CONVEX $\mathbb{RP}^2$ STRUCTURES AND CUBIC DIFFERENTIALS UNDER NECK SEPARATION

John Loftin

Abstract

Let S be a closed oriented surface of genus at least two. Labourie and the author have independently used the theory of hyperbolic affine spheres to find a natural correspondence between convex $\mathbb{RP}^2$ structures on S and pairs (Σ, U) consisting of a conformal structure Σ on S and a holomorphic cubic differential U over Σ. We consider geometric limits of convex $\mathbb{RP}^2$ structures on S in which the $\mathbb{RP}^2$ structure degenerates only along a set of simple, non-intersecting, nontrivial, non-homotopic loops c. We classify the resulting $\mathbb{RP}^2$ structures on $S - c$ and call them regular convex $\mathbb{RP}^2$ structures. Under a natural topology on the moduli space of all regular convex $\mathbb{RP}^2$ structures on S, this space is homeomorphic to the total space of the vector bundle over $\overline{\mathcal{M}}_g$ each of whose fibers over a noded Riemann surface is the space of regular cubic differentials. The proof relies on previous techniques of the author, Benoist-Hulin, and Dumas-Wolf, as well as some details due to Wolpert of the geometry of hyperbolic metrics on conformal surfaces in $\overline{\mathcal{M}}_g$.

1. Introduction

A convex $\mathbb{RP}^2$ surface is given as a quotient $\Gamma \backslash \Omega$, where Ω is a convex domain in $\mathbb{R}^2 \subset \mathbb{RP}^2$ and Γ is a discrete subgroup of $\mathbf{PGL}(3,\mathbb{R})$ acting discretely and properly discontinuously on Ω. For all convex $\mathbb{RP}^2$ surfaces of negative Euler characteristic, we may also assume that $\Omega \subset \mathbb{R}^2$ is bounded (and the surface is *properly convex*). We assume our convex $\mathbb{RP}^2$ surfaces are oriented, and it is natural in this case to lift the action of $\mathbf{PGL}(3,\mathbb{R})$ to an action of $\mathbf{SL}(3,\mathbb{R})$ acting on the convex cone over Ω in $\mathbb{R}^3$. We will not be careful in distinguishing between the groups $\mathbf{PGL}(3,\mathbb{R})$ and $\mathbf{SL}(3,\mathbb{R})$. Labourie and the author independently showed that a marked convex $\mathbb{RP}^2$ structure on a closed oriented surface S of genus g at least two is equivalent to a pair (Σ, U), where Σ is a marked conformal structure on S and U is a holomorphic cubic differential [**37**, **38**, **46**]. This result relies on the geometry of hyperbolic affine spheres, in particular on results of C.P. Wang [**71**] and deep geometric and analytic results of Cheng-Yau [**12**, **13**]. This provides a complex structure on the deformation space $\mathcal{G}_S$ of marked convex $\mathbb{RP}^2$ structures on S. Moreover, we can mod out by the mapping class group to find that the moduli space $\mathcal{R}_S$ of unmarked oriented convex $\mathbb{RP}^2$ structures is given by the total space of the bundle of holomorphic cubic differentials over the moduli space $\mathcal{M}_g$. One may naturally extend the bundle of holomorphic cubic differentials to a (V-manifold) holomorphic vector bundle whose fiber is the space of *regular cubic differentials* over each noded Riemann surface Σ in the boundary divisor in the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$.

Recall that a (compact) noded Riemann surface Σ consists of an open Riemann surface Σ^{reg}, together with a finite number of nodes. Each node has a neighborhood of the form $\{(z,w)\in\mathbb{C}^2 : zw=0, |z|,|w|<1\}$. Such a neighborhood consists of two holomorphic disks glued together at their centers, with the gluing point as the node. A regular cubic differential U on Σ is a holomorphic cubic differential on Σ^{reg} with a special pole structure allowed at each node: Given a standard neighborhood of the node with coordinates z,w, the cubic differential is allowed to have poles of up to order three at $z=0$ and $w=0$. Moreover, there is a complex constant R, the *residue* of U at the node, so that $U = Rz^{-3}\,dz^3 + O(z^{-2})$ and $U = -Rw^{-3}\,dw^3 + O(w^{-2})$ near $z=0$ and $w=0$ respectively.

It is natural to ask why we focus in particular on regular cubic differentials with their third-order poles, and not on more general singular situations such as those in [**58**]. First of all, regular cubic differentials are natural from the point of view of the algebraic geometry of $\overline{\mathcal{M}}_g$. Over a point in the Teichmüller space of a closed surface of genus $g\geq 2$, consider the associated marked Riemann surface Σ. Over a noded Riemann surface Σ, the analog of the canonical bundle is the dualizing sheaf, whose local sections are regular differentials, which are allowed poles of order 1 at the cusps with residues which match up as in the case of cubic differentials above. Then the third tensor power of the dualizing sheaf is the sheaf of regular cubic differentials. The regular cubic differentials naturally extend cubic differentials over the boundary of $\overline{\mathcal{M}}_g$, in the strong sense that the space of all regular cubic differentials forms a vector bundle (up to finite group actions) over $\overline{\mathcal{M}}_g$.

Second, to each cubic differential and related affine sphere structure, there is a natural corresponding Higgs bundle [**38**, **3**] for which the cubic differential is a component of the Higgs field. Over a punctured Riemann surface, cubic differentials of pole order at most 3 then correspond to regular singularities of parabolic Higgs bundles as studied by Simpson [**65**].

A third reason to study regular cubic differentials follows *a posteriori* from our proof. In Theorem 2.6.1, we characterize those degenerations of convex $\mathbb{RP}^2$ surfaces which separate along a neck given by a simple closed curve, and call these degenerations regular convex $\mathbb{RP}^2$ structures. The regular convex $\mathbb{RP}^2$ structures correspond exactly to regular cubic differentials over noded Riemann surfaces by [**51**, **6**, **58**] and Subsection 5.1 below. Thus the regular cubic differentials naturally correspond to degenerations of convex $\mathbb{RP}^2$ surfaces along necks.

In [**50**], for each pair (Σ, U) of noded Riemann surface Σ and regular cubic differential U over Σ, we construct a corresponding $\mathbb{RP}^2$ structure on the nonsingular locus Σ^{reg}, and specify the geometry near each node by the residue of the cubic differential there. In this way, we may define a *regular convex* $\mathbb{RP}^2$ *structure* on Σ. There is a standard topology on the total space of the bundle of regular cubic differentials, and we define a topology on the space of regular convex $\mathbb{RP}^2$ structures under which geometric limits are continuous and which is similar in spirit to Harvey's use of the Chabauty topology to describe the Deligne-Mumford compactification [**31**]. Our main result is then

Theorem 1.0.1. *Let S be a closed oriented surface of genus $g \geq 2$. There is a natural homeomorphism Φ from the total space $\mathcal{V}_g$ of the bundle of regular cubic differentials over the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ to the moduli space $\mathcal{R}_S^{\mathrm{aug}}$ of regular convex $\mathbb{RP}^2$ structures on S.*

We define $\mathcal{V}_g$ and its topology in Subsection 2.9 below, while $\mathcal{R}_S^{\mathrm{aug}}$ and its topology are defined in Subsections 2.6, 2.7 and 2.8 below.

In [**50**], we constructed regular convex $\mathbb{RP}^2$ structures corresponding to regular cubic differentials over a noded Riemann surface, and gave some local analysis of the families of regular convex $\mathbb{RP}^2$ structures in the limit. In particular, we introduced the map Φ and showed it is injective. In passing from regular convex $\mathbb{RP}^2$ structures to regular cubic differentials, Benoist-Hulin show that finite-volume convex $\mathbb{RP}^2$ structures correspond to regular cubic differentials of residue zero [**6**]. Quite recently, as this paper was being finalized, Xin Nie has classified all convex $\mathbb{RP}^2$ structures corresponding to meromorphic cubic differentials on a Riemann surface [**58**], which shows Φ is a one-to-one correspondence. In the present work, we only consider cubic differentials of pole order at most three (as these are the only ones which appear under neck separation), and the $\mathbb{RP}^2$ geometry of each end is determined by the residue R, where $U = R\frac{dz^3}{z^3} + \cdots$. It should be interesting to determine how Nie's higher-order poles relate to degenerating $\mathbb{RP}^2$ structures.

For poles of order 3, there are three cases to consider, as determined by the residue R. If $R = 0$, then the ends are parabolic, which is locally the same structure as a parabolic element of a Fuchsian group. If $\operatorname{Re} R \neq 0$, then the holonomy of the end is hyperbolic, while if $R \neq 0$ but $\operatorname{Re} R = 0$, the holonomy is quasi-hyperbolic. These two cases are not present in the theory of Fuchsian groups. In particular, the associated

Blaschke metric is complete, asymptotically flat, and of finite diameter at these ends. The geometry of $\mathbb{RP}^2$ surfaces thus contains both flat and hyperbolic geometry as limits.

The proof of the main theorem involves several analytic and geometric prior results. First of all, the principal new estimates in the proof are to find sub- and super-solutions to an equation of C.P. Wang [**71**] which are uniform for convergent families (Σ_j, U_j) of noded Riemann surfaces and regular cubic differentials. These will allow us to take limits along the families. A uniqueness result of Dumas-Wolf [**22**] then shows that the limits we find are the ones predicted in [**50**]. To analyze limits of regular $\mathbb{RP}^2$ structures, we use a powerful technique of Benoist-Hulin, which shows that natural projectively-invariant tensors on convex domains converge in $C^\infty_{\rm loc}$ when the domains converge in the Hausdorff topology [**6**]. We also use many details about the structure of the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$, and in particular, the analytic framework due to Masur and refined by Wolpert to relate the hyperbolic metric and with the plumbing construction near the noded Riemann surfaces in $\partial\mathcal{M}_g$.

There are two major compactifications for spaces of hyperbolic (or conformal) structures on a closed oriented surface S of genus $g \ge 2$. For marked hyperbolic structures on a surface, the deformation space is Teichmüller space $\mathcal{T}_g$, and the most prominent compactification is due to Thurston. For unmarked hyperbolic structures, the moduli space $\mathcal{M}_g$ is the quotient of $\mathcal{T}_g$ by the action of the mapping class group. The primary compactification $\overline{\mathcal{M}}_g$ is due to Deligne and Mumford, and all limits in $\partial\mathcal{M}_g$ can be described by a single type of "neck pinch" degeneration, in which the hyperbolic structure degenerates as the length of a geodesic loop goes to zero while the transverse hyperbolic distance goes to infinity. The present work classifies all neck-separating degenerations of unmarked convex $\mathbb{RP}^2$ structures on such an S and supplies a natural bordification of the moduli space $\mathcal{R}_S$, and gives an analog of the well-known theorem that one can describe $\overline{\mathcal{M}}_g$ either in terms of its complex structure or the hyperbolic geometry induced by Fenchel-Nielsen coordinates. See e.g.[**73**].

1.1. $\mathbb{RP}^2$ structures and higher Teichmüller theory. It follows from work of Weil [**72**] and Chuckrow [**20**] that the Teichmüller space of conformal structures on a closed oriented marked surface S of genus at least 2 is homeomorphic to a connected component of the space of representations $\pi_1 S \to \mathbf{PSL}(2,\mathbb{R})$ modulo conjugation in $\mathbf{PSL}(2,\mathbb{R})$. The study of representations of $\pi_1 S$ to higher-order Lie groups is then known as higher Teichmüller theory. Choi-Goldman [**18**] show that the deformation space $\mathcal{G}_S$ of convex $\mathbb{RP}^2$ structures on S is homeomorphic to the Hitchin component of representations $\pi_1 S \to \mathbf{PSL}(3,\mathbb{R})$ [**32**]. Goldman provided in [**27**] the analog of Fenchel-Nielsen coordinates on

$\mathcal{G}_S$. Fenchel-Nielsen coordinates play an important role in analyzing $\overline{\mathcal{M}}_g$ going back to Bers [**9**] and Abikoff [**1**]. In particular, Wolf-Wolpert [**73**] determine the real-analytic relationship between the between complex-analytic coordinates on $\overline{\mathcal{M}}_g$ as given by Masur [**54**] and the Fenchel-Nielsen coordinates. In our present work, we have related the complex-analytic data of regular cubic differentials to the projective geometry of the convex $\mathbb{RP}^2$ structures, but we have not addressed Goldman's Fenchel-Nielsen coordinates. It should be possible to do so, as Marquis has already extended Goldman's coordinates to pairs of pants with non-hyperbolic holonomy [**52**].

