2 = \frac{1}{(1-2|w|^2)^2}\left(2|dw|^2 + (wd\bar{w} - \bar{w}dw)^2\right).$$

The term θ_2, which is the induced metric on Δ, is just the Klein model for the hyperbolic plane and reflects the fact that the fibres of $\exp^\perp$ are totally geodesic. The first term θ_1 is the expression at p for the metric induced by the immersion $\varphi_w : \mathcal{D} \to \mathbb{CH}^2$, $\varphi_w(z) = \Theta(z, w)$. We can think of each vector $(-\bar{w}, w)$ as determining a section of $T\mathcal{D}^\perp$, and φ_w is the image of this under $\exp^\perp$. We claim that there is a constant $\varepsilon_1 > 0$ for which, for every w,

$$\varphi_w^* g(X, X) \geq \varepsilon_1 \gamma(X, X), \quad \forall X \in T_p\mathcal{D}.$$

It follows that $\Theta^* g = \theta_1 + \theta_2$ is bounded below by $\varepsilon_2\gamma + \theta_2$. The latter is a product of complete metrics on $\mathcal{D} \times \Delta$ and therefore $\Theta^* g$ is also complete.

To prove the claim, write $w = w_1 + iw_2$ and $\phi = \phi_1 + i\phi_2$ for real and imaginary parts, and set $r^2 = |w|^2 < 1/2$. Then

$$\theta_1 = \frac{1}{(1-2r^2)^2}\begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}\begin{pmatrix}2-4w_1^2 & 4w_1w_2 \\ 4w_1w_2 & 2-4w_2^2\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}. \tag{4.4}$$

The eigenvalues of the matrix are 2 and $2 - 4r^2$. Therefore, using the smaller eigenvalue $2 - 4r^2$,

$$\theta_1 \geq \frac{2}{1-2r^2}|\phi|^2 \geq 2|\phi|^2.$$

It now suffices to show that

$$|\phi|^2 \geq \varepsilon_2\gamma = \varepsilon_2(dx^2 + dy^2),$$

for a constant $\varepsilon_2 > 0$ independent of w, where $z = x + iy$. For this, write $k = k_1 + ik_2$ so that

$$\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} = B\begin{pmatrix}dx \\ dy\end{pmatrix} \text{ for } B = \begin{pmatrix}l + k_1 & k_2 \\ k_2 & l - k_1\end{pmatrix}. \tag{4.5}$$

The components of the metric $|\phi|^2$ are the entries of $B^tB = B^2$, and the eigenvalues are the roots of

$$\lambda^2 - 2(l^2 + |k|^2)\lambda + (l^2 - |k|^2)^2 = 0,$$

and are therefore $(l \pm |k|)^2$. Thus for any unit vector $X \in T_p\mathcal{D}$,

$$|\phi|^2(X,X) \geq (l - |k|)^2.$$

Now

$$l - |k| = \frac{l^2 - |k|^2}{l + |k|},$$

and $l + |k|$ is clearly bounded above, so it suffices to show that $l^2 - |k|^2$ is bounded below by a positive constant independent of w. We compute

$$\begin{aligned} l^2 - |k|^2 &= \tfrac{1}{2}(1+r^2)^2 - (2Q_0 w + \tfrac{1}{\sqrt{2}}\bar{w}^2)(2Q_0\bar{w} + \tfrac{1}{\sqrt{2}}w^2) \\ &= r^2(1 - 4Q_0^2) + \tfrac{1}{2} - Q_0(\sqrt{2}r)^3\cos(3\alpha), \end{aligned} \tag{4.6}$$

where $w = re^{i\alpha}$. Now $Q_0 < \frac{1}{2}$ and $\sqrt{2}r < 1$, so $1 - 4Q_0^2 > 0$ and $\frac{1}{2} - 2\sqrt{2}Q_0 r^3\cos(3\alpha) > 0$. With $\mathcal{Q}$ fixed we get a uniform positive lower bound over r, α. Thus the metric $\Theta^* g$ is complete on $\mathcal{D} \times \Delta$. We conclude, as in [36], that Θ is a proper map and a diffeomorphism, whence f is an embedding and ρ is almost $\mathbb{R}$-Fuchsian. □

To prove that f is unique we first need a result which can be given in greater generality than our current situation. Let (N, g) be a complete Riemannian manifold.

Proposition 4.5. *Let $f : M \to (N, g)$ be a compact embedded minimal submanifold for which $\exp^\perp : TM^\perp \to N$ is a diffeomorphism. For a local section η of $TM^\perp$ of unit length and a positive constant r, set $\varphi_r = \exp^\perp(r\eta)$ and let v_r be the volume form for the metric $\varphi_r^* g$ on an open subset of M. Suppose $dv_r/dr > 0$ for all r and for every η. Then f is the unique minimal immersion of M transverse to the fibres of $\exp^\perp$.*

The proof is given in Appendix A.

Now we can complete the proof of Theorem 4.3. Since ρ is quasi-Fuchsian the quotient $\mathbb{CH}^2/\rho$ is a manifold, and by the previous lemma $f : \Sigma \to \mathbb{CH}^2/\rho$ is a minimal embedding such that $\exp^\perp : T\Sigma^\perp \to \mathbb{CH}^2/\rho$ is a diffeomorphism. Now if $\varphi : \mathcal{D} \to \mathbb{CH}^2$ is any ρ-equivariant immersion then it must be transverse to the fibres of $\exp^\perp$, since $d\varphi \circ d\delta = d\rho(\delta) \circ d\varphi$ for every $\delta \in \pi_1\Sigma$ and the action of ρ is transverse to the fibres. Therefore the uniqueness claim in Theorem 4.3 follows from the previous proposition and the following lemma.

Lemma 4.6. *Under the assumptions of Theorem 4.3, if v_r is the area form for the metric $\varphi_r^* g$ induced by any local immersion of the form $\varphi_r = \exp^\perp(r\eta)$ for a local section η of $TM^\perp$ of unit length and a positive constant r, then $dv_r/dr > 0$.*

Proof. Since $\varphi_r(z) = \Theta(z, re^{i\alpha})$ for some fixed r and α, the induced metric $\varphi_r^* g$ at a point $p \in \Sigma$ is given by θ_1 from the proof of Lemma 4.4. Using the expression (4.4) we can write

$$\varphi_r^* g = \frac{1}{(1-2r^2)^2}\begin{pmatrix} dx & dy \end{pmatrix} B^t \begin{pmatrix} 2 - 4w_1^2 & 4w_1w_2 \\ 4w_1w_2 & 2 - 4w_2^2 \end{pmatrix} B \begin{pmatrix} dx \\ dy \end{pmatrix},$$

where B is given by (4.5). The determinant of the matrix in the middle is $2(2 - 4r^2)$, so that $v_r = a(r) dx \wedge dy$ where

$$\begin{aligned} a(r) &= \frac{1}{(1-2r^2)^2} 2\sqrt{1-2r^2}(l^2 - |k|^2) \\ &= \frac{1}{(1-2r^2)^{3/2}}\left(1 + 2r^2(1 - 4Q_0^2) - 4\sqrt{2}Q_0 r^3\cos(3\alpha)\right), \end{aligned}$$

using (4.6). A calculation shows that

$$\frac{da}{dr} = \frac{4r}{(1-2r^2)^{3/2}}\left[(1+r^2)(1-4Q_0^2) + \tfrac{3}{2}(1-2\sqrt{2}Q_0 r\cos(3\alpha))\right].$$

Now $Q_0 < \frac{1}{2}$, $r < 1/\sqrt{2}$ and $\cos(3\alpha) \leq 1$, so $da/dr > 0$. □

4.1. **Families of solutions to the Gauss equation.** Theorem 4.3 shows that the norm $\|\mathcal{Q}\|_\gamma$ gives control over uniqueness of minimal Lagrangian immersions, but at present we have no clear way of relating it to the parametrisation by the extension class ξ. Moreover, since this norm depends upon the solution to (4.2) it is difficult to control *a priori*. On the other hand, the combined results of [36] and [30] show that a bound on $\|\mathcal{Q}\|_\mu$ must be combined with a condition the solution of (4.2) to get existence and uniqueness. One knows that the zero solution $u \equiv 0$ is the unique solution for $\mathcal{Q} = 0$, and that for $\|\mathcal{Q}\|_\mu$ small and non-zero there are always solutions [36], but these are not unique [30]. The challenge is to understand how solutions behave as one moves along a ray $t\mathcal{Q}_0$, $t \geq 0$, given a fixed cubic holomorphic differential $\mathcal{Q}_0$. To study solutions along such rays, Huang et. al [30] introduced the following terminology.

