FLAT METRICS, CUBIC DIFFERENTIALS AND LIMITS OF PROJECTIVE HOLONOMIES

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1. Introduction

In [13] and Labourie [12, 11], it is shown that on a closed oriented surface S of genus $g > 1$, there is a one-to-one correspondence between convex $\mathbb{RP}^2$ structures on S and pairs (Σ, U), where Σ is a conformal structure on S and U is a holomorphic cubic differential. In this note, we compute the asymptotic values of the holonomy of the $\mathbb{RP}^2$ structure corresponding to $(\Sigma, \lambda U_0)$ as $\lambda \to \infty$ around geodesic loops of the flat metric $|U_0|^{\frac{2}{3}}$ which do not touch any zeros of the fixed cubic differential U_0. (One may hope to achieve the same thing for general geodesics by exploiting a standard form of the metric near the zeros.) Such asymptotic holonomies are related to the compactification of the deformation space of convex $\mathbb{RP}^2$ structures on S due to Anne Parreau [17] and Inkang Kim [10] (see Section 2 below).

Theorem 1. *Let Σ be a closed Riemann surface of genus $g > 1$ and let U_0 be a holomorphic cubic differential on Σ. Consider a closed oriented geodesic $\mathcal{L}$ of the flat metric $|U_0|^{\frac{2}{3}}$ on Σ which does not touch any of the zeros of U_0. In terms of the flat coordinate z in which $U_0 = 2\,dz^3$, represent the deck transformation corresponding to $\mathcal{L}$ as a displacement $z \mapsto z + Le^{i\theta}$ for $L > 0$. Then there is a constant $\kappa > 0$ so that the eigenvalues $\xi_1 > \xi_2 > \xi_3$ of the $\mathbf{SL}(3, \mathbb{R})$ holonomy along $\mathcal{L}$ for the $\mathbb{RP}^2$ structure determined by the pair $(\Sigma, \lambda U_0)$ for $\lambda > 0$ satisfy*

$$\kappa \xi_i > e^{\lambda^{\frac{1}{3}} \mu_i L} > \kappa^{-1} \xi_i$$

for $\mu_1 \geq \mu_2 \geq \mu_3$ the roots of the equation

$$\mu^3 - 3\mu - 2\cos 3\theta = 0.$$

The techniques involved in the proof are similar to the analysis of the harmonic map equation between hyperbolic surfaces, as discussed by Mike Wolf [21], Y. Minsky [15] and Z.C. Han [7], and some new results on asymptotics of linear systems of ODEs.

The present paper may be thought of as something of sequel to [14], which studies the behavior of $\mathbb{RP}^2$ surfaces corresponding to (Σ, U) as Σ

approaches the boundary of the Deligne-Mumford compactification of the moduli space of Riemann surfaces, and U degenerates to a regular cubic differential.

In future work, we hope to extend this analysis to all geodesics with respect to the singular flat metric $|U_0|^{\frac{2}{3}}$, including those which are singular at the zeros of U_0. This will allow a full description of the data Kim prescribes for the boundary of the deformation space of convex $\mathbb{RP}^2$ structures. It will also be interesting to relate the present work to harmonic maps to $\mathbb{R}$-buildings, as an extension of Wolf's work on harmonic maps to $\mathbb{R}$-trees [22].

I would like to thank Mike Wolf, for, some years ago, pointing out the similarities between the analytic theories of convex $\mathbb{RP}^2$ structures and harmonic maps between hyperbolic surfaces. I also thank Bill Goldman for his encouragement and many fruitful discussions about $\mathbb{RP}^2$ structures, and Lee Mosher for useful discussions. The author is partially supported by NSF Grant DMS0405873.

2. The boundary of the deformation space of convex $\mathbb{RP}^2$ structures

It is well known that a closed hyperbolic surface is determined by its *length spectrum*, which consists of the hyperbolic lengths of the unique geodesic in each free homotopy class of curves (see e.g. [5]). More concretely, hyperbolic lengths of geodesics provide an embedding of Teichmüller space into $\mathbb{R}^{\mathcal{C}}$, where $\mathcal{C}$ is the set of all nontrivial conjugacy classes in $\pi_1(S)$ for a closed surface S of genus $g > 1$. Then Thurston's boundary of Teichmüller space can be described as the set of limit points of sequences in Teichmüller space $\subset \mathbb{R}^{\mathcal{C}}$, when projected to the projective space $\mathbb{PR}^{\mathcal{C}}$ [16, 1, 18].