There have also been many other works on limits of convex $\mathbb{RP}^2$ structures. Anne Parreau has analyzed limits of group representations into Lie groups in terms of group actions on $\mathbb{R}$-buildings [**61**, **62**]. Parreau thus provides an analog of Thurston's boundary of Teichmüller space. In influential lectures several years ago, Daryl Cooper and Kelly Delp analyzed limits of convex $\mathbb{RP}^2$ structures using the hyperreals to produce singular hex structures on the surface. Inkang Kim [**35**] applies Parreau's theory to construct a compactification of the deformation space of convex $\mathbb{RP}^2$ structure $\mathcal{G}_S$. Alessandrini [**2**] also constructs a compactification of $\mathcal{G}_S$ by using Maslov dequantization techniques. Limits of cubic differentials were related to Parreau's picture in [**47**] and recently in [**57**]. Dumas-Wolf have recently studied polynomial cubic differentials on $\mathbb{C}$ [**22**], and they show that the space of polynomial cubic differentials up to holomorphic equivalence is isomorphic via the affine sphere construction to the space of bounded convex polygons in $\mathbb{R}^2 \subset \mathbb{RP}^2$ up to to projective equivalence. Their construction has been used by Nie [**58**] to analyze the $\mathbb{RP}^2$ geometry related to higher-order poles of cubic differentials, and should be useful in other contexts as well. Benoist-Hulin have also studied cubic differentials on the Poincaré disk, and have shown that the Hilbert metric on a convex domain is Gromov-hyperbolic if and only if it arises from a cubic differential on the Poincaré disk with bounded norm with respect to the Poincaré metric [**7**].

Tengren Zhang has considered degenerating families of convex $\mathbb{RP}^2$ structures with natural constraints on Goldman's parameters [**79**]. Ludovic Marquis has studied convex $\mathbb{RP}^2$ structures and their ends from a different point of view from this paper [**52**, **53**]. Recently Choi has analyzed ends of $\mathbb{RP}^n$ orbifolds in any dimension [**17**, **16**].

Fix a conformal structure Σ on a closed oriented surface of S of genus at least two. Let G be a split real simple Lie group with trivial center and rank r. Hitchin uses Higgs bundles to parametrize the Hitchin component of the representation space from $\pi_1 S$ to G by the set of r holomorphic differentials, which always includes a quadratic differential [**32**]. For $\mathbf{PSL}(3,\mathbb{R})$, Hitchin specifies a quadratic and a cubic differential. If the quadratic differential vanishes in this case, then Labourie has shown that we can parametrize the Hitchin component by the affine sphere construction (Σ, U) for U Hitchin's cubic differential (up to a constant factor) [**38**]. Labourie has also recently shown that Hitchin representations for other split real Lie groups of rank 2 ($\mathbf{PSp}(4,\mathbb{R})$ and split real G_2) can be parametrized by pairs (Σ, V), where Σ varies in Teichmüller space and V is a holomorphic differential of an appropriate order [**40**]. It would be interesting to analyze these Hitchin representations similarly as Σ approaches a noded Riemann surface and V is a regular differential. The relationship between the Higgs bundles and the relevant geometric structures is not as well developed in this case. See [**30**, **3**].

One can also follow [**38**] to view the present work in terms of limits of Higgs bundles. Many of these works involve various limits of Higgs bundles over a fixed Riemann surface (see for example Mazzeo-Swoboda-Weiss-Witt [**55**] and Mochizuki [**56**]). These papers may be seen as analogs not of the present work but of [**47**]. The present paper involves different sorts of degenerations. The Riemann surface is not fixed, but there are families degenerating to noded Riemann surfaces, each paired with a Higgs bundle which degenerates also in a prescribed way (to a parabolic bundle on the noded Riemann surface). Swoboda studies a similar problem of degenerating pairs of Riemann surfaces and $\mathbf{SL}(2,\mathbb{C})$ Higgs bundles [**67**]. See also [**66**] for a comparison of these two sorts of limits.

1.2. Outline. Sections 2 and 3 include largely definitions and background material, with a few new results. The remaining Sections 4 and 5 are devoted to the bulk of the proof, which is to show that a natural one-to-one correspondence $\Phi\colon \mathcal{V}_g \to \mathcal{R}_S^{\mathrm{aug}}$ is a homeomorphism. Section 4 shows Φ is continuous by a direct proof involving uniform estimates and the theory of ordinary differential equations with parameters, while Section 5 contains a more indirect and involved proof of the continuity of Φ^{-1} using among other things the compactness of $\overline{\mathcal{M}}_g$.

Section 2 begins by defining the topological space of regular convex $\mathbb{RP}^2$ structures. First of all, we recount Goldman's theory of building convex $\mathbb{RP}^2$ surfaces by gluing together simpler surfaces along principal geodesic boundary components. Then we use a few lemmas from general topology to show that the topology of regular convex $\mathbb{RP}^2$ structures is first countable. Thus the topology can be described in terms of convergent sequences, which is a natural approach given our underlying tools in differential equations. Then we define regular separated necks and show in Theorem 2.6.1 that these regular separated necks encompass all geometric limits of convex $\mathbb{RP}^2$ structures on S which degenerate to convex $\mathbb{RP}^2$ structures on $S-\ell$, for ℓ a simple non-peripheral loop in S. Next, we define the augmented Goldman space of marked regular convex $\mathbb{RP}^2$ structures on S, which, similarly to augmented Teichmüller space, is a non-locally-compact bordification of the Goldman space (the deformation space of marked convex $\mathbb{RP}^2$ structures). The definition is based on pairs (Ω, Γ) and encodes both the Hausdorff limits of convex domains of Ω_j and also the convergence of representations Γ_j of subgroups of the fundamental group, all modulo a natural action of $\mathbf{SL}(3, \mathbb{R})$. Then we take a quotient by the mapping class group to define the augmented moduli space of convex $\mathbb{RP}^2$ structures $\mathcal{R}_S^{\mathrm{aug}}$.

In the final part of Section 2, we recall the plumbing construction for neighborhoods of noded Riemann surfaces in the boundary of moduli space, largely following Wolpert, and its relation to the complete hyperbolic metric on the regular part of each surface. We then use these constructions to construct the standard topology on the total space of the bundle of regular cubic differentials over $\overline{\mathcal{M}}_g$. Roughly, we define a metric m on each noded Riemann surface Σ which is equal to the hyperbolic metric on the thick part of Σ and a flat cylindrical metric on the collar and cusp neighborhoods making up the thin part of Σ^{reg}. Then convergence of a sequence (Σ_j, U_j) is defined as convergence of Σ_i in $\overline{\mathcal{M}}_g$, together with $L^\infty_{m_j,\mathrm{loc}}$ convergence of the cubic differentials U_j.

Then in Section 3, we discuss the basics of hyperbolic affine spheres [**12**, **13**]. Let $\mathcal{H}$ be a hyperbolic affine sphere, which is a surface in $\mathbb{R}^3$ asymptotic to a cone over a bounded convex domain Ω. $\mathcal{H}$ is diffeomorphic to Ω under projection to $\mathbb{RP}^2$, and any projective action on Ω lift to a special linear action on $\mathcal{H}$. Two basic invariant tensors, the Blaschke metric and the cubic tensor, thus descend to Ω. The Blaschke metric induces an invariant conformal structure on Ω, and thus on the quotient $\Gamma\backslash\Omega$. The cubic tensor is equivalent to a holomorphic cubic differential U.

Starting from a pair (Σ, U), we can recover the picture of (Ω, Γ) by introducing a background metric g and solving Wang's integrability condition (21) for a conformal factor e^u [**71**]. Then $e^u g$ is the Blaschke metric, and if it is complete, we recover the global hyperbolic affine sphere $\mathcal{H}$ and (Ω, Γ).

The hyperbolic affine sphere over Ω can be defined as the radial graph of $-\frac{1}{v}$ for v a convex solution to the Monge-Ampère equation

$$\det v_{ij} = \left(-\frac{1}{v}\right)^4$$

with zero Dirichlet boundary condition at $\partial\Omega$. We recall and give a proof of Benoist-Hulin's result that the Blaschke metrics and cubic tensors converge in C^∞_{loc} on bounded convex domains converging in the Hausdorff sense [**6**]. We also prove a new result, Proposition 3.3.2, concerning sequences of pairs of points $x_j, y_j \in \Omega_j$, and show that if the Blaschke distance between x_j and y_j diverges to infinity, then any Hausdorff limits of the pointed spaces $\rho_j(\Omega_j, x_j)$ and $\sigma_j(\Omega_j, y_j)$ for any sequences $\rho_j, \sigma_j \in \mathbf{SL}(3,\mathbb{R})$, must be disjoint (in a sense made precise below).

Finally, we begin the proof of Theorem 1.0.1 in Section 4. In this section, we show that a convergent regular sequence $(\Sigma_j, U_j) \to (\Sigma_\infty, U_\infty)$ of pairs of noded Riemann surfaces and regular cubic differentials produces regular convex $\mathbb{RP}^2$ structures which converge in the limit to the convex $\mathbb{RP}^2$ structure corresponding to $(\Sigma_\infty, U_\infty)$. The proof follows by the method of sub- and super-solutions. We produced a locally bounded family of sub- and super-solutions to (21) uniform over $\overline{\mathcal{C}}_g^{\mathrm{reg}}$, which is the universal curve $\overline{\mathcal{C}}_g$ over the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ minus the set of nodes. This allows us to solve the equation (21) and to take the limit in C^∞_{loc} of the Blaschke metrics on Σ_j^{reg} as $j \to \infty$. We then use a uniqueness theorem of Dumas-Wolf [**22**] to show that this limit is the complete Blaschke metric on $(\Sigma_\infty^{\mathrm{reg}}, U_\infty)$. We use techniques of ordinary differential equations to show the holonomy and developing maps converge, and thus that the map Φ is continuous.

Some of the arguments in Section 5 are indirect in order to address an essential difficulty—that there is not yet, for our purposes, a workable analog of the thick-thin decomposition for convex $\mathbb{RP}^2$ structures. In particular, we would like to have a geometric way of determining when a family of convex $\mathbb{RP}^2$ structures on a surface is separating across a simple loop (a neck, in our parlance). There is a good theory developed by Lee-Zhang [**41**] for determining when the Hilbert metric (or equivalently the Blaschke metric) becomes pinched in such a family, but we must consider as well other types of degenerations in which the Hilbert circumference of the neck remains bounded away from 0. We circumvent this difficulty via an indirect argument to prove Φ^{-1} is continuous. In fact, the conformal structures induced by the Blaschke metrics do become pinched, as follows from Theorem 5.2.2 below.

We prove Theorem 5.2.2 by showing that we can pass from convergent sequences of regular convex $\mathbb{RP}^2$ structures to convergent sequences of regular cubic differentials over noded Riemann surfaces. The proof depends on the thick-thin decomposition of hyperbolic surfaces. In particular, we use Proposition 3.3.2 and lower bounds on the Blaschke metric in terms of the hyperbolic metric, to rule out Benzécri sequences of pointed convex domains (Ω_j, x_j) modulo $\mathbf{SL}(3,\mathbb{R})$ in which the point approaches a node in the universal curve $\overline{\mathcal{C}}_g$. Conversely, if we have points converging in the same component of the thick part of moduli, we use the uniform bounds on the diameter and the ODE theory from Section 4 to show the limit of the domains must be the same up to an $\mathbf{SL}(3,\mathbb{R})$ action.

For the structures on surfaces we study, there are two complementary points of view: 1) to require a basepoint to define the structure or 2) to define the structure on a connected component of surface. From the point of view of the Blaschke metric and cubic differential, the holonomy and developing map of the corresponding $\mathbb{RP}^2$ structure are constructed via parallel transport of a flat connection on a vector bundle. As such, a basepoint is essential. On the other hand, our definition of marked convex $\mathbb{RP}^2$ structure depends on the basepoint only weakly, in terms of the fundamental group, and we note the standard fact in Lemma 2.5.1 that the structure depends only on the connected component. Under the neck separations we study, the number of connected components increases. Mindful of this, we primarily relate the two points of view in terms of the thick-thin decomposition of hyperbolic (and conformal) structures. In particular, if a given neck is thin, we consider it naturally as part of a family in which the neck pinches hyperbolically, and if where we have a sequence of basepoints, we must ensure that they converge to the thick part of the surface, and indeed to the correct component of the thick part.

We give a brief rundown of the various geometric constructions in this paper in terms of the basepoint and component points of view. The basic definitions of convex $\mathbb{RP}^2$ structures, including the augmented moduli space $\mathcal{R}_S^{\rm aug}$ are defined in terms of basepoints only weakly via the fundamental groups, and thus are largely based on the connected components of the surface, as is the thick/thin decomposition of hyperbolic and conformal structures in Subsection 2.9. The affine differential geometry in Section 3 and the ODE theory developed in Section 4 use basepoints heavily. Finally, in Section 5, we use another basepoint construction, Benzécri's theory of convergence of bounded convex pointed domains, and then carefully integrate the two points of view, which is complicated by the difficulties in relating the neck separation of the $\mathbb{RP}^2$ structures to the neck pinching of the induced hyperbolic structures. See the outline of proof for Theorem 5.2.2 below.