Definition 4.7. *A solution u to* (4.2) *is* $\mathcal{F}$-stable *if the linearised operator*

$$\mathcal{L} = -\Delta_\mu + 2e^u - 4\|\mathcal{Q}\|_\mu^2 e^{-2u}, \tag{4.7}$$

is positive.

This condition ensures, by the Implicit Function Theorem in the appropriate Sobolov spaces, that there is locally a smooth curve $u(t)$ of solutions to

$$\mathcal{H}(u,t) = \Delta_\mu u - 2\|t\mathcal{Q}_0\|_\mu^2 e^{-2u} - 2e^u + 2 = 0, \tag{4.8}$$

nearby any $\mathcal{F}$-stable solution $u(t_0)$. Our aim here is to show that, given $\mathcal{Q}_0$, the $\mathcal{F}$-stable solutions form a continuous curve terminated at one end by the zero solution and at the other end by the first solution which is not $\mathcal{F}$-stable. This, together with the results of [36, 30], gives the following summary of the behaviour of solutions to (4.2) as the cubic differential is scaled.

Theorem 4.8. *Fix a non-zero cubic holomorphic differential $\mathcal{Q}_0$ on Σ_c. Set*

$$T_0 = \sqrt{4/27}(\sup_\Sigma \|\mathcal{Q}_0\|_\mu)^{-1}.$$

Then:

(i) *there exists a $T_2 > T_0$ such that* (4.8) *has no solutions for $t \geq T_2$;*
(ii) *there exists $T_0 \leq T_1 < T_2$ such that for $t < T_1$ there is a unique continuous family of $\mathcal{F}$-stable solutions to* (4.8)*. All $\mathcal{F}$-stable solutions lie on this family and the limiting solution $u(T_1)$ is not $\mathcal{F}$-stable;*
(iii) *for $t < T_0$ the $\mathcal{F}$-stable solutions yield almost Fuchsian embeddings;*
(iv) *for $0 < t < T_1$ there is at least one solution which is not $\mathcal{F}$-stable.*

Remark 4.2. This result is analogous to Uhlenbeck's description of the bifurcation in families of solutions to the Gauss equation for minimal surfaces in $\mathbb{RH}^3$ [43, Thm 4.4]. We note that for Uhlenbeck the right notion of stability was stability with respect to the area functional. In our case that gives no extra control, since all minimal Lagrangian surfaces in $\mathbb{CH}^2/\rho$ are stable by a theorem of Oh [38].

Part (i) comes from [36], while (iii) comes from [36] and Theorem 4.3 above. Part (iv) and the existence of a local family of unique $\mathcal{F}$-stable solutions for $t < T_1$ come from [30]. Here we provide the rest of (ii) via the following lemma.

Lemma 4.9. *Let $\mathcal{Q}_0$ be a holomorphic cubic differential on Σ_c, and let $\tau > 0$ be such that $u(\tau)$ is an $\mathcal{F}$-stable solution. Let $u(t)$ be the local family of $\mathcal{F}$-stable solutions to* (4.8) *through $u(\tau)$. Then $\dot{u}(\tau) \leq 0$ on all of Σ_c.*

Proof. By differentiating (4.8) we find that $\dot{u}$ satisfies

$$-\mathcal{L}_\tau \dot{u} = 4\tau \|\mathcal{Q}_0\|_\mu^2 e^{-2u}.$$

Elliptic regularity implies $\dot{u}$ is C^∞. Now define $\dot{u}^+ = \max\{\dot{u}(\tau), 0\}$.

Then $\dot{u}^+$ is in the Sobolev space $H(\Sigma)$ and $d\dot{u}^+ = 0$ wherever $\dot{u} \leq 0$ (see e.g.[17]). Recall we define

$$\langle -\Delta_\mu \dot{u}^+, \dot{u}^+ \rangle = \int_\Sigma \|d\dot{u}^+\|_\mu^2 \, dA_\mu$$

in this case. Let $v = 4\tau\|\mathcal{Q}_0\|_\mu^2 e^{-2u}$, and let $\epsilon_j \searrow 0$ be regular values of $\dot{u}$ (as guaranteed by Sard's Theorem). Thus we can integrate by parts, as each $\{\dot{u} = \epsilon_j\} = \partial\{\dot{u} > \epsilon_j\}$ is a smooth 1-manifold. Let $w_j = \max\{\dot{u} - \epsilon_j, 0\}$. Now since $\mathcal{L} > 0$,

$$\begin{aligned}
0 &\geq \langle -\mathcal{L}\dot{u}^+, \dot{u}^+ \rangle \\
&= \int_\Sigma \left[-\|d\dot{u}^+\|_\mu^2 + (-2e^u + 4\|\mathcal{Q}_0\|_\mu^2 e^{-2u})(\dot{u}^+)^2 \right] dA_\mu \\
&= \int_{\{\dot{u}>0\}} \left[-\|d\dot{u}\|_\mu^2 + (-2e^u + 4\|\mathcal{Q}_0\|_\mu^2 e^{-2u})\dot{u}^2 \right] dA_\mu \\
&= \lim_{j\to\infty} \int_{\{\dot{u}>\epsilon_j\}} \left[-\|dw_j\|_\mu^2 + (-2e^u + 4\|\mathcal{Q}_0\|_\mu^2 e^{-2u})\dot{u}^2 \right] dA_\mu \\
&= \lim_{j\to\infty} \int_{\{\dot{u}>\epsilon_j\}} \left[w_j \Delta_\mu w_j + (-2e^u + 4\|\mathcal{Q}_0\|_\mu^2 e^{-2u})\dot{u}^2 \right] dA_\mu \\
&= \lim_{j\to\infty} \int_{\{\dot{u}>\epsilon_j\}} \left[(\dot{u} - \epsilon_j)\Delta_\mu \dot{u} + (-2e^u + 4\|\mathcal{Q}_0\|_\mu^2 e^{-2u})\dot{u}^2 \right] dA_\mu \\
&= \int_{\{\dot{u}>0\}} (-\mathcal{L}\dot{u})\dot{u} \, dA_\mu \\
&= \int_{\{\dot{u}>0\}} v\dot{u} \, dA_\mu \geq 0.
\end{aligned}$$

(The limits above are valid by the Dominated Convergence Theorem.)

Since v is positive almost everywhere, the last inequality is strict if $\dot{u} > 0$ anywhere on Σ. Thus by contradiction $\dot{u}(\tau) \leq 0$ everywhere on Σ. □

Proof of Thm 4.8(ii). Let $u(\tau)$ be an $\mathcal{F}$-stable solution, with local family $u(t)$ and let $\mathcal{L}_t$ be the corresponding family of linearised operators (4.7). Now

$$\dot{\mathcal{L}} = 2\dot{u}e^u + 8t\|\mathcal{Q}\|_\mu^2 e^{-2u}(\dot{u} - t),$$

which is nonpositive for $t > 0$ by the previous Lemma. Thus $\mathcal{L}_\tau > 0$ implies $\mathcal{L}_t > 0$ for all $t \in [0, \tau]$ in the path of solutions.