There is an analog of this theory to convex $\mathbb{RP}^2$ surfaces due to Paulin [19], Parreau [17] and Inkang Kim [9, 10] (these authors address more general structures as well). Recall a (properly) convex $\mathbb{RP}^2$ surface S is given by $S = \Omega/\Gamma$, where $\Omega \Subset \mathbb{R}^2 \subset \mathbb{RP}^2$ is a convex set and $\Gamma \subset \mathbf{PGL}(3,\mathbb{R})$. Then each element $\gamma \in \Gamma$ may be represented as a matrix in $\mathbf{SL}(3,\mathbb{R})$. The eigenvalues of this matrix are then analogs of the hyperbolic length (see in particular Goldman [6] for a detailed analog of the Fenchel-Nielsen theory of Teichmüller space for the case of convex $\mathbb{RP}^2$ structures). In particular, for a given $\gamma \in \Gamma$ with eigenvalues $\nu_1 > \nu_2 > \nu_3 > 0$ (the eigenvalues have this structure by [8]), the set of logarithms

$$(\ell_1, \ell_2, \ell_3) = (\log \nu_1, \log \nu_2, \log \nu_3)$$

is naturally an element of the positive Weyl chamber of the maximal torus $\mathfrak{t}$ of the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$. Kim [9] shows that a normalized version of map of logarithms of eigenvalues (into $\mathfrak{t}^{\mathcal{C}}$) determines the $\mathbb{RP}^2$ structure. (The normalization is an analog of projectivization of $\mathbb{R}^{\mathcal{C}}$ mentioned above for Teichmüller space.) Then in [19, 17, 10], the boundary of the deformation space of convex $\mathbb{RP}^2$ structures may be defined to be the boundary in $\mathfrak{t}^{\mathcal{C}}$ of the image of the deformation space of all convex $\mathbb{RP}^2$ structures.

The limiting spectra in the case of both hyperbolic lengths and $\mathbb{RP}^2$ structures can be seen as naturally arising in the context of $\pi_1(S)$ actions on $\mathbb{R}$-buildings. Consider Teichmüller space as the quotient of the space of hyperbolic metrics on S by the group of diffeomorphisms isotopic to the identity. So a background hyperbolic structure on S induces a conformal structure Σ_0 on S, and Teichmüller space can be thought of as parametrized by the unique harmonic map homotopic to the identity from Σ_0 to the target hyperbolic structure [21, Wolf]. In turn, these harmonic maps are uniquely determined by a holomorphic quadratic differential Ψ on Σ_0. The key equation to solve to construct the harmonic map is

$$\Delta v + 4e^{-v}\|\Psi\|^2 - 2e^v + 2 = 0,$$

where Δ and $\|\cdot\|$ are determined by the hyperbolic metric on Σ_0. Wolf then essentially studies solutions to this equation to reproduce Thurston's compactification of Teichmüller space as limits of hyperbolic structures for quadratic differentials $\lambda\Psi_0$ as $\lambda \to \infty$ [21], and later uses the same estimates to produce a $\pi_1(S)$ equivariant harmonic map to an appropriate $\mathbb{R}$-tree [22].

The equation we use (due to C.P. Wang [20]) to produce $\mathbb{RP}^2$ structures,

$$\Delta u + 4e^{-2u}\|U\|^2 - 2e^u - 2\kappa = 0,$$

is very similar to Wolf's equation, with a cubic differential U replacing the quadratic differential Ψ. There should also be an analogous theory: The limiting structures for $U = \lambda U_0$ for $\lambda \to \infty$ should be realized as an action of $\pi_1(S)$ on a $\mathbb{R}$-building, together with a $\pi_1(S)$ equivariant map. We hope to address this problem in future work.

3. Hyperbolic affine spheres and convex $\mathbb{RP}^n$ structures

Recall the standard definition of $\mathbb{RP}^n$ as the set of lines through 0 in $\mathbb{R}^{n+1}$. Consider $\pi : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$ with fiber $\mathbb{R}^*$. For a convex domain $\Omega \subset \mathbb{R}^n \subset \mathbb{RP}^n$ as above, then $\pi^{-1}(\Omega)$ has two connected components. Call one such component $\mathcal{C}(\Omega)$, the *cone over* Ω. Then any representation of a group Γ into $\mathbf{PGL}(n+1,\mathbb{R})$ so that Γ acts discretely and properly discontinously on Ω lifts to a representation into

$$\mathbf{SL}^{\pm}(n+1,\mathbb{R}) = \{A \in \mathbf{GL}(n+1,\mathbb{R}) : \det A = \pm 1\}$$

which acts on $\mathcal{C}(\Omega)$. See e.g. [13].