1.3. Acknowledgments. I am grateful to acknowledge stimulating conversations with Zeno Huang, Jane Gilman, Sara Maloni, Ian McIntosh, and Bill Abikoff, and I would like to give special thanks to Mike Wolf and David Dumas for many illuminating discussions, and to Scott Wolpert for many interesting discussions related to this work. I also thank Bill Goldman for many enlightening discussions about convex $\mathbb{RP}^2$ structures over the years, and S.T. Yau for introducing me to the beautiful subject of affine differential geometry. I would also like to thank Suhyoung Choi and two anonymous referees for many helpful and detailed comments to improve the manuscript. This work was supported in part by a Simons Collaboration Grant for Mathematicians 210124; U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 "RNMS: GEometric structures And Representation varieties" (the GEAR Network); and the IMS in at the National University of Singapore, where part of this work was completed.

2. Definitions and topology

2.1. A note on terminology. We consider an oriented surface S of negative Euler characteristic and finite topological type. S will, depending on the context, carry a convex real projective structure, a hyperbolic metric, or a conformal structure. To each free homotopy class c of simple nontrivial non-peripheral loops in S, we consider the *neck across* c to be an associated equivalence class of annular domains homotopic to c. If S has a hyperbolic or convex $\mathbb{RP}^2$ structure, there is a unique geodesic representative ℓ of c. Annular neighborhoods of ℓ in this case are known as *collars*.

We study degenerations in which a surface S is separated along c, and consider the limiting convex $\mathbb{RP}^2$ structure on $S - c$. Each convex $\mathbb{RP}^2$ structure on S (and $S-c$) carries a canonical complete Riemannian metric, the Blaschke metric, which in turn induces a canonical conformal structure and, by the Uniformization Theorem, a hyperbolic metric on S. The degenerations of convex $\mathbb{RP}^2$ structures we consider all induce a "neck pinch" degeneration of the induced conformal and hyperbolic structures, in which the hyperbolic circumference along c limits to zero. At such a limit of conformal or hyperbolic structures, each of the two ends of $S - c$ induced by removing c has a *cusp* neighborhood, which is conformal to a punctured disc, or is isometric to the Poincaré metric on the punctured disc respectively. In the conformal case, these two cusps neighborhoods naturally form the two sheets of the regular part of a neighborhood of a node $\{(z, w) \in \mathbb{C}^2 : zw = 0,\ |z|, |w| < 1\}$. The relationship between the conformal and hyperbolic pictures is well-developed, as a neck is conformally pinched if and only if it is hyperbolically pinched. We recall the relationship between the conformal and hyperbolic theories below in Subsection 2.9.

For most of the degenerations of convex $\mathbb{RP}^2$ structures on S along c, however, the circumference along c with respect to the projectively-invariant Blaschke (or Hilbert) metric does not go to zero. For this reason, we decline to call the ends of $S - c$ formed by removing c cusp neighborhoods, and we also decline to say the neck across c is being pinched. Instead, we refer to the two ends together as a *separated neck*, and below in Subsection 2.6 we will introduce a more technical definition of *regular separated neck* to specify the details of the convex $\mathbb{RP}^2$ structure on $S - c$ at these ends. With respect to the induced conformal structure, each pair of ends forming a separated neck can be joined together by adding a single node. With respect to the hyperbolic or convex real projective geometries then, each node can be considered as an identification of the appropriate pair of ends of $S - c$.

We assume in this paper that for an open oriented surface of finite type, we orient each simple peripheral loop by the boundary orientation of a hypothetical S^1 boundary compactifying the end. In other words, on a punctured Poincaré disk, the orientation of a loop around the puncture is clockwise.

2.2. An outline of the topology of the space of regular convex $\mathbb{RP}^2$ structures. In this subsection, we quickly give the definition of the augmented moduli space $\mathcal{R}_S^{\rm aug}$ and its topology, with full explanations and related results appearing below in Subsections 2.5 - 2.8.

First of all, given a connected surface S, we define the Goldman (deformation) space $\mathcal{G}_S$ of convex $\mathbb{RP}^2$ structures on S as the space $(\Omega,\Gamma)/\sim$, where Ω is a convex domain in $\mathbb{RP}^2$, Γ is a representation of $\pi_1 S$ into $\mathbf{SL}(3,\mathbb{R})$ acting on Ω to induce a convex $\mathbb{RP}^2$ structure on S, and $\sim$ is the equivalence relation induced by the action of $\mathbf{SL}(3,\mathbb{R})$ by ρ: $(\Omega,\Gamma)\mapsto(\rho\Omega,\rho\Gamma\rho^{-1})$. The topology is given by the Hausdorff topology on domains $\Omega\subset\mathbb{RP}^2$ measured with the Fubini-Study metric on $\mathbb{RP}^2$, together with convergence of a fixed set of generators of Γ in $\mathbf{SL}(3,\mathbb{R})$, and then taking the quotient topology under the equivalence relation $\sim$. For a disconnected surface S with components $S_1,\dots,S_n$, define $\mathcal{G}_S$ to be the Cartesian product of the $\mathcal{G}_{S_i}$ with the product topology.

The augmented Goldman space $\mathcal{G}_S^{\rm aug}$ is a stratified space with one stratum $\mathcal{G}_S^c$ for each multi-curve c on S. (In other words, c is a collection of free homotopy classes of simple nonoriented loops on S which are nonperipheral, nonintersecting and nontrivial.) If $S-c$ has connected components $S_1,\dots,S_n$, the pulling map $\mathrm{Pull}_{S,c}$ from $\mathcal{G}_S$ to $\mathcal{G}_S^c$ is induced by the map $(\Omega,\Gamma)\mapsto\oplus_{i=1}^n(\Omega,\Gamma|_{S_i})$. Consider a convex $\mathbb{RP}^2$ structure X on $S-c$ which is in the closure of the image of $\mathrm{Pull}_{S,c}$. On X each loop $\ell\in c$ is a *separated neck*, and is called a *trivial* if there is some $d\subset c-\ell$ so that $X\in\mathrm{Pull}_{S-d,c-d}\mathcal{G}_{S-d}$. All other loops in c are called *regular separated necks*, which represent nontrivial limits of the $\mathbb{RP}^2$ structure along the necks. Then $\mathcal{G}_S^c$ is the set of all convex $\mathbb{RP}^2$ structures on $S-c$ for which all loops in c are regular separated necks.

Each neighborhood in $\mathcal{G}_S^{\rm aug}$ of $X\in\mathcal{G}_S^c$ intersects $\mathcal{G}_S^d$ for all $d\subset c$. Let $\mathcal{O}$ be an open subset of the subspace of $\mathcal{G}_{S-c}$ for which every loop in c is a separated neck. Then $\mathcal{O}$ is stratified by the regular separated necks in c. Let $\mathcal{O}_{{\rm triv},d}\subset\mathcal{O}$ be the set of $\mathbb{RP}^2$ structures with trivial separated necks along d. Then the sets

$$\tilde{O}=\bigsqcup_{d\subset c}\mathcal{O}_{{\rm triv},d},$$

for all $\mathcal{O}$, form a basis for the topology on $\mathcal{G}_S^{\rm aug}$.

Finally, the augmented moduli space $\mathcal{R}_S^{\rm aug}$ is given by the quotient of $\mathcal{G}_S^{\rm aug}$ by the natural action of the mapping class group, and is equipped with the quotient topology.

2.3. Goldman's attaching across a principal geodesic. We recount some of the basic facts about $\mathbb{RP}^n$ manifolds. An $\mathbb{RP}^n$ manifold has by definition a maximal atlas of coordinate charts in $\mathbb{RP}^n$

with gluing maps in $\mathbf{PGL}(n+1,\mathbb{R})$; in other words, there is an (X,G) structure in the sense of Thurston and Ehresmann for $X=\mathbb{RP}^n$ and $G=\mathbf{PGL}(n+1,\mathbb{R})$. A *geodesic* in an $\mathbb{RP}^n$ manifold is a path which is a straight line segment in each $\mathbb{RP}^n$ coordinate chart.

See e.g. Goldman [**27**] for details. An $\mathbb{RP}^n$ structure on an n-manifold M can also be described in terms of the development-holonomy pair. Choose a basepoint $p\in M$. The developing map is a local diffeomorphism from $\mathrm{dev}\colon \tilde M\to\mathbb{RP}^n$, while the holonomy $\mathrm{hol}\colon \pi_1M\to \mathbf{PGL}(n+1,\mathbb{R})$. Dev and hol are related by the following equivariance condition: if $\gamma\in\pi_1M$, then

$$\mathrm{dev}\circ\gamma=\mathrm{hol}(\gamma)\circ\mathrm{dev}.$$

The developing map is defined in terms of a choice of $\mathbb{RP}^n$ coordinate chart around $p\in M$. First lift this chart to a neighborhood in $\tilde M$, and then analytically continue to define dev on all of $\tilde M$. For any other choice of coordinate chart and/or basepoint, there is a map $g\in\mathbf{PGL}(n+1,\mathbb{R})$ so that

$$\mathrm{dev}'=\mathrm{dev}\circ g,\qquad \mathrm{hol}'(\gamma)=g^{-1}\circ\mathrm{hol}(\gamma)\circ g.$$

For oriented convex $\mathbb{RP}^n$ manifolds, we may naturally lift the holonomy $\mathrm{hol}\colon\pi_1M\to\mathbf{PGL}(n+1,\mathbb{R})$ to lie in $\mathbf{SL}(n+1,\mathbb{R})$. We will consider the holonomy to lie in $\mathbf{SL}(n+1,\mathbb{R})$ for the remainder of this paper.

An $\mathbb{RP}^n$ manifold X is called *convex* if the image of the developing map is a convex domain Ω in an inhomogeneous $\mathbb{R}^n\subset\mathbb{RP}^n$, and X is a quotient $\mathrm{hol}(\pi_1X)\backslash\Omega$. X is *properly convex* if Ω is in addition bounded in an inhomogeneous $\mathbb{R}^n\subset\mathbb{RP}^n$. All the manifolds we study in this paper are properly convex, and we often simply call them convex.

On any closed oriented convex $\mathbb{RP}^2$ surface of genus at least 2, the $\mathbb{RP}^2$ holonomy (in $\mathbf{SL}(3,\mathbb{R})$) around any nontrivial simple loop is hyperbolic, in that it is conjugate to a diagonal matrix $D(\lambda,\mu,\nu)$, where $\lambda>\mu>\nu>0$ and $\lambda\mu\nu=1$. Choose coordinates in $\mathbb{RP}^2$ so that this holonomy action is given by $H=D(\lambda,\mu,\nu)$. The three fixed points of this action are the attracting fixed point $[1,0,0]$, the repelling fixed point $[0,0,1]$, and the saddle fixed point $[0,1,0]$. Define the *principal triangle* T as the projection onto $\mathbb{RP}^2$ of the first octant in $\mathbb{R}^3$. The *principal geodesic* $\tilde\ell$ associated to this holonomy matrix is the straight line in the boundary of T from the repelling to the attracting fixed point. Let $\bar T$ denote the triangle given by the reflection of T across the principal geodesic given by the matrix $J=D(1,-1,1)$. The vertices of the principal triangle are the fixed points of the holonomy matrix. The quotient $T/\langle H\rangle$ is called the *principal half-annulus*, while the quotient of $(T\sqcup\tilde\ell\sqcup\bar T)/\langle H\rangle$ is called the π*-annulus.*

We recall Goldman's theory of attaching $\mathbb{RP}^2$ surfaces across a principal geodesic. On a properly convex $\mathbb{RP}^2$ surface S^a with principal geodesic boundary, an annular neighborhood of a principal geodesic boundary component ℓ is called a *principal collar neighborhood*. We may choose coordinates so that a lift $\tilde\ell$ of ℓ is the standard principal geodesic mentioned above. Assume the image Ω^a of the developing map is then a subset of the principal triangle $\bar T$. The principal collar neighborhood then develops to be a neighborhood $\mathcal{N}$ of $\tilde\ell$ in $\bar T$ which is invariant under the action of the holonomy matrix $H = D(\lambda,\mu,\nu)$. Then the quotient $\mathcal{N}^a = \langle H\rangle\backslash\mathcal{N}$ is the principal collar neighborhood.

Now consider a second convex $\mathbb{RP}^2$ surface S^b with principal geodesic boundary, together with a principal geodesic boundary component. Choose local $\mathbb{RP}^2$ coordinates so that the lift of this geodesic boundary loop is $-\tilde\ell$ (the minus sign denoting the opposite orientation), and the image Ω^b of the developing map of S^b is contained in T. If the holonomy around $-\tilde\ell$ is H^{-1}, then H acts on $\Omega^a \sqcup \tilde\ell \sqcup \Omega^b$. (In order to glue the surfaces along ℓ, we need to glue across all the lifts of ℓ, which we may describe as $\mathrm{hol}(\beta)\circ\tilde\ell$ for β in the coset space $\pi_1(S^a)/\langle\gamma\rangle$, where γ is the element in π_1 determined by the loop ℓ.)