The Maximum Principle shows that every solution to (4.8) is nonpositive. Thus for any t in an interval of the form $(\tau_0, \tau]$ the proposition implies $0 \geq u_t \geq u_\tau$, and thus we have uniform L^∞ bounds on u_t for all $t \in (\tau_0, \tau]$. Then the L^p theory, applied to (4.8), and standard bootstrapping show that the limit

$$\lim_{t \to \tau_0^+} u_t$$

exists and is a solution u_{τ_0} to the equation. This provides a closedness argument for the continuity method. On the other hand the $\mathcal{L}_t > 0$ condition, verified in the previous paragraph, provides openness as well, and thus we can extend the solution space back down to $t = 0$. □

5. Surfaces with zero cubic holomorphic differential.

Minimal (possibly branched) immersions for which $\mathcal{Q} = 0$ have particularly important properties. They include all the holomorphic and anti-holomorphic immersions and, by Theorem 3.1, the extension class zero cases when f is not holomorphic or anti-holomorphic. In all such cases, the Higgs bundle E is a *Hodge bundle* (or *variation of Hodge structure*), i.e., $E = \oplus_{i=1}^m E_i$ for proper sub-bundles E_i for which $\Phi : E_i \to E_{i+1} \otimes K$, with $E_{m+1} = 0$. For $PU(2,1)$ the *length* m of the Hodge bundle must be either two or three [21]. We will show below that the length-two Hodge bundles correspond to holomorphic or anti-holomorphic immersions, while immersions which arise from Theorem 3.1 with $\xi = 0$ give length-three Hodge bundles.

Hodge bundles play the central role in calculating the Betti numbers of the smooth components of the moduli space $\mathcal{H}(\Sigma_c, G)$ of polystable Higgs bundles, and therefore the Betti numbers of the representation space $\mathcal{R}(G)$. For on smooth components the *Hitchin function* $\|\Phi\|_{L^2}^2 : \mathcal{H}(\Sigma_c, G) \to \mathbb{R}$, is a perfect Morse function, whose critical points are the Hodge bundles. The length-two Hodge bundles are minima, while the length-three Hodge bundles have non-zero Morse index [21].

5.1. Holomorphic and anti-holomorphic surfaces.

By Toledo's theorem [42] every *maximal representation* (those for which $\tau(\rho) = \pm\chi(\Sigma)$) leaves invariant a complex line and acts on that line as a Fuchsian representation. Such representations are reducible. To understand the non-maximal ρ-equivariant holomorphic or anti-holomorphic immersions, we will start by describing their Hodge bundles. First we note that for any holomorphic ρ-equivariant immersion $f : \mathcal{D} \to \mathbb{CH}^2$ the area form v_γ for the induced metric equals $f^*\omega$. It follows from the definition (2.9) that $\tau(\rho) > 0$. For anti-holomorphic immersions $f^*\omega = -v_\gamma$ so that $\tau(\rho) < 0$. The next lemma completely characterises the Higgs bundle data for representations which admit either holomorphic or anti-holomorphic branched immersions.

Theorem 5.1. *An irreducible representation ρ admits a branched holomorphic ρ-equivariant immersion if and only if it corresponds to a Hodge bundle $(V \oplus 1, \Phi)$ with $\Phi = (\Phi_1, 0)$ and V a non-trivial extension bundle of the form*

$$0 \to K^{-1}(B) \stackrel{\Phi_1}{\to} V \to K^{-2}L \to 0.$$

Here B is a non-negative divisor of degree $b \geq 0$ (the divisor of branch points of the immersion) and L is a line bundle of degree l, satisfying the inequalities

$$3(g-1) + \tfrac{1}{2}b < l < 6(g-1) - b, \quad 0 \leq b < 2(g-1). \tag{5.1}$$

In particular, $3(1-g) < \deg(V) < 0$, and $\tau(\rho) = \frac{2}{3}(6g - 6 - b - l)$.

Moreover, (E, Φ) corresponds to a branched anti-holomorphic immersion f if and only if $\bar{f}$ is the branched holomorphic immersion determined by (E^*, Φ^t).

Note that by $\bar{f}$ we mean the post-composition of f with the natural anti-holomorphic involution on $\mathbb{CH}^2$ which descends from complex conjugation on $\mathbb{C}^{2,1}$. Clearly f is ρ-equivariant precisely when $\bar{f}$ is $\bar{\rho}$-equivariant. The map $\rho \mapsto \bar{\rho}$ is an involution on $\mathrm{Hom}(\pi_\Sigma, G)/G$ for which $\tau(\bar{\rho}) = -\tau(\rho)$: it fixes the representations with values in $PO(2,1)$.

Proof. First suppose (E, Φ), with $E = V \oplus 1$ and $\Phi = (\Phi_1, \Phi_2)$, is a length-two Hodge bundle with $\deg(V) < 0$. In this case by [21, §3] we have $\Phi_2 = 0$. The corresponding representation ρ admits a ρ-equivariant harmonic map $f : \mathcal{D} \to \mathbb{CH}^2$ determined by the sub-bundle 1 as a section $\mathbb{P}E_-$ with $df' = \Phi$. Therefore $\bar{\partial} f' = 0$ and f is holomorphic. Conversely, suppose ρ admits a holomorphic immersion f. Taking $V = f^{-1}T'\mathbb{CH}^2$ and $\Phi = \partial f' : 1 \to KV$ gives a length-two Hodge bundle with $\deg(V) < 0$. The involution $(E, \Phi) \to (E^*, \Phi^t)$ on Higgs bundles maps length-two Hodge bundles with $\Phi_2 = 0$ to those with $\Phi_1 = 0$. In the latter case $V \simeq f^{-1}T''\mathbb{CH}^2$ with the opposite complex structure, and the map f is anti-holomorphic.

Now the structure of $(V \oplus 1, \Phi)$ follows a simplified version of the argument in the proof of Theorem 2.3. We can think of Φ_1 as a holomorphic section of KV, with divisor $B \geq 0$ corresponding to the branch divisor of f. The bundle injection $\Phi_1 : K^{-1}(B) \to V$ has image V_1 and quotient line bundle $V_2 = V/V_1$. Provided V is not the direct sum $V_1 \oplus V_2$ the Φ-invariant sub-bundles of E are $V_1, V_1 \oplus 1$ and V. The stability inequalities are therefore

$$\deg(V_1) < \tfrac{1}{3}\deg(V), \quad \tfrac{1}{2}\deg(V_1) < \tfrac{1}{3}\deg(V), \quad \tfrac{1}{2}\deg(V) < \tfrac{1}{3}\deg(V).$$

On the other hand, if V is the direct sum then V_2 is also Φ-invariant and stability requires the additional inequality

$$\deg(V_2) < \tfrac{1}{3}\deg(V), \text{ i.e., } \quad \tfrac{2}{3}\deg(V) < \deg(V_1),$$

so this is not possible. For later convenience we write $V_2 = K^{-2}L$ and the inequalities (5.1) follow from $\deg(V_1) = b - 2(g-1)$ and $\deg(V_2) = l - 4(g-1)$. □

Note that while the splitting of $V \simeq f^{-1}T'\mathbb{CH}^2/\rho$ into $T\Sigma \oplus T\Sigma^\perp$ is J-invariant, the sub-bundle $T\Sigma^\perp$ is not $\bar{\partial}_E$-invariant unless ρ is reducible. Indeed, the normal bundle is ∂_E-invariant (since it is paired with $T\Sigma$ by the Hermitan metric) so the induced structure of this splitting makes the normal bundle *anti-holomorphic.*

Remark 5.1 (Reducible representations.). Although $E = V \oplus 1$ cannot be stable when V is a trivial extension, it can be polystable. This corresponds to a reducible reductive representation. Such representations either:

(i) factor through a maximal compact subgroup, or,
(ii) factor through $P(U(1,1) \times U(1))$.