For a properly convex Ω, there is a unique hypersurface asymptotic to the boundary of the cone $\mathcal{C}(\Omega)$ called the hyperbolic affine sphere [2, 3, 4]. This hyperbolic affine sphere $H \subset \mathcal{C}(\Omega)$ is invariant under automorphisms of $\mathcal{C}(\Omega)$ in $\mathbf{SL}^{\pm}(n+1,\mathbb{R})$. The projection map π induces a diffeomorphism of H onto Ω. Affine differential geometry provides $\mathbf{SL}^{\pm}(n+1,\mathbb{R})$-invariant structure on H which then descends to $M = \Omega/\Gamma$. In particular, both the affine metric, which is a Riemannian metric conformal to the (Euclidean) second fundamental form of H, and a projectively flat connection whose geodesics are the $\mathbb{RP}^n$ geodesics on M, descend to M. See [13] for details. A fundamental fact about hyperbolic affine spheres is due to Cheng-Yau [4] and Calabi-Nirenberg (unpublished):

Theorem 2. *If the affine metric on a hyperbolic affine sphere H is complete, then H is properly embedded in $\mathbb{R}^{n+1}$ and is asymptotic to a convex cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ which contains no line. By a volume-preserving affine change of coordinates in $\mathbb{R}^{n+1}$, we may assume $\mathcal{C} = \mathcal{C}(\Omega)$ for some properly convex domain Ω in $\mathbb{RP}^n$.*

Note that if $S = \Omega/\Gamma$ is compact, then Cheng-Yau's completeness condition on the affine metric is satisfied by *any* appropriate affine metric on S.

4. Wang's developing map

In dimension 2, there is a local theory due to C.P. Wang [20] which exploits the elliptic PDE nature of the problem of finding hyperbolic affine spheres to relate oriented convex $\mathbb{RP}^2$ surfaces to holomorphic data on Riemann surfaces. See also Labourie [11] and Loftin [14]. In particular, the affine metric of a 2-dimensional hyperbolic affine sphere induces a conformal structure on the surface, and, moreover, there is a holomorphic cubic differential U (which is essentially the difference between the Levi-Civita connection of the affine metric and the projectively flat connection of the $\mathbb{RP}^2$ structure) induced by the affine sphere. All this structure descends to projective quotients of the hyperbolic affine sphere. In particular, we have the following

Theorem 3. *On a compact oriented surface S, a convex $\mathbb{RP}^2$ structure is equivalent to the pair of a conformal structure Σ and a holomorphic cubic differential U.*

Locally, the structure equations of a 2-dimensional hyperbolic affine sphere may be expressed in terms of a embedding map $f : \Omega \to \mathbb{R}^3$, where $\Omega \subset \mathbb{C}$ is a simply-connected domain. We take f to be conformal with respect to the affine metric $e^\psi|dz|^2$ and U to be a holomorphic function. Then f satisfies

$$(1) \qquad \begin{cases} f_{zz} = \psi_z f_z + U e^{-\psi} f_{\bar z} \\ f_{\bar z\bar z} = \bar U e^{-\psi} f_z + \psi_{\bar z} f_{\bar z} \\ f_{z\bar z} = \frac12 e^\psi f \end{cases}$$

The conformal factor e^ψ must satisfy the following integrability condition,

$$(2) \qquad \psi_{z\bar z} + |U|^2 e^{-2\psi} - \tfrac12 e^\psi = 0,$$

which we call Wang's equation. On a Riemann surface U transforms as a cubic differential, and (2) becomes, with respect to a conformal background metric h,

$$(3) \qquad \Delta u + 4e^{-2u}\|U\|^2 - 2e^u - 2\kappa = 0,$$

where Δ is the Laplacian of h, $\|U\|^2$ is the norm-squared of U with respect to the metric h, κ is the Gauss curvature of h, and the metric $e^u h = e^\psi|dz|^2$ locally for ψ given by (2).

We now study solutions to (3) for $U = \lambda U_0$ as $\lambda \to \infty$.