We say an $\mathbb{RP}^2$ surface has *principal geodesic boundary* if its boundary is compact and each component of the boundary is a principal geodesic loop around which the holonomy is hyperbolic, as above. We also say a disconnected $\mathbb{RP}^2$ surface is properly convex if each of its connected components is. We are now ready to state Goldman's gluing theorem in the form we will need.

Theorem 2.3.1 (Goldman)**.** *Let M be a properly convex $\mathbb{RP}^2$ surface with principal geodesic boundary. M is not assumed to be either connected or compact. Let B_1, B_2 be two boundary components, and assume that they have principal collar neighborhoods N_1, N_2 respectively which are projectively isomorphic under an orientation-reversing projective map J across the boundary. This induces a projective structure on a full neighborhood of the geodesic formed from gluing B_1 and B_2. The resulting $\mathbb{RP}^2$ surface $\bar M$ is also convex. We say the resulting surface $\bar M$ is formed from M by* gluing *along B_1, B_2 via the orientation-reversing map J. $\bar M$ is properly convex except in the case of gluing two principal half-annuli together to make a π-annulus.*

Remark. Goldman states this theorem (Theorem 3.7 in [**27**]) a bit differently, in that the hypothesis is that M is compact as an $\mathbb{RP}^2$ surface with boundary. However, proper convexity, rather than compactness, is the criterion used in Goldman's proof. The image of the developing map of a properly convex M with a hyperbolic holonomy along the principal boundary geodesic must be properly contained in a principal triangle. *Proper* convexity is essential for us, as the π-annulus, which has principal boundary geodesics and is convex but not properly convex, cannot be glued to another $\mathbb{RP}^2$ surface while maintaining convexity. In fact, Choi [**14**, **15**] has cut non-convex closed $\mathbb{RP}^2$ surfaces along geodesics into a disjoint union of properly convex pieces and π-annuli. Choi-Goldman use this construction to classify all closed $\mathbb{RP}^2$ surfaces [**19**].

REMARK. In this work, we stay in the category of properly convex $\mathbb{RP}^2$ surfaces, as we never glue two principal half-annuli together. But we will below often consider convex $\mathbb{RP}^2$ surfaces formed by gluing a principal half-annulus to a properly convex $\mathbb{RP}^2$ surface with principal geodesic boundary and of negative Euler characteristic (this is the case we call "bulge ∞" below).

Corollary 2.3.2. *Let M be a properly convex $\mathbb{RP}^2$ surface with principal geodesic boundary. Assume the hyperbolic holonomies along two boundary components B_1, B_2 are, up to conjugation, inverses of each other. Then we may glue B_1 to B_2 as above to make the $\mathbb{RP}^2$ surface $\bar{M}$ properly convex.*

Proof. Choose local $\mathbb{RP}^2$ coordinates near B_1, B_2 so that there is a lift of each to the standard principal geodesic $\tilde{\ell}$, so that the neighborhoods of B_1, B_2 are respectively on opposite sides of the $\tilde{\ell}$, and so that the holonomies around B_1, B_2 are diagonal. Since they are both in canonical form, they must actually be inverses of each other, say H and H^{-1}. Now for a point p in T close enough to the interior of $\tilde{\ell}$, we may form a neighborhood of ℓ by moving p by the one-parameter group H^t corresponding to the holonomy. The region between $\{H^t p\}$ and ℓ is then a principal collar neighborhood $\mathcal{N}^a$. But now we can do the same thing on the other side of ℓ to find a principal collar neighborhood $\mathcal{N}^b$. Since their holonomies are inverses of each other, we see that, after possibly shrinking the collar neighborhoods, $\mathcal{N}^a = J\mathcal{N}^b$ with the holonomy equivariant under the action of J. This means that J descends to the quotient, and the hypotheses in Theorem 2.3.1 are satisfied. Note J commutes with the holonomy matrix. q.e.d.

Goldman's construction in Theorem 2.3.1 involves a choice of a orientation-reversing projective map J to glue the collar neighborhoods across the principal geodesic boundary components. If standard coordinates are chosen on $\mathbb{R}^3$ as above, then we may choose $J = D(1, -1, 1)$. Note J commutes with the holonomy H. But there are other possible choices determined by generalized twist parameters σ, τ. For $M_{\sigma,\tau} = D(e^{-\tau-\sigma}, e^{2\sigma}, e^{\tau-\sigma})$, consider the projective involution $J_{\sigma,\tau} = M_{\sigma,\tau}J$, which still commutes with H. τ is called the *twist parameter*, as it corresponds to the usual twist parameter on a hyperbolic surface. We call σ the *bulge parameter*. (In [**50**], these are called the horizontal and vertical twist parameters respectively.) Our choice of J is not canonical, as it depends on a choice of coordinates; the twist and bulge parameters then are relative to this choice of J. Below we define limiting cases of bulge $\pm\infty$ independently of the choice of J.

On a neck with hyperbolic holonomy, with coordinates on $\mathbb{R}^3$ so that the holonomy is given by the diagonal matrix $H = D(\lambda, \mu, \nu)$ with $\lambda > \mu > \nu > 0$ and $\lambda\mu\nu = 1$. A *Dehn twist* is a generalized twist which corresponds exactly to the holonomy along the geodesic loop. The Dehn twist is transverse to the family of bulge parameters, but is not typically strictly a twist parameter as defined above.

Finally we define the geometry of hyperbolic ends with *bulge* $\pm\infty$. An end of a convex $\mathbb{RP}^2$ surface with hyperbolic holonomy has *bulge* $+\infty$ if the image of the developing map contains a principal triangle T. Similarly, an end of a convex $\mathbb{RP}^2$ surface with hyperbolic holonomy has *bulge* $-\infty$ if the image Ω of the developing map has the principal geodesic $\tilde{\ell}$ in its boundary. In this case, we choose T and Ω to be disjoint, and the $\mathbb{RP}^2$ surface has natural principal geodesic boundary at the end. If we then attach a principal half-annulus across this geodesic, it changes the $\mathbb{RP}^2$ structure at the end from bulge $-\infty$ to bulge $+\infty$.

2.4. General topology of orbit spaces. We will need a few lemmas about first countability and spaces of orbits of homeomorphisms. The proofs follow closely from the definitions.

Lemma 2.4.1. *Let X be a first countable topological space, and let Φ be a set of homeomorphisms acting on X. Then the quotient space $\Phi\backslash X$ is first countable with respect to the quotient topology.*

Lemma 2.4.2. *Let X be a first countable topological space, let Φ be a set of homeomorphisms acting on X, and let f be the projection to the quotient space. Then $y_i \to y$ in $\Phi\backslash X$ if and only if there is a sequence $x_i \to x$ in X with $y_i = f(x_i)$ and $y = f(x)$.*

2.5. Markings on convex $\mathbb{RP}^2$ surfaces. In this subsection, we consider a connected oriented surface S of negative Euler characteristic. A marked convex $\mathbb{RP}^2$ structure on S is given by the quotient $\{(\Omega, \Gamma)\}/\sim$, where Ω is a properly convex domain in $\mathbb{RP}^2$ and, for a basepoint $x_0 \in S$,

$$\Gamma \colon \pi_1(S, x_0) \to \mathbf{SL}(3, \mathbb{R})$$

is a discrete embedding which acts on Ω so that $\Gamma\backslash\Omega$ is diffeomorphic to S. The equivalence relation $\sim$ is given by $(\Omega, \Gamma) \sim (A\Omega, A\Gamma A^{-1})$ for $A \in \mathbf{SL}(3, \mathbb{R})$. Note that Ω is the image of a developing map for this $\mathbb{RP}^2$ structure on S, and Γ is the corresponding holonomy representation.

We have the following standard lemma which allows us to ignore the basepoint for connected S.

Lemma 2.5.1. *On a connected surface S, all choices of basepoint for the fundamental group give rise to the same element in the quotient space $\{(\Omega, \Gamma)\}/\sim$.*

Proof. If x is another basepoint, choose a path from x_0 to x and develop the $\mathbb{RP}^2$ structure along the path. In this case Ω remains fixed.

For two non-homotopic paths p_1, p_2 from x_0 to x, the resulting pairs $(\Omega,\Gamma)_{p_i}$, $i = 1,2$ can be seen to differ by the action of an element in $\mathbf{SL}(3,\mathbb{R})$. q.e.d.

Remark. It is perhaps more usual to address the deformation space of convex $\mathbb{RP}^2$ structures directly without using a basepoint for the fundamental group, as in e.g. Goldman [**26**]. We still find specifying the basepoint useful, however, as it is natural from the point of view of developing the convex $\mathbb{RP}^2$ structure by parallel transport on a flat vector bundle, as in Subsection 3.4 below.

Define the *Goldman space* of S by $\mathcal{G}_S = \{(\Omega,\Gamma)\}/\sim$, where as above, Ω is a bounded convex domain, Γ is a faithful representation of $\pi_1(S,s_0) \to \mathbf{SL}(3,\mathbb{R})$ which acts discretely and properly discontinuously on Ω so that S is diffeomorphic to $\Gamma\backslash\Omega$. The equivalence relation $\sim$ is given by the action of $\mathbf{SL}(3,\mathbb{R})$ described above. We provide $\mathcal{G}_S$ with the following quotient topology. First of all, for convex domains in $\mathbb{RP}^2$, consider the Hausdorff topology with respect to the Fubini-Study metric in $\mathbb{RP}^2$. For the space of representations Γ, use the product topology of one copy of $\mathbf{SL}(3,\mathbb{R})$ for each element of $\Gamma(\gamma)$ for $\gamma \in \pi_1(S,x_0)$ (note we consider only surfaces of finite type, for which $\pi_1(S,x_0)$ is finitely generated). Since Γ is countable, this topology is first countable. Now the equivalence relation $\sim$ represents the orbits of a group action of $\mathbf{SL}(3,\mathbb{R})$, which acts by homeomorphisms on the space of all (Ω,Γ), equipped with the product of the two topologies described above. Then Lemma 2.4.1 shows this quotient topology on $\mathcal{G}_S$ is also first countable.

In the case that S is a closed surface of genus $g \geq 2$, we may define Goldman space $\mathcal{G}_g = \mathcal{G}_S$. This deformation space is the analog of Teichmüller space for convex $\mathbb{RP}^2$ structures on S. Goldman [**27**] proved that $\mathcal{G}_g$ is homeomorphic to $\mathbb{R}^{16g-16}$. For augmented Goldman space, which we define below, we will need the more general theory described above, which also applies to noncompact S.

It will be useful for us to allow the case in which $S = \sqcup_{i=1}^n S_i$ has finitely many connected components. In this case, define $\mathcal{G}_S$ as the Cartesian product $\mathcal{G}_{S_1} \times \cdots \times \mathcal{G}_{S_n}$ with the product topology.

Remark. The topology we consider is related to the Chabauty topology consider by Harvey [**31**]. See also Wolpert [**76**]. Harvey's work is concerned with limits of Fuchsian groups under the Chabauty topology. In particular, the image of the developing map for a Fuchsian group is always the hyperbolic plane, while our analogous domains Ω_j can and do vary. Moreover, for noncompact convex $\mathbb{RP}^2$ surfaces which naturally appear as limits in our case (regular convex $\mathbb{RP}^2$ surfaces), the holonomy representations Γ_j do not determine the geometry. Distinct pairs of convex $\mathbb{RP}^2$ surfaces with isomorphic holonomy naturally arise: consider a surface with a simple end, put hyperbolic holonomy on the end, and vary allow the bulge parameter to be either $-\infty$ or $+\infty$. (For compact convex $\mathbb{RP}^2$ surfaces, a rigidity theorem for the holonomy spectrum holds [**21**, **34**, **36**].) Our definition is in a sense a little less general than Harvey's, as we specify the loops along which the degeneration occurs.

Our topology is also analogous to the geometric topology on hyperbolic manifolds (see e.g. [**5**]), in which sequences of pairs of points whose hyperbolic distance diverges to infinity cannot reside in the same geometric limit space. Although our definitions are phrased differently, we do see below in Proposition 3.3.2 that a similar property holds with respect to the projectively-invariant Blaschke metric on Ω.

2.6. Separated necks and the pulling map. Let S be a connected oriented surface of finite hyperbolic type. Define $C(S)$ to be the set whose elements consist of sets of nontrivial free homotopy classes of simple loops on S so that each loop is nonperipheral and no two loops intersect ($C(S)$ may be identified with the set of simplices of the complex of curves on S). Let $c \in C(S)$. Denote the connected components of $S - c$ by $S_1, \dots, S_n$. Note that each surface S_i admits a finite-area hyperbolic metric. (In the notation below, we will not be careful to distinguish between $c \in C(S)$ as a collection of nonintersecting loops in S as opposed to a collection of homotopy classes.)