This is easy to see. We may simplify things, by replacing ρ by $\bar{\rho}$ if necessary, to assume that $\tau(\rho) \geq 0$ and thus $\Phi_2 = 0$. To be strictly polystable (E, Φ) must decompose into either

$$(i)\ (V, 0) \oplus (1, 0), \text{ or } \quad (ii)\ (1 \oplus V_1, \Phi_1) \oplus (V_2, 0),$$

where $V = V_1 \oplus V_2$ and $\Phi_1 : 1 \to V_1$. In the first case V must be a stable rank two bundle of degree zero (to have the same slope as 1) and therefore the representation lies in a maximal compact subgroup and has $\tau(\rho) = 0$. We note that the corresponding harmonic map f is constant. In the second case $(1 \oplus V_1, \Phi_1)$ corresponds to a representation into $U(1,1)$. In this case polystability requires

$$\deg(V_1) < \tfrac{1}{2}\deg(V_1 \oplus 1) = \tfrac{1}{2}\deg(V_1), \text{ i.e., } \quad \deg(V_1) < 0,$$

together with the "same slope" condition $\frac{1}{2}\deg(V_1) = \deg(V_2)$. When we write $V_1 = K^{-1}(B)$ as above we deduce that

$$b < 2(g-1), \quad \deg(V_2) = \tfrac{1}{2}b - g + 1.$$

Thus b is even and

$$\tau(\rho) = -\tfrac{2}{3}\deg(V) = -\deg(V_1) = 2g - 2 - b \in 2\mathbb{Z}.$$

In particular, such ρ can only admit an unbranched holomorphic map f when $\tau(\rho)$ is maximal, i.e., when ρ factors through a Fuchsian representation. The map $f : \mathcal{D} \to \mathbb{CH}^2$ is a totally geodesic embedding onto a complex line. More generally, the $PU(1,1)$ representation corresponding to the rank two Higgs bundle $(1 \oplus V_1, \Phi_1)$ has Toledo invariant $-\deg(\mathrm{Hom}(1, V_1))$, which therefore equals $\tau(\rho)$. Every irreducible representation into $PU(1,1)$ of even Toledo invariant lifts to $SU(1,1)$, and therefore provides a representation into $P(U(1,1) \times U(1))$. Thus the whole structure of Higgs bundles for irreducible representations in $SU(1,1)$ [28] lifts up to provide reducible representations into $PU(2,1)$, and this is what we are seeing above. Note that those which are non-maximal cannot be convex cocompact, since they preserve a complex line but act non-cocompactly on this line.

Each Higgs bundle in Theorem 5.1 corresponds to a quadruple (Σ_c, B, L, η) where $\eta \neq 0$ is the extension class, $\eta \in H^1(\Sigma_c, KL^{-1}(B))$. As in §3, we can understand the relationship between η and geometric invariants of f via the Dolbeault isomorphism. First we need to introduce the tensor

$$S \in \mathcal{E}^0(\Sigma_c, K^2\bar{K}^2), \quad S = h(\pi^\perp \nabla' \partial f', \pi^\perp \nabla' \partial f'),$$

where $\pi^\perp : f^{-1}T'\mathbb{CH}^2/\rho \to T\Sigma^\perp$ is projection onto the normal bundle.

Theorem 5.2. *Let $f : \mathcal{D} \to \mathbb{CH}^2$ be a ρ-equivariant holomorphic immersion corresponding to the data (Σ_c, B, L, η) above. Then under the Dolbeault isomorphism η maps to the cohomology class*

$$-[S/\gamma] \in H^{0,1}(\Sigma_c, KL^{-1}(B)).$$

Moreover, $\|S\|_\gamma = \frac{1}{4}\|\mathrm{I\!I}_f\|_\gamma^2$, where $\mathrm{I\!I}_f$ is the second fundamental form of f.

Proof. We follow the steps in the proofs of Theorem 3.1 but using a local frame more suited to holomorphic maps. As before, use $\ell_0 \subset E$ to denote 1 and write its $\partial f'$ transform as $\ell_1 \otimes K$. But now take the further transform of ℓ_1, so we have a harmonic sequence

$$\ell_0 \overset{\pi_0^\perp Z}{\to} \ell_1 \overset{\pi_1^\perp Z}{\to} \ell_2 \overset{\pi_2^\perp Z}{\to} 0,$$

in each chart (U, z). The last step terminates the sequence because $\pi_2^\perp Z : \ell_2 \to \ell_0$ is the adjoint to $\pi_0^\perp \bar{Z} : \ell_0 \to \ell_2$, which represents $\bar{\partial} f' = 0$. As before, let $f_0 \in \Gamma(\ell_0)$ be global and parallel with $\langle f_0, f_0 \rangle = -1$. Set

$$\sigma_1 = \pi_0^\perp Z f_0 = Z f_0, \quad \sigma_2 = \pi_1^\perp Z \sigma_1.$$

In this holomorphic case the induced metric is $u_1^2|dz|^2$ and $u_1 = |\sigma_1|$. Set $s_2 = |\sigma_2|$, then in a chart U in which neither u_1 nor s_2 vanishes we have a local $U(2,1)$ frame given by f_1, f_2, f_0 where $f_1 = \sigma_1/u_1$, $f_2 = \sigma_2/s_2$. Straighforward calculations as before give

$$\begin{aligned} Zf_1 &= (Z\log u_1)f_1 + u_1^{-1}s_2f_2, \\ Zf_2 &= (Z\log s_2)f_2, \\ Zf_0 &= u_1f_1. \end{aligned} \tag{5.2}$$

From this we can read off the connexion 1-form for the projectively flat connexion ∇ in this frame. Now let $\varphi : V|U \to U \times \mathbb{C}^2$ be the local trivialisation corresponding to f_1, f_2, then it follows that

$$\varphi \circ \bar\partial_E \circ \varphi^{-1} = d\bar z \left[\frac{\partial}{\partial \bar z} - \begin{pmatrix} \bar Z \log u_1 & s_2/u_1 \\ 0 & \bar Z \log s_2 \end{pmatrix} \right].$$

To deal with zeroes of u_1 and s_2, we may assume U only has these at $z = 0$, to order p and q respectively. In such a chart we take the local frame $\tilde f_1 = w^{-p}f_1$, $\tilde f_2 = w^{-q}f_2$, where $w = z/|z|$, and in the corresponding trivialisation $\tilde\varphi$ we have

$$\tilde\varphi \circ \nabla^{0,1} \circ \tilde\varphi^{-1} = d\bar z \left[\frac{\partial}{\partial \bar z} - \begin{pmatrix} \bar Z \log(u_1/|z|^p) & -s_2w^{p-q}/u_1 \\ 0 & \bar Z \log(s_2/|z|^q) \end{pmatrix} \right].$$

We obtain a local holomorphic trivialisation, with respect to $\bar\partial_E$, by applying a gauge transformation of the form

$$R = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} |z|^p/u_1 & 0 \\ 0 & |z|^q/s_2 \end{pmatrix},$$

i.e., for $\chi = R\tilde\varphi$ we have $\chi \circ \bar\partial_E \circ \chi^{-1} = d\bar z(\partial/\partial \bar z)$. This requires

$$\partial a/\partial \bar z = -s_2^2 z^{p-q}/u_1^2. \tag{5.3}$$

When U is contractible this equation has a solution since $s_2^2/u_1^2 \sim |z|^{2(q-p)}$ near $z = 0$, therefore the right hand side of (5.3) is smooth throughout U.