5. Limits of the conformal metrics

Let U_0 be a holomorphic cubic differential on Σ which is not identically zero. We study the limiting behavior of solutions to Wang's equation (3) for solutions u_λ as $U = \lambda U_0$ for λ a real parameter approaching ∞. In his work on harmonic maps between hyperbolic surfaces and Thurston's boundary of Teichmüller space, Mike Wolf has studied a similar equation to (3) with a holomorphic quadradic differential instead of a cubic differential [21]. The proof below is similar to the one in Han [7].

Proposition 1. *Let Σ be a closed Riemann surface of genus $g > 1$ equipped with a background metric h and a holomorphic cubic differential U_0 which is not identically zero. Let $\lambda > 0$ and let $u = u_\lambda$ be the solution to (3) for $U = \lambda U_0$. Let K be a compact subset of Σ which does not contain any of the zeroes of U_0. Then there is a constant $C = C(\Sigma, U_0, K)$ so that*

$$\frac12 \ge \|U\|^2 e^{-3u_\lambda} \ge \frac12 - C\lambda^{-\frac23}.$$

Proof. We prove this proposition by the use of barriers. The key observation is that the singular flat conformal metric $2^{\frac{1}{3}}|U|^{\frac{2}{3}}$ provides a solution to (2) away from the zeros of U.

Consider a smooth background metric g by requiring $g = 2^{\frac{1}{3}}|U_0|^{\frac{2}{3}}$ on K and $\|U_0\|_g^2 \le \frac{1}{2}$ on all Σ. (This is possible since $\|U_0\|_g^2 = \frac{1}{2}$ on K and $\|U_0\|_g^2 = 0$ at the zeros of U_0.)

Now for $U = \lambda U_0$, define $s = s_\lambda$ by

$$ge^s = 2^{\frac{1}{3}}|U|^{\frac{2}{3}} = 2^{\frac{1}{3}}\lambda^{\frac{2}{3}}|U_0|^{\frac{2}{3}}.$$

Note that $s = \frac{2}{3}\log\lambda$ on K. We may also check that s solves (3) away from the zeros of U, and is equal to $-\infty$ at the zeros of U. By applying the comparison principle to (3), we find that $u \ge s$, and so s is a subsolution of (3).

Now let $S = S_\lambda$ be equal to $\log r$ for $r = r_\lambda$ the positive root of

$$p(x) = x^3 - \sigma x^2 - \lambda^2 = 0, \qquad \sigma = \max_\Sigma(-\kappa_g),$$

for κ_g the Gauss curvature of g. Then S is a supersolution of (3): At a maximum point of u,

$$\begin{aligned} 0 &\ge \Delta_g u = 2e^u + 2\kappa - 4e^{-2u}\|U\|_g^2, \\ &\ge 2e^{-2u}(e^{3u} - \sigma e^{2u} - \lambda^2). \end{aligned}$$

The largest value of u for which this inequality can be true occurs when $p(e^u) = 0$.

On K then, $\frac{2}{3}\log\lambda \le u \le S$, and so

$$\frac{1}{2} \ge \|U\|_g^2 e^{-3u} \ge \frac{1}{2}\lambda^2 e^{-3S}.$$

Now we note that $\tilde{x} = \lambda^{-\frac{2}{3}}e^S$ solves

$$\tilde{x}^3 - \sigma\lambda^{-\frac{2}{3}}\tilde{x}^2 - 1 = 0,$$

and so for large values of λ, $\tilde{x} = 1 + O(\lambda^{-\frac{2}{3}})$. This proves the proposition. □

Corollary 2. *There is another constant $C = C(\Sigma, U_0, K)$ so that $|\psi_z| \le C\lambda^{-\frac{1}{3}}$ on K, where z is a local coordinate so that $U_0 = 2\,dz^3$.*

Proof. Note that in the proof and below, different uniform constants may be referred to by the same letter C depending on the context.

For $p \in K$, choose the local coordinate z so that $z(p) = 0$ and let consider

$$\alpha(w) = \psi(\lambda^{-\frac{1}{3}}w) - \tfrac{2}{3}\log\lambda - \tfrac{1}{3}\log 2.$$

Then

$$\alpha_{w\bar{w}} = \lambda^{-\frac{2}{3}}\psi_{z\bar{z}} = 2^{-\frac{2}{3}}(e^{-2\alpha} - e^{\alpha}).$$

Proposition (1) implies that there is a constant C so that

$$0 \le \alpha(\lambda^{\frac{1}{3}} z) = \psi(z) - \tfrac{2}{3}\log\lambda - \tfrac{1}{3}\log 2 \le C\lambda^{-\frac{2}{3}}$$

for all z in a neighborhood of K.