If $c \in C(S)$, we define the *pulling map*

$$\mathrm{Pull}_{S,c} \colon \mathcal{G}_S \to \mathcal{G}_{S-c}$$

as follows. Let $S - c = \sqcup_{i=1}^n S_i$. Then for $X \in \mathcal{G}_S$, take a representative (Ω, Γ) in the equivalence relation for X. Represent $\mathrm{Pull}_{S,c}(X)$ by the ordered n-tuple with i^{th} element represented by (Ω, Γ^i), where $\Gamma^i = \Gamma|_{S_i}$ is a sub-representation of Γ corresponding to $\pi_1(S_i, x_i)$ for a basepoint x_i. (Recall Lemma 2.5.1 shows the marked convex $\mathbb{RP}^2$ structure is unchanged if the choice of basepoint changes.) To be precise, for the subsurface $S_i \subset S$, the fundamental group of S_i is naturally a conjugacy class of subgroups of $\pi_1(S, x_i)$. We choose Γ^i to be the composition of the injection $\pi_1(S_i, x_i) \to \pi_1(S, x_0)$ with Γ. The element of $\mathcal{G}_S$ is independent of the conjugacy class though: Let $\eta = \Gamma(\gamma)$ for $\gamma \in \pi_1(S, x_i)$, and consider $(\Omega, \Gamma^i) \sim \eta(\Omega, \Gamma^i) = (\Omega, \eta\Gamma^i\eta^{-1})$. Thus choosing a particular sub-representation in the conjugacy class to identify as Γ^i is harmless.

Note for each $\mathbb{RP}^2$ surface $X_i = \Gamma^i \backslash \Omega$, the developing map still has image equal to all of Ω, while the group of deck transformations $\Gamma^i(\pi_1 S_i)$ is smaller than $\Gamma(\pi_1 S)$. In terms of Goldman's attaching map, the domain attached across the principal geodesic still remains attached under the pulling map. This map is called *pulling* in part because it not simply *cutting* along the principal geodesic. Instead, one can imagine a viscous liquid being pulled apart, and the material on either side of the principal geodesic remains attached to the other side after the neck is pulled.

For S an oriented connected surface of negative Euler characteristic, let ℓ be a non-peripheral non-trivial simple loop. Let $\tilde{X}$ be a (possibly disconnected) convex $\mathbb{RP}^2$ structure on $S - \ell$. Then we say the pair of ends formed by removing ℓ from S form a *trivial separated neck* if there is a convex $\mathbb{RP}^2$ structure X on S so that $\tilde{X} = \mathrm{Pull}_{S,\ell}(X)$.

A pair of ends of a (possibly disconnected) convex $\mathbb{RP}^2$ surface forms a *regular separated neck* in these three cases:

- The holonomy around each end is parabolic.
- The holonomy around each end is quasi-hyperbolic, and the oriented holonomies around each end are, up to conjugation in $\mathbf{SL}(3,\mathbb{R})$, inverses of each other.
- The holonomy around each end is hyperbolic; the oriented holonomies around each end are, up to conjugation, inverses of each other; and the $\mathbb{RP}^2$ structure about one of the two ends has bulge $+\infty$, while the other end has bulge $-\infty$.

A simple end of a convex $\mathbb{RP}^2$ surface is *regular* if it forms half of a regular separated neck.

Theorem 2.6.1. *Let S be a surface each of whose components has negative Euler characteristic, and let $c \in C(S)$. Then under the topology defined above, the closure in $\mathcal{G}_{S-c}$ of the image $\mathrm{Pull}_{S,c}(\mathcal{G}_S)$ consists of convex $\mathbb{RP}^2$ structures on $S - c$ for which the neck across each loop in c is either a regular or a trivial separated neck.*

Proof. Marquis showed that since each component of S has negative Euler characteristic, the holonomy around each loop $\ell \in c$ is hyperbolic [**52**].

Recall that a hyperbolic element in $\mathbf{SL}(3,\mathbb{R})$ has three distinct positive eigenvalues. Any nonhyperbolic limit A of such holonomies must still have all positive eigenvalues. Moreover, it must have maximal Jordan blocks (the other cases are ruled out by Choi [**15**] and the author [**51**]; see also [**58**]). These nonhyperbolic limits are exactly the quasi-hyperbolic and parabolic cases, which are regular. Note also that in the nonhyperbolic cases, there is no ambiguity about the developing image of the end, as there is in the hyperbolic case. For the quasi-hyperbolic case, the inverse property follows from the fact that the holonomy around the two ends of a neck are inverses of each other (since these loops are freely homotopic in S with opposite orientations).

Now we consider the cases of hyperbolic limits, and show that any limits which are not trivial must have bulge $\pm\infty$. In order to do this, consider a sequence $X_i \in \mathcal{G}_S$.

There are two cases to consider. First of all, assume that $S - \ell = S_1 \sqcup S_2$ is disconnected. Then the hypothesis shows that there are sequences

(Ω_k, Γ_k) and $\rho_k, \sigma_k \in \mathbf{SL}(3,\mathbb{R})$ so that $\rho_k(\Omega_k, \Gamma_k|_{S_1}) \to (\mathcal{O}, H)$ and $\sigma_k(\Omega_k, \Gamma_k|_{S_2}) \to (\mathcal{U}, G)$. The quotient $H\backslash\mathcal{O}$ gives the $\mathbb{RP}^2$ structure on S_1, while $G\backslash\mathcal{U}$ gives the structure on S_2. Now pick a based loop ℓ_0 in S which is freely homotopic to ℓ, and let γ_k be the corresponding element $\Gamma_k(\ell_0)$. Recall we assume ℓ is oriented in the same direction as the boundary of S_1. This implies ℓ is oriented in the opposite direction to the boundary of S_2. Let $\gamma_\mathcal{O}$ and $\gamma_\mathcal{U}$ be the limits of $\rho_k\gamma_k\rho_k^{-1}$ and $\sigma_k\gamma_k\sigma_k^{-1}$ respectively. We may choose coordinates (and modify ρ_k and σ_k) so that the limiting hyperbolic holonomies around ℓ satisfy

$$\gamma_\mathcal{O} = D(\lambda,\mu,\nu), \quad \gamma_\mathcal{U} = \gamma_\mathcal{O}^{-1}, \quad \lambda>\mu>\nu>0,$$

where D represents the diagonal matrix. In other words, for $\gamma_\mathcal{O}$, the principal geodesic is the line segment from $[1,0,0]$ to $[0,0,1]$ with non-negative entries. Denote this principal geodesic by $\tilde\ell$.

We also can make a further normalization to assume that

$$\sigma_k\gamma_k\sigma_k^{-1} = \rho_k\gamma_k\rho_k^{-1} = D(\lambda_k,\mu_k,\nu_k). \tag{1}$$

Here is how to justify this normalization: The eigenvalues λ_k,μ_k,ν_k of $\sigma_k\gamma_k\sigma_k^{-1}$ approach λ,μ,ν. For k large enough, λ_k,μ_k,ν_k are uniformly bounded, positive and separated from each other. We may choose a matrix ϕ_k of eigenvectors of $\sigma_k\gamma_k\sigma_k^{-1}$ which approaches the identity matrix as $k\to\infty$ (for example, we may choose eigenvectors of unit length; note the identity matrix is a matrix of unit eigenvectors of the limit $\gamma_\mathcal{O}$ of $\sigma_k\gamma_k\sigma_k^{-1}$). Then $(\sigma_k\Omega_k, \sigma_k\Gamma_k\sigma_k^{-1}) \to (\mathcal{O},H)$ if and only if $\phi_k^{-1}(\sigma_k\Omega_k, \sigma_k\Gamma_k\sigma_k^{-1}) \to (\mathcal{O},H)$. Note our construction implies

$$\phi_k^{-1}\sigma_k\gamma_k\sigma_k^{-1}\phi_k = D(\lambda_k,\mu_k,\nu_k).$$

Thus we may replace σ_k by $\phi_k^{-1}\sigma_k$, and we may assume (1).

Now Equation (1) implies the diagonal matrix $D(\lambda_k,\mu_k,\nu_k)$ commutes with $\sigma_k\rho_k^{-1}$, and so $\sigma_k\rho_k^{-1}$ is diagonal as well. Define $\alpha_k = \sigma_k\rho_k^{-1}$. Thus we write $\alpha_k = D(\lambda_k^{t_k},\mu_k^{t_k},\nu_k^{t_k})\cdot D(e^{-s_k},e^{2s_k},e^{-s_k})$ as a product of holonomy and bulge matrices. Now if $\alpha_k=\sigma_k\rho_k^{-1}$ has a subsequential finite limit α modulo Dehn twists, we have $\mathcal{O} = \lim_{j\to\infty}\alpha_{k_j}\rho_{k_j}(\Omega_{k_j}) = \alpha\mathcal{U}$. Moreover, if we let $\hat\alpha_k$ be the matrix given by the product of α_k by an integral power of a Dehn twist so that

$$\hat\alpha_k = D(\lambda_k^{\hat t_k},\mu_k^{\hat t_k},\nu_k^{\hat t_k})\cdot D(e^{-s_k},e^{2s_k},e^{-s_k})$$

for $\hat t_k\in[0,1)$, then $\sigma_{k_j}\Gamma_{k_j}\sigma_{k_j}^{-1} = \hat\alpha_{k_j}\rho_k\Gamma_{k_j}\rho_k^{-1}\hat\alpha_{k_j}^{-1}$ converges to a limit L. Apply Goldman's Theorem 2.3.1 above to show the neck is trivial. Thus we may assume the s_k converge to $\pm\infty$.

(Here is how to apply Theorem 2.3.1 to the present situation. Consider $\hat S_1$ as the convex $\mathbb{RP}^2$ surface homeomorphic to S_1 formed by cutting along the principal geodesic at the end ℓ. This can be constructed by letting $\hat{\mathcal{O}}$ be the convex domain formed by cutting along $h\tilde\ell$ for all

$h \in H(\pi_1 S_1)$. Then $\hat{S}_1$ is the quotient $H\backslash\hat{\mathcal{O}}$. We can similarly form $\hat{S}_2$ with image $\hat{\mathcal{U}}$ of the developing map. Then since $\alpha^{-1}\mathcal{U} = \alpha\mathcal{O}$, the domains $\hat{\mathcal{O}}$ and $\alpha^{-1}\hat{\mathcal{U}}$ are disjoint subsets of $\mathcal{O}$ with common boundary segment $\tilde{\ell}$. Since the group actions also match up on all of $\mathcal{O}$, there are principal collar neighborhoods in $\hat{\mathcal{O}}$ and $\alpha^{-1}\hat{\mathcal{U}}$ which are invariant under holonomy along $\tilde{\ell}$ and are projectively equivalent via the principal reflection across $\tilde{\ell}$. Thus Theorem 2.3.1 applies, and we may glue $\hat{S}_1$ and $\hat{S}_2$ together via this identification to form a convex $\mathbb{RP}^2$ surface. This surface must be identical to the quotient $L\backslash\mathcal{O}$ by analytic continuation, and therefore uniqueness, of the developing map.)

Assume without loss of generality that $s_k \to +\infty$. To show the bulge must be infinite in the limit, we recall the principal triangles with principal geodesic $\tilde{\ell}$. Let T be the open triangle in $\mathbb{RP}^2$ given by the projection of the first octant in $\mathbb{R}^3$, and let $\bar{T}$ be the reflection of T across the principal geodesic. Since the surface S is separated along this principal geodesic, we may assume that the universal covers $\mathcal{O}$ and $\mathcal{U}$ of S_1 and S_2 respectively are in part on opposite sides of $\tilde{\ell}$. Without loss of generality, assume that $\mathcal{O}\cap\bar{T} \neq \emptyset$ and $\mathcal{U}\cap T \neq \emptyset$. From the point of view of the end on S_1, recall we say the bulge is $+\infty$ if the principal triangle $T \subset \mathcal{O}$, and the bulge is $-\infty$ if the principal geodesic $\tilde{\ell} \subset \partial\mathcal{O}$.

Let $q \in \mathcal{U} \cap T$. Then for large k, $q \in \rho_k\Omega_k$ and so $\alpha_k q \in \sigma_k\Omega_k$. This shows that the limit of $\alpha_k q$ is in the closure of $\mathcal{O}$. But since we know α_k has bulge parameter s_k going to $+\infty$, $\alpha_k q \to [0,1,0]$ from within T. Since the principal geodesic $\tilde{\ell} \subset \bar{\mathcal{O}}$, we see by convexity that $T \subset \mathcal{O}$, and thus that the bulge of this end of S_1 is $+\infty$.