Now cover Σ_c by charts (U_j, z_j) of the type used above, and index the local objects living over U_j by j. Thus for V we have transition relations $\tilde\varphi_j = \tilde c_{jk}\tilde\varphi_k$ where

$$\tilde c_{jk} = \begin{pmatrix} \frac{dz_j/dz_k}{|dz_j/dz_k|}\frac{t_{jk}}{|t_{jk}|} & 0 \\ 0 & \frac{(dz_j/dz_k)^2}{|dz_j/dz_k|^2}\frac{t_{jk}}{|t_{jk}|} \end{pmatrix} \begin{pmatrix} w_j^{p_j}w_k^{-p_k} & 0 \\ 0 & w_j^{q_j}w_k^{-q_k} \end{pmatrix}.$$

Therefore $\chi_j = b_{jk}\chi_k$ where

$$b_{jk} = R_j\tilde c_{jk}R_k^{-1} = t_{jk} \begin{pmatrix} z_j^{p_j}z_k^{-p_k}dz_j/dz_k & \lambda_{jk} \\ 0 & z_j^{q_j}z_k^{-q_k}(dz_j/dz_k)^2 \end{pmatrix}, \tag{5.4}$$

for

$$\lambda_{jk} = a_jz_j^{q_j}z_k^{-q_k}(dz_j/dz_k)^2 - a_kz_j^{p_j}z_k^{-p_k}dz_j/dz_k, \tag{5.5}$$

and we have used the fact that

$$u_{1k}/u_{1j} = |dz_j/dz_k|, \quad s_{2k}/s_{2j} = |dz_j/dz_k|^2.$$

As earlier, this means that the extension class of the bundle V is given by the Cech cohomology class η of $\{(\eta_{jk}, U_j, U_k)\}$ where

$$\begin{aligned}\eta_{jk} &= z_j^{-p_j} z_k^{p_k}(dz_k/dz_j)\lambda_{jk}dz_k \\ &= a_j z_j^{q_j-p_j}dz_j - a_k z^{q_k-p_k}dz_k.\end{aligned}$$

This is plainly a coboundary for Cech cohomology in smooth sections of $KL^{-1}(B)$ of the form $\delta\tau$ where $\{(\tau_j, U_j)\}$ has $\tau_j = a_j z^{q_j-p_j}dz_j$. Then

$$\bar\partial\tau_j = -\frac{s_{2j}^2}{u_{1j}^2}dz_j d\bar z_j.$$

Now we claim that in a chart (U, z)

$$s_2^2 = S(Z, Z, \bar Z, \bar Z) = h(\pi^\perp(\nabla_Z^{\mathbb{CH}^2}Z), \pi^\perp(\nabla_Z^{\mathbb{CH}^2}Z)).$$

To see this, we note first that under the isomorphism $f^{-1}T'\mathbb{CH}^2/\rho \simeq \mathrm{Hom}(1, V) \simeq V$ the holomorphic line sub-bundle $\partial f'(T^{1,0}\Sigma)$ is identified with ℓ_1. Recall that

$$(\nabla_Z Z)f_0 = \pi_0^\perp Z\pi_0^\perp Zf_0 - \pi_0^\perp Z\pi_0 Zf_0.$$

But $\pi_0 Zf_0 = 0$ so that

$$[\pi^\perp(\nabla_Z Z)]f_0 = \pi_1^\perp Z\pi_0^\perp f_0 = \sigma_2.$$

It follows that

$$S(Z, Z, \bar Z, \bar Z) = \langle\pi_1^\perp Z\pi_0^\perp Zf_0, \pi_1^\perp Z\pi_0^\perp Zf_0\rangle = |\sigma_2|^2 = s_2^2.$$

Therefore $\|S\|_\gamma = s_2^2/u_1^4$ since $\|f_*Z\|_\gamma = u_1$ by (2.4). On the other hand, since f is holomorphic $Jf_*X = Y$ where $X = \partial/\partial x$ and $Y = \partial/\partial y$. Since $\nabla^{\mathbb{CH}^2}$ is Kähler we have, under the identification $T'\mathbb{CH}^2 \simeq T\mathbb{CH}^2$,

$$\nabla_Z^{\mathbb{CH}^2}Z = \tfrac{1}{4}(\nabla_{X-JY}^{\mathbb{CH}^2}(X - JY)) = \nabla_X X.$$

So

$$s_2^2 = S(Z, Z, \bar Z, \bar Z) = g(\mathrm{I\!I}_f(X, X), \mathrm{I\!I}_f(X, X)).$$

Now take an oriented orthonormal basis X_1, X_2, N_1, N_2 of $f^{-1}T\mathbb{CH}^2$ for which $X_1 = u_1^{-1}f_*X$, $X_2 = JX_1 = u_1^{-1}f_*Y$, and $N_2 = JN_1$. Let $\mathrm{I\!I}_{ij}^k = g(\nabla_{X_i}^{\mathbb{CH}^2}X_j, N_k)$ be the components of $\mathrm{I\!I}_f$ in this frame. Then we have the symmetries

$$\mathrm{I\!I}_{11}^k = -\mathrm{I\!I}_{22}^k, \quad \mathrm{I\!I}_{11}^1 = \mathrm{I\!I}_{12}^2 = \mathrm{I\!I}_{21}^2, \quad \mathrm{I\!I}_{12}^1 = \mathrm{I\!I}_{21}^1 = -\mathrm{I\!I}_{11}^2. \tag{5.6}$$

It follows that

$$\begin{aligned}\|\mathrm{I\!I}_f\|_\gamma^2 &= 4[(\mathrm{I\!I}_{11}^1)^2 + (\mathrm{I\!I}_{11}^2)^2] \\ &= \tfrac{4}{u_1^4}g(\mathrm{I\!I}_f(X, X), \mathrm{I\!I}_f(X, X)) \\ &= 4\|S\|_\gamma.\end{aligned}$$

□

Remark 5.2. We note that, by the symmetries (5.6), when the shape operator $A_f : T\mathcal{D}^\perp \to \mathrm{End}(T\mathcal{D})$ is applied to any unit normal vector field $N = (a + bJ)N_1$ it can be expressed with respect to the frame X_1, X_2 in the form

$$A_f(N) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} \mathrm{I\!I}_{11}^1 & \mathrm{I\!I}_{12}^1 \\ \mathrm{I\!I}_{12}^1 & -\mathrm{I\!I}_{11}^1 \end{pmatrix}.$$

Therefore the operator norm of $A_f(N)$ satisfies

$$\|A_f(N)\|^2 = \tfrac{1}{2}\operatorname{tr} A_f(N)^2 = -\det(A_f(N)) = \tfrac{1}{4}\|\mathrm{I\!I}_f\|_\gamma^2 = \|S\|_\gamma.$$

This is independent of N, so we simply write it as $\|A_f\|^2$ for holomorphic immersions. Moreover, the symmetries (5.6) also imply that the Gauss equation for f is

$$-4 = \kappa_\gamma + \|\mathrm{I\!I}_{11}\|^2 + \|\mathrm{I\!I}_{12}\|^2 = \kappa_\gamma + 2\|A_f\|^2. \tag{5.7}$$

5.2. **Surfaces arising from zero extension class.** Theorems 2.3 and 3.1 imply that Higgs bundles for non-holomorphic minimal surfaces with $\mathcal{Q} = 0$ are exactly the length-three Hodge bundles. When $\mathcal{Q} = 0$ we have a trivial extension bundle and can take

$$E_1 \oplus E_2 \oplus E_3 = K(-D_2) \oplus 1 \oplus K^{-1}(D_1),$$

to get $\Phi : E_i \to KE_{i+1}$. Conversely, any length-three Hodge bundle is projectively equivalent to one of this form and has $\operatorname{tr}\Phi^2 = 0$. We will show that f is still related to a holomorphic map, via its harmonic sequence (in the sense of Erdem & Glazebrook [15], these maps are isotropic but non-holomorphic). To explain this, we first recall (from e.g., [4, 14]), the notion of the *Gauss transforms* φ_1, φ_2 of f.