This implies that in any disk in the w-plane centered at 0, there is a constant C independent of $p \in K$ and λ large so that

$$|\alpha|\,, |\alpha_{w\bar w}| \le C\lambda^{-\frac{2}{3}}.$$

Then the L^p theory implies that on a slightly smaller disk, that $\|\alpha\|_{W^{2,p}} \le C\lambda^{-\frac{2}{3}}$. Then, for $p > 2$, Sobolev embedding implies similar bounds for the C^1 norm of α:

$$|\alpha_w| \le C\lambda^{-\frac{2}{3}}.$$

Now simply compute $\psi_z = \lambda^{\frac{1}{3}}\alpha_w$. □

6. ODE estimates

Now the structure equations (1) can be recast in terms of the frame $\langle f, \lambda^{-\frac{1}{3}} f_z, \lambda^{-\frac{1}{3}} f_{\bar z}\rangle$ to read

$$(4)\quad \begin{pmatrix} f \\ \lambda^{-\frac{1}{3}} f_z \\ \lambda^{-\frac{1}{3}} f_{\bar z} \end{pmatrix}_z = \begin{pmatrix} 0 & \lambda^{\frac{1}{3}} & 0 \\ 0 & \psi_z & Ue^{-\psi} \\ \frac{1}{2}\lambda^{-\frac{1}{3}} e^{\psi} & 0 & 0 \end{pmatrix} \begin{pmatrix} f \\ \lambda^{-\frac{1}{3}} f_z \\ \lambda^{-\frac{1}{3}} f_{\bar z} \end{pmatrix},$$

$$(5)\quad \begin{pmatrix} f \\ \lambda^{-\frac{1}{3}} f_z \\ \lambda^{-\frac{1}{3}} f_{\bar z} \end{pmatrix}_{\bar z} = \begin{pmatrix} 0 & 0 & \lambda^{\frac{1}{3}} \\ \frac{1}{2}\lambda^{-\frac{1}{3}} e^{\psi} & 0 & 0 \\ 0 & \bar U e^{-\psi} & \psi_{\bar z} \end{pmatrix} \begin{pmatrix} f \\ \lambda^{-\frac{1}{3}} f_z \\ \lambda^{-\frac{1}{3}} f_{\bar z} \end{pmatrix}.$$

Away from the zeros of U_0, choose a local coordinate z so that $U_0 = 2\,dz^3$, and $U = \lambda U_0 = 2\lambda\,dz^3$. Proposition 1 and Corollary 2 then show that the matrices in the structure equations above have the form

$$(6)\, P = \begin{pmatrix} 0 & \lambda^{\frac{1}{3}} & 0 \\ 0 & \psi_z & Ue^{-\psi} \\ \frac{1}{2}\lambda^{-\frac{1}{3}} e^{\psi} & 0 & 0 \end{pmatrix} = \lambda^{\frac{1}{3}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + O(\lambda^{-\frac{1}{3}}),$$

$$(7)\, Q = \begin{pmatrix} 0 & 0 & \lambda^{\frac{1}{3}} \\ \frac{1}{2}\lambda^{-\frac{1}{3}} e^{\psi} & 0 & 0 \\ 0 & \bar U e^{-\psi} & \psi_{\bar z} \end{pmatrix} = \lambda^{\frac{1}{3}} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + O(\lambda^{-\frac{1}{3}}),$$

where $O(\lambda^{-\frac{1}{3}})$ is as $\lambda \to \infty$ for all points in K a compact set not containing any zero of U_0.

We will integrate the initial value problem along a geodesic path with respect to the metric $|U_0|^{\frac{2}{3}}$ which avoids the zeroes of U_0. These paths are simply straight lines in each local complex coordinate chart with coordinate z satisfying $U_0 = 2\,dz^3$. In the particular case of a geodesic loop, the system of ODEs (4-5) will compute the real projective holonomy around such a loop: along such a loop, the coordinate z can be analytically continued in the universal cover, and the corresponding deck transformation corresponds to $z \mapsto z + c$ for a complex constant c. Therefore, the frame $\langle f, \lambda^{-\frac{1}{3}} f_z, \lambda^{-\frac{1}{3}} f_{\bar z}\rangle$ is the frame of a rank-3 vector bundle on the quotient whose holonomy in $\mathbf{GL}(3,\mathbb{R})$ projects to $\mathbf{PGL}(3,\mathbb{R})$ to compute the real projective holonomy of the geodesic loop. For more details of this argument, see e.g. Proposition 2 of [14].