On the other hand, the same argument shows that if there is a $p \in \mathcal{U} \cap \bar{T}$, then $\bar{T} \subset \mathcal{O}$. This is impossible, as then $\mathcal{O} \supset \bar{T} \cup \tilde{\ell} \cup T$, which contains the coordinate line with infinite point $[0,1,0]$. This contradicts the proper convexity of $\mathcal{O}$. Thus $\mathcal{U} \cap \bar{T} = \emptyset$, which means that the principal geodesic $\tilde{\ell} \subset \mathcal{U}$, and thus the bulge of this end of S_2 is $-\infty$. The limit then satisfies the condition for a separated neck with hyperbolic holonomy to be regular.

For the second case, assume $S - \ell = S_1$ is connected. In this case, we have a sequence of marked $\mathbb{RP}^2$ structures (Ω_k, Γ_k) and distinguished hyperbolic elements $\gamma_k, \delta_k \in \Gamma_k(\pi_1 S)$, together with attaching maps $T_k \in \operatorname{Aut}\Omega_k$ so that $T_k\gamma_k T_k^{-1} = \delta_k^{-1}$ (see for example Harvey [**31**]). The hyperbolic elements γ_k and δ_k both represent the holonomy (with opposite orientations) of the neck to be separated. We assume that $(\Omega_k, \Gamma_k|_{S_1}) \to (\mathcal{O}, H)$. (In this case, since there is a single limit domain, we absorb the $\rho_k \in \mathbf{SL}(3,\mathbb{R})$ into the definitions of Ω_k and Γ_k. In the present case, T_k will diverge instead of ρ_k.) See Figure 1.

If $\gamma = \lim \gamma_k$ is parabolic or quasi-hyperbolic, then the holonomy is regular, and the holonomy type of δ_k is the inverse of that of γ_k.

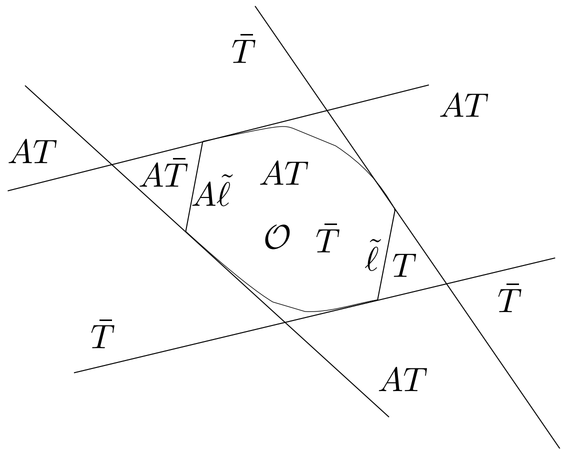

Figure 1

If on the other hand, γ is hyperbolic, we proceed as above. Choose coordinates so that $\gamma = D(\lambda, \mu, \nu)$. By the same arguments as above, we may slightly modify the Ω_k for k large so that $\gamma_k = D(\lambda_k, \mu_k, \nu_k)$ as well. Let $\delta = \lim \delta_k$ and fix $A \in \mathbf{SL}(3, \mathbb{R})$ so that $A\gamma A^{-1} = \delta^{-1}$. Since $\delta_k \to \delta$, we find

$$A^{-1}T_k \cdot D(\lambda_k, \mu_k, \nu_k) \cdot T_k^{-1}A \to D(\lambda, \mu, \nu).$$

As above, there is a matrix ϕ_k of eigenvectors of $A^{-1}T_k \cdot D(\lambda_k, \mu_k, \nu_k) \cdot T_k^{-1}A$ so that $\phi_k \to I$ and

$$\phi_k^{-1}A^{-1}T_k \cdot D(\lambda_k, \mu_k, \nu_k) \cdot T_k^{-1}A\phi_k = D(\lambda_k, \mu_k, \nu_k).$$

Thus $T_k^{-1}A\phi_k$ is diagonal, and may be written as the product of Dehn twist and bulge matrices $D(\lambda_k^{t_k}, \mu_k^{t_k}, \nu_k^{t_k}) \cdot D(e^{-s_k}, e^{2s_k}, e^{-s_k})$. If s_k has a finite limit, then we can show, as in the case above in which $S - \ell$ is not connected, that the separated neck is trivial. (Theorem 2.3.1 applies in a similar way.) The remaining cases to analyze are $s_k \to \pm\infty$. Assume without loss of generality that $s_k \to +\infty$. Recall the definitions of the principal geodesic $\tilde{\ell}$ and principal triangles $T, \bar{T}$ as above. Let $\tilde{\ell}$ be the line segment from $[1, 0, 0]$ to $[0, 0, 1]$ with nonnegative coordinate entries, and let T be the open triangle with vertices $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$ all of whose coordinates are nonnegative. Finally, let $\bar{T}$ be the reflection of T across $\tilde{\ell}$. By the limiting attaching map A, we may assume that $\mathcal{O} \cap \bar{T}$ and $\mathcal{O} \cap AT$ are not empty. For $q \in \mathcal{O} \cap \bar{T}$, we see that for large k, $q \in \Omega_k \cap \bar{T}$. Then $\phi_k A^{-1} T_k q \to [0, 1, 0]$ from within $\bar{T}$, which shows $T_k q \to A[0, 1, 0]$ from within $A\bar{T}$. Since $\bar{\mathcal{O}} \supset A\tilde{\ell}$, we see by convexity that $\mathcal{O} \supset A\bar{T}$, and so the bulge of this end is $+\infty$.

On the other hand, if there is a $p \in \mathcal{O} \cap T$, then the same argument shows $\mathcal{O} \supset AT$ as well. This contradicts the proper convexity of $\mathcal{O}$, and so we see that $\mathcal{O} \cap T = \emptyset$, and thus the bulge of this end is $-\infty$. This picture satisfies the definition of a regular separated neck. q.e.d.

2.7. Augmented Goldman space. In order to introduce the augmented Goldman space, we first form, as a warmup, a bordification of $\mathcal{G}_S$ by attaching singular $\mathbb{RP}^2$ structures which degenerate only along a single simple, nonperipheral, homotopically nontrivial loop ℓ. Define $\mathcal{G}_S^\ell$ as the set of all properly convex $\mathbb{RP}^2$ structures on $S - \ell$ which form regular separated necks across ℓ. We produce a topology on the bordification

$$\mathcal{G}_S \sqcup \mathcal{G}_S^\ell.$$

First of all, if $X \in \mathcal{G}_S$, then we declare all open neighborhoods in $\mathcal{G}_S$ to form a neighborhood basis in the bordification.

Now let $X \in \mathcal{G}_S^\ell$. First of all, consider open sets among $\mathbb{RP}^2$ structures on $S - \ell$ which form separated necks across ℓ. Each such open set $\mathcal{O} \subset \mathcal{G}_{S-\ell}$ contains both regular and trivial necks across ℓ. Now we construct from $\mathcal{O}$ a subset $\tilde{\mathcal{O}}$ of $\mathcal{G}_S \sqcup \mathcal{G}_S^\ell$. Let $\mathcal{O}_{\rm reg}$ consist of the $\mathbb{RP}^2$ structures with regular necks across ℓ, and let $\mathcal{O}_{\rm triv} = \mathcal{O} - \mathcal{O}_{\rm reg}$ consist of the $\mathbb{RP}^2$ structures with trivial necks across ℓ. Then define

$$\tilde{\mathcal{O}} = \mathcal{O}_{\rm reg} \sqcup {\rm Pull}_{S,\ell}^{-1} \mathcal{O}_{\rm triv}.$$

In other words, for each $\mathbb{RP}^2$ structure with a trivial neck in a neighborhood of X, we attach the neck by taking the inverse image of the pulling map. All such $\tilde{\mathcal{O}}$ form a neighborhood basis for the topology of augmented Goldman space near X. Note that this topology on $\mathcal{G}_S \sqcup \mathcal{G}_S^\ell$ is not locally compact, since the pulling map is unchanged under each Dehn twist around ℓ. But, by Subsection 2.4, we may choose a countable collection of such neighborhoods, and so the topology is first countable.

If $c \in C(S)$, define $\mathcal{G}_S^c$ to be the set of all properly convex $\mathbb{RP}^2$ structures on $S - c$ with regular necks across each loop in c. (If $c = \emptyset$, $\mathcal{G}_S^\emptyset = \mathcal{G}_S$.) As a set, augmented Goldman space

$$\mathcal{G}_S^{\rm aug} = \bigsqcup_{c \in C(S)} \mathcal{G}_S^c.$$

If $X \in \mathcal{G}_S^{\rm aug}$, then there is a unique $c \in C(S)$ so that $X \in \mathcal{G}_S^c$, and thus X has a regular separated neck across each loop in c. As we deform X, some of these necks may remain separated, while others may be glued together. As above, let $\mathcal{O}$ be a neighborhood of X in the subset of $\mathcal{G}_{S-c}$ consisting of those $\mathbb{RP}^2$ structures which have separated necks across each loop in c. Each $Y \in \mathcal{O}$ has either trivial or regular separated necks across each loop in c. Then

$$\mathcal{O} = \bigsqcup_{d \subset c} \mathcal{O}_{{\rm triv},d},$$

where $Y \in \mathcal{O}_{\mathrm{triv},d}$ if and only d is the set of loops across which the separated neck is trivial in Y (thus the necks across the loops in $c - d$ are the regular separated necks). Now define

$$\tilde{\mathcal{O}} = \bigsqcup_{d \subset c} \mathrm{Pull}_{S,d}^{-1} \mathcal{O}_{\mathrm{triv},d},$$

where $\mathrm{Pull}_{S,\emptyset}$ is the identity map. The set of such $\tilde{\mathcal{O}}$ forms a neighborhood basis for the topology of $\mathcal{G}_S^{\mathrm{aug}}$ around X.

Remark. It is instructive to compare the construction of $\mathcal{G}_S^{\mathrm{aug}}$ to the construction of augmented Teichmüller space; see e.g. [**1**]. Given a free simple loop c in a surface closed S of genus at least two, we may take the hyperbolic length parameter around c to be zero. Then no neighborhood of this point in the augmented Teichmüller space has compact closure, as the associated twist parameters around c take all real values in the neighborhood.

As we must keep track of the developing map of a surface pulled across a loop c, each point in $\mathcal{G}_{S-c}$ *a priori* has a neighborhood in $\mathcal{G}_S^{\mathrm{aug}}$ which contains all integral powers of Dehn twists along c. This shows $\mathcal{G}_S^{\mathrm{aug}}$ is not locally compact. We can say more, however. For the regular cases, which are of primary interest, one may check that each neighborhood of an $\mathbb{RP}^2$ structure all of whose separated necks are regular includes $\mathbb{RP}^2$ structures on the glued necks twisted by the one-parameter group of all real powers of the holonomy.

We have defined augmented Goldman space essentially in terms of the dev-hol pair (Ω, Γ) of convex $\mathbb{RP}^2$ structures, as opposed to the Fenchel-Nielsen parameters commonly used in study of augmented Teichmüller space. It should be interesting to try to use Goldman's analog of Fenchel-Nielsen parameters on convex $\mathbb{RP}^2$ structures [**27**] to put coordinates on augmented Goldman space. Goldman's parameters have been extended by Marquis to the cases of parabolic and quasi-hyperbolic holonomy [**52**].

2.8. Augmented moduli space. Our main space of interest is in the quotient of augmented Goldman space by the mapping class group, which we call the augmented moduli space of convex $\mathbb{RP}^2$ structures. Recall the mapping class group is the group of orientation-preserving homeomorphism modulo diffeomorphisms isotopic to the identity $MCG(S) = \mathrm{Diff}^+(S)/\mathrm{Diff}^0(S)$.

Consider a diffeomorphism ϕ of S. If $x_0 \in S$ is a basepoint, then ϕ induces a map $\phi_* : \pi_1(S, x_0) \to \pi_1(S, \phi(x_0))$. We fix a holonomy representation $\Gamma \colon \pi_1(S, x_0) \to \mathbf{SL}(3, \mathbb{R})$. We assume that for the image of the developing map Ω, that the quotient $\Gamma \backslash \Omega$ is diffeomorphic to S.

Proposition 2.8.1. *$MCG(S)$ acts on $\mathcal{G}_S$ by homeomorphisms.*

This is quite standard, at least in the case S is closed. For example, Labourie showed that the action of $MCG(S)$ on $\mathcal{G}_S$ is proper and also that the quotient is Hausdorff [**39**].