The line bundles ℓ_1, ℓ_2 defined by (3.3) both lie inside the subset E_+ consisting of fibre vectors which have positive length with respect to the $\mathbb{C}^{2,1}$ metric. Let

$$\mathbb{C}^{2,1}_+ = \{v \in \mathbb{C}^{2,1} : \langle v, v\rangle > 0\}.$$

Then its projective space $\mathbb{P}\mathbb{C}^{2,1}_+$ is an open submanifold of $\mathbb{CP}^2$ on which $G = PU(2,1)$ acts transitively. We identify it with the orbit of the line $[e_2]$, with isotropy subgroup $H_2 \simeq P(U(1,1) \times U(1))$, so that $\mathbb{P}\mathbb{C}^{2,1}_+ \simeq G/H_2$. In fact we can think of it as the complex version of two dimensional de Sitter space, and will henceforth denote it by $\mathbb{C}\mathrm{dS}^2$. Its tangent space at the base point $[e_2]$ is identified with the orthogonal complement $\mathfrak{m}_2 = \mathfrak{h}_2^\perp \subset \mathfrak{su}(2,1)$ with respect to the Killing form $-\frac{1}{2}\operatorname{tr}(AB)$, and the latter gives $\mathfrak{m}_2$ an indefinite Hermitian metric. This extends to the tangent bundle, isomorphic to $G \times_{H_2} \mathfrak{m}_2$, and makes $\mathbb{C}\mathrm{dS}^2$ a pseudo-Hermitian symmetric space. Clearly $\mathbb{P}E_+ \simeq \mathcal{D} \times_\rho \mathbb{C}\mathrm{dS}^2$ and therefore ℓ_1, ℓ_2 each determine a smooth ρ-equivariant map $\varphi_1, \varphi_2 : \mathcal{D} \to \mathbb{C}\mathrm{dS}^2$, and these will be conformal harmonic with respect to the pseudo-Hermitian metric (they are isotropic in the sense of [15]). Following the terminology of harmonic sequences, we call φ_1 the *∂-Gauss transform* of f, and φ_2 the *$\bar\partial$-Gauss transform* of f. An immersion into $\mathbb{C}\mathrm{dS}^2$ is *timelike* when its induced metric is negative definite (away from branch points).

Proposition 5.3. *Let $f : \mathcal{D} \to \mathbb{CH}^2$ be ρ-equivariant and not $\pm$-holomorphic. Then $\mathcal{Q} = 0$ if and only if its $\bar\partial$-transform $\varphi_2 : \mathcal{D} \to \mathbb{C}\mathrm{dS}^2$ is a timelike ρ-equivariant holomorphic map, branched at the divisor of complex points D_2 of f.*

Proof. Let $\varphi_2 : \mathcal{D} \to \mathbb{C}\mathrm{dS}^2$ be the $\bar\partial$-Gauss transform f. Write the differential of φ_2 as $d\varphi_2' = \partial\varphi_2' + \bar\partial\varphi_2'$ in terms of the type decomposition

$$\varphi_2^{-1}T_{\ell_2}\mathbb{C}\mathrm{dS}^2 = \mathrm{Hom}(\ell_2, \ell_2^\perp) \oplus \mathrm{Hom}(\ell_2^\perp, \ell_2).$$

In local coordinates

$$\bar\partial\varphi_2'(\bar Z) = \pi_2^\perp \bar Z.$$

But from (3.7) and the fact that ℓ_0, ℓ_1, ℓ_2 are mutually orthogonal we have

$$\begin{aligned} Q &= \langle \pi_0^\perp Z \pi_0^\perp Z f_0, \pi_0^\perp \bar{Z} f_0 \rangle, \\ &= -\langle \pi_0^\perp Z f_0, \pi_2^\perp \bar{Z} \pi_0^\perp \bar{Z} f_0 \rangle. \end{aligned}$$

Therefore if neither $\pi_0^\perp Z f_0$ nor $\pi_0^\perp \bar{Z} f_0$ is zero, $\mathcal{Q}$ vanishes if and only if $\pi_2^\perp \bar{Z}$ is identically zero on ℓ_2. Hence $\bar\partial \varphi_2'$ vanishes and φ_2 is holomorphic.

Finally, we claim that when $\mathcal{Q} = 0$ the induced metric for φ_2 is $-u_2^2|dz|^2$. To see this, let (f_0, f_1, f_2) be a local Toda frame for f. Then by definition φ_2 is given locally by the family of lines $[f_2]$. To calculate the differential we use (3.8) to deduce that in this frame $d\varphi_2$ is represented by

$$\begin{pmatrix} & 0 & \\ Q/u_1u_2 & & 0 \\ & u_2 & \end{pmatrix} dz + \begin{pmatrix} & -\bar{Q}/u_1u_2 & \\ 0 & & u_2 \\ & 0 & \end{pmatrix} d\bar{z}.$$

Here we use blank spaces to indicate the Lie subalgebra $\mathfrak{h}_2 \subset \mathfrak{g}$ of isotropy group H_2: relative to the frame $d\varphi_2$ takes values in $\mathfrak{h}_2^\perp$. Therefore the induced metric is

$$-\tfrac{1}{2}\operatorname{tr}(d\varphi_2)^2 = (\frac{|Q|^2}{u_1^2u_2^2} - u_2^2)|dz|^2.$$

□

Note that, since $\partial\varphi_2' : \ell_2 \to K\ell_0$, f is the ∂-Gauss transform of φ_2.

6. Moduli

Theorems 2.3 and 5.1 provide parameterisations for different components of the set

$$\mathcal{V} = \{(\rho, f) : \rho \text{ irreducible}, f \text{ branched minimal}\}/G,$$

where the quotient is by the simultaneous action of G as conjugation of ρ and ambient isometry of f. By those theorems it is natural to write $\mathcal{V}$ as a disjoint union of the sets

$$\mathcal{V}(d_1, d_2) = \{(\rho, f) \in \mathcal{V} : f \text{ has } d_1 \text{ anti-complex points and } d_2 \text{ complex points}\}$$

and

$$\begin{aligned} \mathcal{W}^+(b, l) &= \{(\rho, f) \in \mathcal{V} : f \text{ holomorphic with } \deg(B) = b,\ \deg(L) = l\}, \\ \mathcal{W}^-(b, l) &= \{(\rho, f) \in \mathcal{V} :\ (\bar\rho, \bar f) \in \mathcal{W}^+(b, l)\}. \end{aligned}$$

These last two spaces are bijective under $(\rho, f) \mapsto (\bar\rho, \bar f)$. We will now show that the parametrisations give each component the structure of a complex manifold.