Any geodesic loop of $|U_0|^{\frac{2}{3}}$ which avoids the zeroes of U_0 may be described by a starting point, at which we set the local coordinate z to be 0, and a finishing point, which we set to be $z = c$ in the analytically continued z coordinate. The geodesic is then the straight line segment between 0 and c. If $c = Le^{i\theta}$ for $L > 0$, then the holonomy with respect to the frame $\mathcal{F} = \langle f, \lambda^{-\frac{1}{3}} f_z, \lambda^{-\frac{1}{3}} f_{\bar z}\rangle$ is $\Phi(L)$, where Φ solves the initial value problem

$$\Phi(0) = I, \qquad \frac{d\Phi}{dt} = (e^{i\theta} P + e^{-i\theta} Q)\Phi.$$

This ODE system is equivalent to

$$\frac{d\Phi}{dt} = \left[\lambda^{\frac{1}{3}} \begin{pmatrix} 0 & e^{i\theta} & e^{-i\theta} \\ e^{-i\theta} & 0 & e^{i\theta} \\ e^{i\theta} & e^{-i\theta} & 0 \end{pmatrix} + O(\lambda^{-\frac{1}{3}})\right] \Phi.$$

As we are primarily interested in the eigenvalues of $\Phi(L)$, we replace the matrix

$$M = \begin{pmatrix} 0 & e^{i\theta} & e^{-i\theta} \\ e^{-i\theta} & 0 & e^{i\theta} \\ e^{i\theta} & e^{-i\theta} & 0 \end{pmatrix}$$

by the conjugate diagonal matrix

$$\begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$$

for μ_i the roots of the characteristic equation

$$\det(\mu I - M) = \mu^3 - 3\mu - 2\cos 3\theta = 0.$$

We note M is diagonalizable and $\mu_i \in \mathbb{R}$. Assume $\mu_1 \geq \mu_2 \geq \mu_3$.

Then, to compute the conjugacy class of the holonomy matrix around this geodesic loop, we compute the solution to

$$\Phi(0) = I, \tag{8}$$

$$\frac{d\Phi}{dt} = \left[\lambda^{\frac{1}{3}}\begin{pmatrix}\mu_1 & 0 & 0\\ 0 & \mu_2 & 0\\ 0 & 0 & \mu_3\end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix}\right]\Phi, \tag{9}$$

where there is a constant C so that each $b_{ij} = b_{ij}(t,\lambda)$ satisfies $|b_{ij}| \le C\lambda^{-\frac{1}{3}}$.

Proposition 3. *The solution Φ to the initial value problem (8-9) has the form*

$$\begin{pmatrix} e^{\lambda^{\frac13}\mu_1 t} + O(\lambda^{-\frac13}e^{\lambda^{\frac13}\mu_1 t}) & O(\lambda^{-\frac13}e^{\lambda^{\frac13}\mu_2 t}) & O(\lambda^{-\frac13}e^{\lambda^{\frac13}\mu_3 t}) \\ O(\lambda^{-\frac13}e^{\lambda^{\frac13}\mu_1 t}) & e^{\lambda^{\frac13}\mu_2 t} + O(\lambda^{-\frac13}e^{\lambda^{\frac13}\mu_2 t}) & O(\lambda^{-\frac13}e^{\lambda^{\frac13}\mu_3 t}) \\ O(\lambda^{-\frac13}e^{\lambda^{\frac13}\mu_1 t}) & O(\lambda^{-\frac13}e^{\lambda^{\frac13}\mu_2 t}) & e^{\lambda^{\frac13}\mu_3 t} + O(\lambda^{-\frac13}e^{\lambda^{\frac13}\mu_3 t}) \end{pmatrix},$$

Where the O notation denotes bounds as $\lambda \to \infty$ that are uniform for $t \in [0, L]$.