Proof. Let $[(\Omega,\Gamma)]\in\mathcal{G}_S$, where $[\cdot]$ denotes the equivalence class under the $\mathbf{SL}(3,\mathbb{R})$ action. In order to consider the action of the diffeomorphism ϕ on Γ, each homotopy class of paths in S from x_0 to $\phi(x_0)$ determines an isomorphism from $\pi_1(S,x_0)$ to $\pi_1(S,\phi(x_0))$. As in Lemma 2.5.1 above, different choices of paths lead to representations equivalent under the $\mathbf{SL}(3,\mathbb{R})$ action. This shows that the action of ϕ on (Ω,Γ) produces an equivalence class in $\mathcal{G}_S$. Moreover, the actions of the diffeomorphism group and $\mathbf{SL}(3,\mathbb{R})$ commute with each other, and so the diffeomorphism group acts on $\mathcal{G}_S$.

All that remains is to show the diffeomorphisms isotopic to the identity act trivially. The argument in the previous paragraph shows that we may assume such a diffeomorphism preserves the basepoint x_0. In this case, an isotopy of diffeomorphisms induces a homotopy of loops based at x_0, and so the elements of $\pi_1(S,x_0)$ are fixed by diffeomorphisms isotopic to the identity.

It is clear from the definition of the topology on $\mathcal{G}_S$ that this action is by homeomorphisms. q.e.d.

We denote the quotient $MCG(S)\backslash\mathcal{G}_S$ by $\mathcal{R}_S$.

To extend this proposition to $\mathcal{G}_S^{\mathrm{aug}}$, we must extend our marking to the case of separated necks. Each $c\in C(S)$ represents a set of separated necks, and $S-c$ has a number of connected components $S_1,\dots,S_n$. First of all, consider the case that $S_1=S-c$ is connected. Then the action of $\Gamma(\pi_1(S,x_0))$ is restricted on S_1 to include only those homotopy classes of loops in $\pi_1(S_1,x_0)$ which have representative loops which do not intersect c. In other words,

$$\Gamma|_{S_1}(\pi_1(S_1,x_0))\subset\Gamma(\pi_1(S,x_0))\subset\mathbf{SL}(3,\mathbb{R})$$

in this case.

In the case $S-c=\sqcup_{i=1}^n S_i$ is not connected, then we consider $\pi_1(S_i,x_i)$ for basepoints $x_i\in S_i$. First of all, we may relate $\pi_1(S,x_0)$ to $\pi_1(S,x_i)$ for each i by choosing a path from x_0 to x_i. As above in Lemma 2.5.1, different choices of a homotopy class of each such path lead to holonomy representations conjugate by elements of $\mathbf{SL}(3,\mathbb{R})$. But our definition of $\mathcal{G}_S^c$ allows for conjugating by one element of $\mathbf{SL}(3,\mathbb{R})$ for each connected component S_i of $S-c$, and so our argument works independently of the choice of paths. Now we restrict only to those elements in $\pi_1(S,x_i)$ which do not intersect c: these are exactly the elements of $\pi_1(S_i,x_i)$.

Now the following proposition follows without much difficulty from the definitions laid out above. The assertion about $\mathcal{R}_S^{\mathrm{aug}}$ being first countable follows from Lemma 2.4.1.

Proposition 2.8.2. *For every $c \in C(S)$, a diffeomorphism ϕ of S induces a homeomorphism of $\mathcal{G}_S^{\mathrm{aug}}$ which sends each stratum $\mathcal{G}_S^c$ homeomorphically onto $\mathcal{G}_S^{\phi(c)}$. The mapping class group acts by homeomorphisms on $\mathcal{G}_S^{\mathrm{aug}}$, and the quotient topology on $\mathcal{R}_S^{\mathrm{aug}} \equiv MCG(S)\backslash\mathcal{G}_S^{\mathrm{aug}}$ is first countable.*

2.9. Plumbing coordinates and the topology of regular cubic differentials. In this subsection, we consider the relevant holomorphic theory, as opposed to the real projective theory above, in order to define the topology on the total space of the bundle of cubic differentials over $\overline{\mathcal{M}}_g$. Below in Subsection 3.4, we will see how the $\mathbb{RP}^2$ structure on a closed surface S is determined by a pair (Σ, U) of conformal structure and cubic differential. Also, we recall some basic facts about the thick/thin decomposition of complete hyperbolic surfaces and its relationship to the holomorphic theory on $\overline{\mathcal{M}}_g$. The hyperbolic and conformal structures we consider will not be primarily considered with respect to basepoints. Instead, later in Sections 4 and 5, we will see how to choose basepoints in the thick part of moduli.

To define the topology of the total space of the bundle $\mathcal{V}_g$ of regular cubic differentials over $\overline{\mathcal{M}}_g$, we start with a heuristic picture of the main construction. Recall that on the regular part Σ^{reg} of a compact stable noded Riemann surface Σ, there is a unique complete conformal finite-area hyperbolic metric k. Each hyperbolic surface can then naturally be decomposed into the thick part and the thin part, as Margulis's Lemma shows that there is a universal positive constant $\tilde{c}$ so that the set of points with injectivity radius less than $\tilde{c}$ is a disjoint union of annular cusp and collar neighborhoods. The noded Riemann surface is smooth (in other words, there are no nodes) if and only if there are no cusp neighborhoods. Each cusp neighborhood is isometric to every other, and a single parameter, the length l of the core geodesic, is the only hyperbolic invariant for collar neighborhoods centered around the geodesic. Each cusp or collar neighborhood is metrically rotationally invariant. Allowing $l \to 0$ changes a collar neighborhood to a pair of cusp neighborhoods, and heuristically provides a path to the boundary of the moduli space of Riemann surfaces. In this setting, we would like to define a related conformal metric m on each Riemann surface given by replacing the hyperbolic metric on the thin part by conformal flat cylindrical metrics of circumference $2\tilde{c}$ so that the resulting metric, hyperbolic on the thick part and flat on the thin part, is continuous. See Figure 2. Then, inspired by Wolpert [**78**], we may define regular cubic differentials as holomorphic cubic differentials on Σ^{reg} which are bounded in the L^∞_m norm and whose residues match up appropriately. Convergence of families of regular cubic differentials over a sequence of noded Riemann surface Σ_i which converge to Σ_∞ in $\overline{\mathcal{M}}_g$ is then defined to be convergence in L^∞_{loc} with respect to the m_i metric.

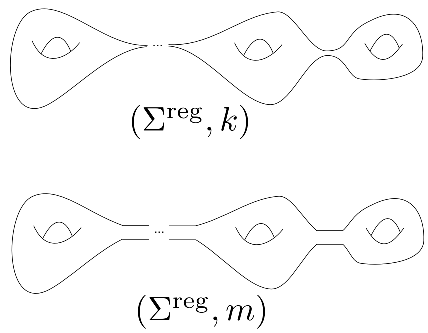

Figure 2

This heuristic picture is imprecise, and using l as a parameter in moduli is not well-suited to the geometry of holomorphic objects such as regular cubic differentials. Instead, we consider Wolpert's hyperbolic metric plumbing coordinates, which describe the holomorphic moduli of noded Riemann surfaces, but also are constructed to be closely related to the hyperbolic metrics.

Consider local V-manifold cover coordinates on $\overline{\mathcal{M}}_g$ near a nodal curve. These are due to Masur [**54**] and refined by Wolpert. See Wolpert [**77**] for an overview and references. Consider a stable noded Riemann surface Σ with n nodes. We think of Σ as representing a point in the boundary of the Deligne-Mumford compactification of the moduli space of closed Riemann surfaces of genus g. For the $i^{\rm th}$ node there is a small *cusp neighborhood* N_i so that:

- The closures of the N_i are disjoint in Σ.
- There are coordinates z_i, w_i on each part of $N_i \cap \Sigma^{\rm reg}$ and a uniform constant $c < 1$ so that

$$N_i^{\rm reg} \equiv N_i \cap \Sigma^{\rm reg} = \{|z_i| \in (0,c)\} \sqcup \{|w_i| \in (0,c)\}$$

and the complete hyperbolic metric on $\Sigma^{\rm reg}$ restricts to $N_i^{\rm reg}$ as

$$\frac{|dx|^2}{(|x|\log|x|)^2}, \qquad x = w_i, z_i. \tag{2}$$

The coordinates z_i, w_i are called hyperbolic cusp coordinates.

Moreover, Wolpert [**75**] has constructed a real-analytic family of Beltrami differentials $\nu(s)$ on $\Sigma^{\rm reg}$ for s in a neighborhood of the origin in $\mathbb{C}^{3g-3-n}$ so that

- $\nu(0) = 0$.
- The support of each $\nu(s)$ is disjoint from the closure of each cusp neighborhood N_i.
- Each $\nu(s)$ is C^∞.

- There is an induced quasiconformal diffeomorphism of Riemann surfaces $\chi^s\colon \Sigma^{\rm reg} \to \Sigma^{s,{\rm reg}}$ satisfying $\bar\partial\chi^s = \nu(s)\partial\chi^s$.
- On each N_i, the restriction of $\chi^s\colon N_i \to N_i$ is a rotation (and thus a hyperbolic isometry).

Each node contributes an additional complex parameter via the plumbing construction. First of all, each cusp neighborhood N_i is biholomorphic as a complex-analytic set to $\{z_iw_i = 0, |z_i| < c, |w_i| < c\} \subset \mathbb{C}^2$. To open the node, let $|t_i| < c^2$ and consider the annulus

$$N_i^{t_i} = \{z_iw_i = t_i, |z_i|, |w_i| \in (\tfrac{|t_i|}{c}, c)\} \subset \mathbb{C}^2.$$

If we choose $t = (t_1, \dots, t_n)$ as above, we may replace N_i with $N_i^{t_i}$ (by using the same z_i, w_i coordinates) in order to form Σ^t. Since the Beltrami differentials are constructed so that the hyperbolic cusp coordinates are essentially preserved, we have

- (s,t) near $(0,0)$ form local V-manifold coordinates, the *hyperbolic metric plumbing coordinates*, for $\overline{\mathcal{M}}_g$.

Given these hyperbolic metric plumbing coordinates, we recall Wolpert's grafting metric $g^{s,t}$. Let $k^{s,t}$ be the complete hyperbolic metric on $\Sigma^{s,t,{\rm reg}} \equiv \Sigma^{s,t} - \{\text{nodes}\}$. We will not use the construction of the grafting metric, but only the following properties [**74**]:

- $g^{s,t}$ is a complete conformal metric on $\Sigma^{s,t,{\rm reg}}$.
- If $t_i = 0$, then $g^{s,t} = k^{s,t}$ on $N_i \cap \Sigma^{s,t,{\rm reg}}$.
- For $t_i \neq 0$, then $g^{s,t}$ is equal to

$$\left(\frac{\pi}{|z_i|\log|t_i|}\csc\left(\pi\frac{\log|z_i|}{\log|t_i|}\right)\right)^2 |dz_i|^2 \tag{3}$$

  on $N_i^{t_i}$.
- Away from $z_i = w_i = 0$, the metrics $g^{s,t}$ on $N_i^{t_i}$ vary real-analytically in $\frac{1}{\log|t_i|}$ for all $|t_i| < c^2$.
- There is a uniform constant C so that

$$\left|\frac{g^{s,t}}{k^{s,t}} - 1\right| \le C\left|\sum_{i=1}^n (\log|t_i|)^{-2}\right|. \tag{4}$$

- There is a uniform constant C' so that the curvature $\kappa_{g^{s,t}}$ satisfies

$$\|\kappa_{g^{s,t}} + 1\|_{C^0} \le C'(\log|t_i|)^{-2}. \tag{5}$$

Recall a regular cubic differential over a noded Riemann surface Σ is given by a holomorphic cubic differential U on $\Sigma^{\rm reg}$ with the following behavior at the nodes: Let each node be given by $z_iw_i = 0$ in local coordinates. In terms of the z_i and w_i coordinates, we first of all require U to have a pole of order at most 3 at the origin. For $x = z_i, w_i$, the *residue* of U is defined to be the dx^3/x^3 coefficient of U. The residue does not depend on the choice of local conformal coordinate. The second condition is that the residues of the z_i and w_i coordinates for each node sum to zero.

It will be useful for us to describe the convergence of cubic differentials in terms of a family of metrics constructed by modifying the grafting metrics $g^{s,t}$. Our construction is to replace the (locally) hyperbolic grafting metrics on the thin part of each surface by a flat conformal cylindrical metric of uniformly constant diameter. For $t_i = 0$, we will replace each hyperbolic cusp end by a complete flat cylinder, while for $t_i \neq 0$ small, we replace the hyperbolic collar by a flat collar. The details are presented below.