6.1. **The structure of $\mathcal{V}(d_1, d_2)$.** As explained at the end of §2, each point of $\mathcal{V}(d_1, d_2)$ is parametrised by a quadruple $(\Sigma_c, D_1, D_2, \xi)$. To understand the space of these quadruples we must understand how $H^1(\Sigma_c, K^{-2}(D_1 + D_2))$ varies with (Σ_c, D_1, D_2). Note that

$$\deg(K^{-2}(D_1 + D_2)) = d_1 + d_2 - 4(g - 1) < 0,$$

by the inequalities (2.16). Whenever a holomorphic line bundle $\mathcal{L}$ over Σ_c has negative degree d its first cohomology has, by the Riemann-Roch theorem, dimension

$$h^1(\mathcal{L}) = g - 1 - d.$$

Therefore as $\mathcal{L}$ moves over $\operatorname{Pic}_d(\Sigma_c)$, the Picard component of degree d line bundles, the dimension of $H^1(\Sigma_c, \mathcal{L})$ is constant. Now $\Sigma_c \times \operatorname{Pic}_d(\Sigma_c)$ carries a tautological line bundle $\mathcal{P}$

(sometimes called a Poincaré line bundle) whose fibre over $(p, \mathcal{L})$ is the fibre of $\mathcal{L}$ at p. The vector spaces $H^1(\Sigma_c, \mathcal{L})$ are the fibres of the higher direct image $R^1\pi_*(\mathcal{P})$ for the projection $\pi : \Sigma_c \times \mathrm{Pic}_d(\Sigma_c) \to \mathrm{Pic}_d(\Sigma_c)$ to the second factor. By a theorem of Grauert [26, III, Cor 12.9] their constant dimension implies they form a vector bundle over $\mathrm{Pic}_d(\Sigma_c)$. In particular, for $d = d_1 + d_2 - 4(g-1)$ this bundle has rank

$$h^1(K^{-2}(D_1 + D_2)) = 5g - 5 - d_1 - d_2.$$

The pairs (D_1, D_2) are parametrised by the product of symmetric products $S^{d_1}\Sigma_c \times S^{d_2}\Sigma_c$ (in which the co-prime pairs occupy an open subvariety). The bundle can be pulled back along the holomorphic map

$$S^{d_1}\Sigma_c \times S^{d_2}\Sigma_c \to \mathrm{Pic}_d(\Sigma_c); \quad (D_1, D_2) \mapsto K^{-2}(D_1 + D_2),$$

and the total space of the pullback parametrises the data (D_1, D_2, ξ). It is a connected non-singular complex manifold of dimension $5g - 5$.

All the objects in this discussion form complex analytic families over Teichmüller space, since $\mathcal{T}_g$ possesses a "universal curve" $\mathcal{C}_g$. It is a complex manifold which admits a fibration $\pi_{\mathcal{C}} : \mathcal{C}_g \to \mathcal{T}_g$ such that the fibre over c is a complex submanifold biholomorphic to Σ_c. This *Teichmüller curve* only has a bundle structure in the smooth category, but about each point it does have a *permanent uniformising local parameter*, i.e., a complex chart $(\mathcal{U}, \mathfrak{z})$ for which $\mathfrak{z} = (z_1, \ldots, z_{3g-3}, \zeta)$ has the properties that: (i) each non-empty intersection $\mathcal{U}_c = \pi_{\mathcal{C}}^{-1}(c) \cap \mathcal{U}$ is such that $(\mathcal{U}_c, \zeta)$ is a chart on Σ_c; (ii) the coordinates z_j are constant on the fibres. The existence of such a chart is an immediate consequence of Bers' construction of $\mathcal{C}_g$ as a quotient of an open submanifold $\mathcal{F}_g \subset \mathcal{T}_g \times \mathbb{C}$ by a properly discontinuous action of $\pi_1\Sigma$ which preserves the fibes over $\mathcal{T}_g$ [2] (this is just the standard picture of Kodaira-Spencer for unobstructed deformations of complex structure [32]). It follows that one can put complex charts on the symmetric fibre-products of $\mathcal{C}_g$ over $\mathcal{T}_g$, whose fibres are $S^d\Sigma_c$, to obtain non-singular analytic families over $\mathcal{T}_g$. The corresponding families of Picard components have been constructed by Earle [12].

Thus for each pair d_1, d_2 satisfying (2.16) we obtain a manifold parametrising the data $(\Sigma_c, D_1, D_2, \xi)$ with $\deg(D_j) = d_j$. Each is clearly a connected non-singular complex manifold of dimension $8g - 8$. Therefore we have proved the following theorem.

Theorem 6.1. *Each set $\mathcal{V}(d_1, d_2)$ can be given the structure of a non-singular complex manifold of dimension $8g-8$. With this structure $\mathcal{V}(d_1, d_2)$ is diffeomorphic to a bundle over Teichmüller space $\mathcal{T}_g$. The fibre of this bundle over $c \in \mathcal{T}_g$ is a complex submanifold biholomorphic to a holomorphic vector bundle over $S^{d_1}\Sigma_c \times S^{d_2}\Sigma_c$ of rank $5(g-1) - d_1 - d_2$.*

6.2. **The structure of $\mathcal{W}^{\pm}(b, l)$.** By Theorem 5.1 the set $\mathcal{W}^+(b, l)$ is parametrised by quadruples (Σ_c, B, L, η) with $\eta \in H^1(\Sigma_c, KL^{-1}(B))$ and $\eta \neq 0$. We want to show this cohomology space has constant dimension as the pair (B, L) moves over $S^b\Sigma_c \times \mathrm{Pic}_l(\Sigma_c)$. Its suffices to see that, by the stability inequalities (5.1),

$$\deg(KL^{-1}(B)) = 2g - 2 + b - l < \tfrac{1}{2}b - (g-1) < 0,$$

and therefore

$$h^1(KL^{-1}(B)) = l + 1 - b - g.$$

By much the same argument as above, for each marked conformal structure $c \in \mathcal{T}_g$ the triple (B, L, η) lies in a holomorphic vector bundle over $S^b\Sigma_c \times \mathrm{Pic}_l(\Sigma_c)$ of rank $l + 1 - b - g$ but we must remove the zero section to avoid $\eta = 0$. The total space of this punctured bundle has dimension

$$(b+g) + l + 1 - b - g = l + 1.$$

As c moves through $\mathcal{T}_g$ we obtain a complex analytic manifold, fibred over $\mathcal{T}_g$, of total dimension $3(g-1) + l + 1$.

Theorem 6.2. *Each set $\mathcal{W}^{\pm}(b,l)$ can be given the structure of a non-singular complex manifold of dimension $3(g-1)+l+1$. It is a smooth fibre bundle over $\mathcal{T}_g$ whose fibre at $c \in \mathcal{T}_g$ is a complex analytic submanifold biholomorphic to a punctured holomorphic vector bundle over $S^b\Sigma_c \times \mathrm{Pic}_l(\Sigma_c)$ (i.e., a bundle with its zero section removed) of rank $l+1-b-g$.*

6.3. **Map from $\mathcal{V}$ to $\mathcal{R}(G)$.** In order to understand when we can use minimal surface data to parametrise representations, we must understand the map

$$R : \mathcal{V} \to \mathcal{R}(G), \quad (\rho, f) \mapsto \rho. \tag{6.1}$$

We can expect this to be smooth. From the results above, this is likely to be most interesting on the components $\mathcal{V}(d_1, d_2)$ since these have the same dimension as $\mathcal{R}(G)$. While a full understanding of this map will require further work, we can at least make some interesting comments about its behaviour on the fibres $\mathcal{V}_c$ of $\mathcal{V}$ over Teichmüller space. With a fixed conformal structure c we can identify $\mathcal{R}(G)$ with the moduli space $\mathcal{H}(\Sigma_c, G)$ of G-Higgs bundles. Then R is injective on $\mathcal{V}_c$, since it amounts to inclusion (equally, this is a consequence of the uniqueness theorem for twisted harmonic maps [9, 10]). Indeed

$$\mathcal{V}_c = \{(E, \Phi) \in \mathcal{H}(\Sigma_c, G) : \mathrm{tr}\, \Phi^2 = 0\},$$

and so it plays the role of the *nilpotent cone* in $\mathcal{H}(\Sigma_c, G)$. Let us consider the structure of this in light of the discussion above. Recall that $\|\Phi\|^2_{L^2}$ is a proper Morse function on $\mathcal{H}(\Sigma_c, G)$ (at least at smooth points): we will normalise this by defining

$$\mathfrak{F}(E, \Phi) = \frac{i}{2} \int_\Sigma \mathrm{tr}(\Phi \wedge \Phi^\dagger).$$