Proof. Write $\Phi = (\phi_{ij})$, and consider the first column $\phi_{11}, \phi_{21}, \phi_{31}$, which satisfies the linear system

$$\begin{aligned} \phi_{11}(0) &= 1, & \tfrac{d}{dt}\phi_{11} &= (\lambda^{\frac13}\mu_1 + b_{11})\phi_{11} + b_{12}\phi_{21} + b_{13}\phi_{31}, \\ \phi_{21}(0) &= 0, & \tfrac{d}{dt}\phi_{21} &= b_{21}\phi_{11} + (\lambda^{\frac13}\mu_2 + b_{22})\phi_{21} + b_{23}\phi_{31}, \\ \phi_{31}(0) &= 0, & \tfrac{d}{dt}\phi_{31} &= b_{31}\phi_{11} + b_{32}\phi_{21} + (\lambda^{\frac13}\mu_3 + b_{33})\phi_{31}. \end{aligned}$$

Each of the above differential equations is first-order linear, and so we must have

$$\begin{aligned} \phi_{11} &= e^{\lambda^{\frac13}\mu_1 t} e^{\int_0^t b_{11}} \left[1 + \int_0^t e^{-\lambda^{\frac13}\mu_1\tau - \int_0^\tau b_{11}} (b_{12}\phi_{21} + b_{13}\phi_{31})\,d\tau\right], \\ \phi_{21} &= e^{\lambda^{\frac13}\mu_2 t} e^{\int_0^t b_{22}} \int_0^t e^{-\lambda^{\frac13}\mu_2\tau - \int_0^\tau b_{22}} (b_{21}\phi_{11} + b_{23}\phi_{31})\,d\tau, \\ \phi_{31} &= e^{\lambda^{\frac13}\mu_3 t} e^{\int_0^t b_{33}} \int_0^t e^{-\lambda^{\frac13}\mu_3\tau - \int_0^\tau b_{33}} (b_{31}\phi_{11} + b_{32}\phi_{21})\,d\tau. \end{aligned}$$

The previous three equations can be seen as a map $\mathcal{M}$ from the $\mathbb{R}^3$-valued function $(\phi_{11}, \phi_{21}, \phi_{31})$ to the right-hand sides.

Now let $N \gg 1$ be a constant independent of λ, and consider the Banach space $\mathcal{B}_\lambda$ of continuous $\mathbb{R}^3$-valued functions with norm

$$\|(f_1, f_2, f_3)\|_{\mathcal{B}_\lambda} = \sup_i \sup_{t\in[0,L]} |f_i(t)| e^{-\lambda^{\frac13}\mu_1 t}.$$

Let $\mathcal{B}_\lambda(N)$ be the closed ball of radius N centered at the origin in $\mathcal{B}_\lambda$. We claim that for λ large enough, $\mathcal{M}$ is a contraction map from $\mathcal{B}_\lambda(N)$ to itself, and thus the solution $(\phi_{11}, \phi_{21}, \phi_{31})$ to the ODE system, which is the fixed point of $\mathcal{M}$, must lie in $\mathcal{B}_\lambda(N)$.

Now consider $F = (f_1, f_2, f_3)$, $G = (g_1, g_2, g_3) \in \mathcal{B}_\lambda(N)$. Then the first component of $\mathcal{M}(F) - \mathcal{M}(G)$ is given by

$$e^{\lambda^{\frac{1}{3}}\mu_1 t} e^{\int_0^t b_{11}} \int_0^t e^{-\lambda^{\frac{1}{3}}\mu_1\tau - \int_0^\tau b_{11}} [b_{12}(f_2 - g_2) + b_{13}(f_3 - g_3)] d\tau.$$

Now assume $|b_{ij}| \le R$ and recall $t \le L$. Then a straightforward calculation shows that the first component of $\mathcal{M}(F) - \mathcal{M}(G)$ is pointwise bounded by

$$e^{\lambda^{\frac{1}{3}}\mu_1 t} e^{2RL} \cdot 2R \cdot L \cdot \|F - G\|_{\mathcal{B}_\lambda},$$

and so if we choose λ large enough so that $R \sim \lambda^{-\frac{1}{3}}$ is small enough, we may assume $e^{2RL} \cdot 2R \cdot L < 1$. Essentially the same calculation shows that $\mathcal{M} \colon \mathcal{B}_\lambda(N) \to \mathcal{B}_\lambda(N)$ for large λ, since $N \gg 1$. The two other components of $\mathcal{M}$ behave the same way. All this shows $\mathcal{M}$ is a contraction map.