Whenever the twisted harmonic map determined by (E, Φ) is conformal, we have

$$\mathfrak{F}(E, \Phi) = \int_\Sigma v_\gamma = \mathrm{Area}_\gamma(\Sigma),$$

for the induced metric γ. Now fix a non-maximal value τ for the Toledo invariant and consider the connected component $\mathcal{H}(\Sigma_c, G)_\tau$. Whenever $d_2 = \frac{3}{2}\tau + d_1$ this component contains $\mathcal{V}_c(d_1, d_2)$. Inside the latter lies the locus $\xi = 0$ consisting of length-three Hodge bundles, and this represents one connected critical manifold of $\mathfrak{F}$ (cf. [21, §3]). Since $\xi = 0$ exactly when $\mathcal{Q} = 0$ we deduce from Prop. 3.5 that this is the level $\mathfrak{F} = (4g - 4 - d_1 - d_2)\pi$. As we move along the fibres of the bundle $\mathcal{V}_c(d_1, d_2)$ Prop. 3.5 tells us that $\mathfrak{F} < (4g - 4 - d_1 - d_2)\pi$. Moreover, the dimension of these fibres equals the Morse index of the critical manifold [21, Prop 3.2]. Indeed, it seems that the fibres of the bundle $\mathcal{V}_c(d_1, d_2)$ agree with the unstable manifold of the Morse flow (at least where that makes sense, on the smooth components $\mathcal{H}(\Sigma_c, G)_\tau$, where $\tau \not\equiv 0 \mod 3$). This seems to be a manifestation of Hausel's theorem [27, Thm 5.2]. He proved that in the moduli space of stable $GL(n, \mathbb{C})$-Higgs bundles of odd degree, the downward Morse flow coincides with the nilpotent cone. Although Hausel only gave the proof for the complex case, the ingredients hold equally well in the case of the smooth components of $\mathcal{H}(\Sigma_c, G)$ for a real form G [22].

Even in $\mathcal{V}_c(0,0)$, which lies in the component of Toledo invariant zero, we can see a manifestation of this relationship. In $\mathcal{V}_c(0,0)$ the inequality in Prop. 3.5 becomes an equality, since we can parametrise this about $\xi = 0$ (which is a single point, representing the unique $\mathbb{R}$-Fuchsian representation corresponding to the marked conformal structure c) by the cubic holomorphic differential [36, Thm 9.2]. In that neighbourhood we have

$$\mathfrak{F} = 4g - 4 - \int_\Sigma \|\mathcal{Q}\|_\gamma^2 v_\gamma,$$

so that $\mathfrak{F}$ decreases along each ray $t\mathcal{Q}$ at least for some interval $0 \leq t < T_\mathcal{Q}$. Note that in the component $\mathcal{H}(\Sigma_c, G)_0$ the minimum value for $\mathfrak{F}$ is zero, which occurs on reducible Higgs bundles for which $E = V \oplus 1$ with V stable and degree zero (Remark 5.1). In that limit, the harmonic maps have collapsed to a constant.

Appendix A. Uniqueness of minimal embeddings.

Here we give the proof of Proposition 4.5. We are assuming that $\exp^\perp : TM^\perp \to N$ is a diffeomorphism, and therefore there is a radial distance function $\rho : N \to \mathbb{R}_0^+$ given by $\rho(p) = |(\exp^\perp)^{-1}(p)|$. The idea of the proof is to show that the condition on dv_r/dr means each $\varphi_r = \exp^\perp(r\eta)$ must have non-zero mean curvature, and that by local comparison every immersion must also have non-zero mean curvature at non-zero maximum values of ρ. Note that this is a local argument: we do not need the existence of global sections of $TM^\perp$ of unit length (which will not, in general, exist).

First recall that for a family of immersions $\varphi_t : M \times \mathbb{R} \to (N, g)$ with variational vector field $V = \varphi_* \partial/\partial_t$ a standard calculation gives

$$\frac{dv_\gamma(t)}{dt}|_0 = d \star g(V, d\varphi) - g(V, H_\varphi) v_\gamma,$$

where $H_\varphi = \operatorname{tr}_\gamma \mathrm{II}_\varphi$ is the mean curvature for $\gamma = \varphi^* g$. In particular, for the mean curvature H_r of the map φ_r,

$$d\rho(H_r) = g(\operatorname{grad}\rho, H_r) = -\frac{1}{v_r}\frac{dv_r}{dr}, \tag{A.1}$$

since $\operatorname{grad}\rho$ is a normal variation.

Next we show that a local embedding φ which comes from an arbitrary non-vanishing local section ν of $TM^\perp$ must have non-zero mean curvature at any point at which $|\nu|$ has a local maximum.

Lemma A.1. *Let $\varphi : \mathcal{U} \to N$ be an embedding of the form $\varphi = \exp^\perp(\nu)$ for some local section ν of $TM^\perp$ which does not vanish on an open subset $\mathcal{U} \subset M$, and suppose $u = \rho \circ \varphi$ has a local maximum at $x \in \mathcal{U}$. For each $r > 0$ set $\varphi_r = \exp^\perp(r\nu/u)$, and let v_r be the volume form for $\varphi_r^* g$. Suppose that $dv_r/dr > 0$ at x for each r, then φ must have non-zero mean curvature H_φ at x.*

Proof. Consider the expressions for the mean curvatures H_φ and H_r, considered as the tension fields $\tau(\varphi)$ and $\tau(\varphi_r)$, in terms the tension fields for $u = \rho \circ \varphi$ and the constant function $r = \rho \circ \varphi_r$. The composition formulas [13, 2.20] give

$$\begin{aligned} \tau(u) &= d\rho(H_\varphi) + \operatorname{tr}_\gamma \nabla d\rho(d\varphi, d\varphi), \\ 0 = \tau(\rho \circ \varphi_r) &= d\rho(H_r) + \operatorname{tr}_{\gamma_r} \nabla d\rho(d\varphi_r, d\varphi_r), \end{aligned}$$

where $\gamma = \varphi^* g$ and $\gamma_r = \varphi_r^* g$. Now $\tau(u) = \mathrm{tr}_\gamma \mathrm{Hess}(u)$ and at the local maximum x we have $du = 0$, which implies $d\varphi = d\varphi_{u(x)}$ and $\gamma = \gamma_{u(x)}$. Therefore at x we have

$$d\rho(H_\varphi)|_x = \mathrm{tr}_\gamma \mathrm{Hess}(u)|_x + d\rho(H_{u(x)})|_x.$$

Since x is a local maximum we have $\mathrm{tr}_\gamma \mathrm{Hess}(u)|_x \leq 0$, and by assumption $d\rho(H_r)|_x < 0$ for all $r > 0$, using (A.1). Thus H_φ cannot vanish at x. □

Proof of Prop 4.5. Suppose $\psi : M \to N$ is any immersion transverse to the fibres of $\exp^\perp$, other than f. The function $r \circ \psi$ must have a non-zero maximum at some $y \in M$. Then there is a local section ν of $TM^\perp$ and a local diffeomorphism α on M for which $\psi = \varphi \circ \alpha$ where $\varphi = \exp^\perp(\nu)$, and $H_\psi|_y = H_\varphi|_{\alpha(y)}$ as elements of $T_{\psi(y)}N$. Now $(\varphi, \alpha(y))$ satisfy the conditions of Lemma A.1, so $H_\psi|_y \neq 0$. □

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