Since $\mathcal{M}$ is a contraction map on the complete metric space $\mathcal{B}_\lambda(N)$, the unique solution $(\phi_{11}, \phi_{21}, \phi_{31})$ to the ODE system is the fixed point, and so must be in $B_\lambda(N)$ for all λ sufficiently large. Now simply apply the bounds

$$|\phi_{11}|,\ |\phi_{21}|,\ |\phi_{31}| \le N e^{\lambda^{\frac{1}{3}}\mu_1 t}$$

to the fixed point equation $(\phi_{11}, \phi_{21}, \phi_{31}) = \mathcal{M}(\phi_{11}, \phi_{21}, \phi_{31})$ to show that

$$\phi_{11} = e^{\lambda^{\frac{1}{3}}\mu_1 t} + O(\lambda^{-\frac{1}{3}} e^{\lambda^{\frac{1}{3}}\mu_1 t}), \quad \phi_{21} = O(\lambda^{-\frac{1}{3}} e^{\lambda^{\frac{1}{3}}\mu_1 t}), \quad \phi_{31} = O(\lambda^{-\frac{1}{3}} e^{\lambda^{\frac{1}{3}}\mu_1 t}).$$

This justifies the first column in the matrix in Proposition 3. The argument for the other two columns is identical. □

Theorem 4. *There is a constant $\kappa > 0$ so that the eigenvalues $\xi_1 \ge \xi_2 \ge \xi_3 > 0$ of the holonomy matrix $\Phi(L)$ satisfy*

$$\kappa \xi_i > e^{\lambda^{\frac{1}{3}}\mu_i L} > \kappa^{-1}\xi_i$$

for $i = 1, 2, 3$.

Proof. Proposition 3 and the fact that $\Phi(L) \in \mathbf{SL}(3, \mathbb{R})$ show that the characteristic polynomial of $\Phi(L)$ is

$$x^3 - (e^{\lambda^{\frac{1}{3}}\mu_1 L} + e^{\lambda^{\frac{1}{3}}\mu_2 L} + e^{\lambda^{\frac{1}{3}}\mu_3 L})[1 + O(\lambda^{-\frac{1}{3}})]x^2 +$$
$$(e^{\lambda^{\frac{1}{3}}(\mu_1+\mu_2)L} + e^{\lambda^{\frac{1}{3}}(\mu_1+\mu_3)L} + e^{\lambda^{\frac{1}{3}}(\mu_2+\mu_3)L})[1 + O(\lambda^{-\frac{1}{3}})]x - 1.$$

Kac-Vinberg showed [8] that the holonomy of any nontrivial loop in a closed oriented convex $\mathbb{RP}^2$ surface of genus $g > 1$ has positive distinct eigenvalues $\xi_1 > \xi_2 > \xi_3 > 0$. Then

$$\xi_1 + \xi_2 + \xi_3 = (e^{\lambda^{\frac{1}{3}}\mu_1 L} + e^{\lambda^{\frac{1}{3}}\mu_2 L} + e^{\lambda^{\frac{1}{3}}\mu_3 L})[1 + O(\lambda^{-\frac{1}{3}})]$$

implies that there is an ϵ which goes to 0 as $\lambda \to \infty$ so that

$$(3+\epsilon)e^{\lambda^{\frac{1}{3}}\mu_1 L} > \xi_1 > (\tfrac{1}{3} - \epsilon)e^{\lambda^{\frac{1}{3}}\mu_1 L}.$$

Now use the bounds on ξ_1 and

$$\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3 = (e^{\lambda^{\frac{1}{3}}(\mu_1+\mu_2)L} + e^{\lambda^{\frac{1}{3}}(\mu_1+\mu_3)L} + e^{\lambda^{\frac{1}{3}}(\mu_2+\mu_3)L})[1 + O(\lambda^{-\frac{1}{3}})]$$

to conclude that there is an $\epsilon' \to 0$ as $\lambda \to \infty$ so that

$$(9+\epsilon')e^{\lambda^{\frac{1}{3}}\mu_2 L} > \xi_2 > (\tfrac{1}{9} - \epsilon')e^{\lambda^{\frac{1}{3}}\mu_2 L}.$$

Then the theorem follows from $\mu_1 + \mu_2 + \mu_3 = 0$ and

$$\xi_1\xi_2\xi_3 = 1.$$

□

Since $\Phi(L)$ is conjugate to the holonomy matrix with respect to the frame $\langle f, \lambda^{-\frac{1}{3}} f_z, \lambda^{-\frac{1}{3}} f_{\bar{z}}\rangle$ around the loop $\mathcal{L}$, this concludes the proof of Theorem 1.

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