Since the boundary of $\mathcal{M}_g$ is compact, it can be covered by a finite number of hyperbolic metric plumbing coordinate neighborhoods V^α, $\alpha = 1, \dots, M$ centered at nodal curves on the boundary. Define the set V^0 to be an open set containing $\overline{\mathcal{M}}_g \setminus \cup_\alpha V^\alpha$ whose closure does not intersect $\partial\mathcal{M}_g$. V^0 lies in the thick part of the moduli space for some $\epsilon > 0$, as it excludes a neighborhood of the boundary. Consider the universal curve $\pi\colon \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$. For each noded Riemann surface $\Sigma^{s,t}$ in $\pi^{-1}V^\alpha$, $\alpha = 0, \dots, M$, define the metric $m^{\alpha,s,t}$ as follows

- Let m^0 be the hyperbolic metric on the (necessarily nonsingular and closed) Riemann surface Σ.
- For a noded Riemann surface $\Sigma = \Sigma^{s,t}$ in V^α, define $m^{\alpha,s,t}$ to be equal to $g^{s,t}$ on $\Sigma \setminus \cup_i N_i^{t_i}$.
- On $N_i^{t_i}$, consider the quasi-coordinate $\ell = \log x$ for $x = z_i, w_i$. Then for $t_i \neq 0$,

$$g^{s,t} = \left(\frac{\pi}{\log|t_i|} \csc\left(\pi \frac{\operatorname{Re}\ell}{\log|t_i|} \right) \right)^2 |d\ell|^2,$$

  for $\log|t_i| - \log c \le \operatorname{Re}\ell \le \log c$. For $t_i = 0$, $g^{s,t} = (\operatorname{Re}\ell)^{-2}\,|d\ell|^2$ for $\operatorname{Re}\ell \le \log c$.
- For the $t_i = 0$ case, consider the half-cylinder $\{x : \operatorname{Re}\ell \le 2\log c\}$ with flat metric $f = (2\log c)^{-2}\,|d\ell|^2$.
- For $0 < |t_i| < c^{2\pi}$, let

$$K = \frac{\log|t_i|}{\pi} \arcsin\left(\frac{\pi}{\log|t_i|} \cdot 2\log c \right),$$

  consider the annulus $\{\operatorname{Re}\ell \in [\log|t_i| - K, K]\}$ with flat metric $f = (2\log c)^{-2}\,|d\ell|^2$. Note this metric is equal to $g^{s,t}$ on the boundary of the annulus.
- Now on each surface we interpolate between the two metrics. Let η be a smooth nonnegative function of $\operatorname{Re}\ell$ which is equal to 1 for $\operatorname{Re}\ell \le 2\log c$ and equal to 0 for $\operatorname{Re}\ell \ge \log c$. On each connected component of N_i^{reg} for $t_i = 0$, define the metric

$$m^{\alpha,s,t} = (g^{s,t})^{1-\eta(\operatorname{Re}\ell)} \cdot f^{\eta(\operatorname{Re}\ell)}. \tag{6}$$

- We make a similar definition for $t_i \neq 0$, shifting the interpolating factor and adjusting for the fact that $N_i^{t_i}$ is connected. Let

$$\phi(\operatorname{Re}\ell) = \eta(\operatorname{Re}\ell - K + 2\log c)\cdot\eta(2\log c + \log|t_i| - K - \operatorname{Re}\ell).$$

  Then the metric $m^{\alpha,s,t}$ restricted to $N_i^{t_i} \subset \Sigma = \Sigma^{s,t}$ is defined to be

$$m^{\alpha,s,t} = (g^{s,t})^{1-\phi(\operatorname{Re}\ell)}\cdot f^{\phi(\operatorname{Re}\ell)}. \tag{7}$$

- For $|t_i| \geq c^{2\pi}$, let $m^{\alpha,s,t} = g^{s,t}$. This definition means that $m^{\alpha,s,t}$ does not smoothly vary at $|t_i| = c^{2\pi}$. Nonetheless, the metrics $m^{\alpha,s,t}$ and their derivatives remain uniformly bounded in terms of each other there. This will be sufficient to derive the estimates we will need below.
- Note that $m^{\alpha,s,t}$ is always a complete conformal metric on Σ^{reg}. It is always equal to $g^{s,t}$ outside cusp and collar neighborhoods, and well inside these small neighborhoods, the metric is flat cylindrical of uniform circumference. These two regions are glued together along annual regions using a uniform partition of unity, and so the metric $m^{\alpha,s,t}$ on these annular regions is smooth and has uniform geometry. In particular, the $m^{\alpha,s,t}$ metrics have uniformly bounded Gauss curvature.
- Moreover, there is a uniform positive constant $C > 0$ so that for on every Riemann surface $\Sigma = \Sigma^{s,t}$ represented in V^α,

$$\frac{m^{\alpha,s,t}}{g^{s,t}} \geq C. \tag{8}$$

  By our construction, $m^{\alpha,s,t}/g^{s,t} \geq 1$ in the region where $m^{\alpha,s,t}$ is flat. The existence of such a bound C on the region of the interpolation follows from compactness considerations, while outside these two regions, the two metrics are equal.

We will also use a basic description of the thick-thin decomposition of hyperbolic surfaces and of the universal curve. See e.g. [**77**]. For positive ϵ small enough, the locus of points Thin_ϵ on a complete hyperbolic surface with injectivity radius less that ϵ is called the thin part of the moduli space, while the complement is the thick part Thick_ϵ. The thin part is a disjoint union of punctured disks (cusps) and annuli (collars). Margulis's Lemma shows there is a fixed $\epsilon_0 > 0$ so that this is true for all $0 < \epsilon < \epsilon_0$, while Mumford's Compactness Theorem shows Thick_ϵ is compact. We will need to relate this to the hyperbolic metric plumbing coordinates. In particular, in each V^α neighborhood, (4) shows that for any sequence of points in $\overline{\mathcal{C}}_g$, the injectivity radius of the hyperbolic metric goes to zero if and only if the plumbing coordinates z_j, w_j for the appropriate collar go to zero.

Lemma 2.9.1. *For any convergent sequence in $\overline{\mathcal{C}}_g$, either it converges to a node (where some z_j, w_j coordinates are 0) or there is an $\epsilon > 0$ so that all but a finite number of elements of the sequence lie in Thick$_\epsilon$.*

For the remainder of this subsection, we recall the topology of the bundle of regular cubic differentials and formulate it in a way that will be useful below.

For $c \in C(S)$, define $\mathcal{T}_S^{\mathrm{aug},c}$ as the Teichmüller space augmented only by pinching loops in c. Let $\Lambda(c)$ be the group of Dehn twists along loops in c. Let $\mathcal{Q} = \mathcal{Q}_S^c = \mathcal{T}_S^{\mathrm{aug},c}/\Lambda(c)$. This quotient space is has a natural complex structure constructed by Hubbard-Koch [**33**]. Let $\mathcal{X} = \mathcal{X}_S^c$ be the proper flat family of noded Riemann surfaces over $\mathcal{Q}_S^c$. Then the relative bundle $\mathcal{K}^{3,\mathrm{reg}}_{\mathcal{X}/\mathcal{Q}}$ of regular cubic differentials is a complex vector bundle of rank $5g-5$ over $\mathcal{Q}$.

Note that each V^α chart discussed above is naturally a manifold coordinate chart on $\mathcal{Q}$: For V^α centered around a noded Riemann surface with a number of nodes, the coordinates naturally lie in $\mathcal{Q}_c$ for c the collection of loops on S pinched to form these nodes.

To see why $\mathcal{K}^{3,\mathrm{reg}}_{\mathcal{X}/\mathcal{Q}}$ is a complex vector bundle, and to describe its topology, we follow [**33**], where the analogous case of quadratic differentials is investigated. First define the *plumbing locus*

$$\mathcal{P} = \{(z,w,t) \in \mathbb{C}^3 : zw = t,\ |z| < 4,\ |w| < 4,\ |t| < 1\},$$

and $\rho \colon \mathcal{P} \to \mathcal{D}$ given by $\rho \colon (z,w,t) \mapsto t$. There is a surjective complex-analytic map $p \colon \mathcal{X} \to \mathcal{Q}$ so that for every $a \in \mathcal{X}$, there are neighborhoods U of a in $\mathcal{X}$ and V of $p(a)$ in $\mathcal{Q}$, together with maps $\psi \colon V \to \mathcal{D}$ and $\tilde\psi \colon U \to \psi^*\mathcal{P}$ so that the following diagram commutes:

$$\begin{array}{ccccc} U & \xrightarrow{\tilde\psi} & \psi^*\mathcal{P} & \longrightarrow & \mathcal{P} \\ & \searrow^{p} & \downarrow & & \downarrow \rho \\ & & V & \xrightarrow{\psi} & \mathcal{D} \end{array}$$

On the smooth part $\mathcal{X}^*$ of $\mathcal{X}$, define a sheaf $\mathcal{F}$ to be the cube of the sheaf of relative differentials on the fiber of $\mathcal{X} \to \mathcal{Q}$. Near each plumbing fixture, $\mathcal{F}$ is generated by multiples of $\tilde\psi^*(\frac{dx}{x} - \frac{dy}{y})^3$ by local holomorphic functions on $\mathcal{X}$. Then as in [**33**], $\mathcal{F}$ is a coherent analytic sheaf over $\mathcal{X}$. Moreover, a cohomology calculation allows us to use Grauert's theorem [**29**] to show the push-forward sheaf $\mathcal{K}^{3,\mathrm{reg}}_{\mathcal{X}/\mathcal{Q}} \equiv p_*\mathcal{F}$ is the sheaf of local sections of a vector bundle of rank $5g-5$ over $\mathcal{Q}$.

The topology on the total space of $\mathcal{K}^{3,\mathrm{reg}}_{\mathcal{X}/\mathcal{Q}}$, when considered as a vector bundle over $\mathcal{Q}$, is given by local analytic frames. In the next proposition, we will formulate a condition for convergence of sequences (Σ_j, U_j), for $\Sigma_j \in \mathcal{Q}$ and U_j a regular cubic differential on Σ_j. First of all, if $(\Sigma_j, U_j) \to (\Sigma, U)$, then we may choose a coordinate neighborhood V^α, and the associated cusp/collar neighborhoods N_i, coordinates $x = z_i, w_i$ and metric m on noded Riemann surfaces in V^α, so that for all j large $\Sigma_j \in V^\alpha$.

Conversely, given $\Sigma_j \to \Sigma \in V^\alpha$ we say that regular cubic differentials U_i on Σ_j *converge in* $L^{\infty,\mathrm{loc}}_m$ if $[(\chi^{s_j})^{-1}]^* U_j \to [(\chi^s)^{-1}]^* U$ in L^∞ with respect to the m^{α,Σ_j} metrics with the additional caveat that on the $N_i^{t_{i,j}}$ regions, the convergence in the hyperbolic metric plumbing coordinates is normal convergence on the domains $\{|x| \in (\frac{|t_{i,j}|}{c}, c)\}$, which vary in size as $t_{i,j}$ varies.

Proposition 2.9.2. *$(\Sigma_j, U_j) \to (\Sigma, U)$ in the total space of the vector bundle $\mathcal{K}^{3,\mathrm{reg}}_{\mathcal{X}/\mathcal{Q}}$ if and only if $\Sigma_j \to \Sigma$ in $\mathcal{Q}$ and $U_j \to U$ in the sense of $L^{\infty,\mathrm{loc}}_m$.*

Proof. First assume $(\Sigma_j, U_j) \to (\Sigma, U)$ in the total space of the vector bundle $\mathcal{K}^{3,\mathrm{reg}}_{\mathcal{X}/\mathcal{Q}}$. Then clearly $\Sigma_j \to \Sigma$ in $\mathcal{Q}$. On a neighborhood of Σ in $\mathcal{Q}$, there is a holomorphic frame $\{\xi^a\}_{a=1}^{5g-5}$ of $\mathcal{K}^{3,\mathrm{reg}}_{\mathcal{X}/\mathcal{Q}}$. The first thing to show is that the sections $\xi^a|_{\Sigma_j}$ converge to $\xi^a|_\Sigma$ in the sense of $L^{\infty,\mathrm{loc}}_m$.

As investigated by Wolpert [**78**], the convergence in the plumbing locus (inside the regions $N_i^{t_{i,j}}$) is given by normal convergence of the holomorphic functions given by the ratio $\xi^a|_{\Sigma_j}/(x^{-3}dx^3)$ in the coordinate x as above. The m^{α,Σ_j} metrics are constructed to be uniformly constant in the $\ell = \log x$ quasi-coordinates near the center of the $N_i^{t_{i,j}}$ neighborhoods, and are uniformly bounded away from zero on the remaining part of $N_i^{t_{i,j}}$. Thus the normal convergence of the $\xi^a|_{\Sigma_j}$ to $\xi^a|_\Sigma$ is in the sense of $L^{\infty,\mathrm{loc}}_m$ in the $N_i^{t_{i,j}}$ regions.

In other regions, $\Sigma_j \setminus \cup_i N_i^{t_{i,j}}$, the deformation of the complex structure there is given by the Beltrami coefficients $\nu(s_j)$, and thus by the pullback of the induced quasiconformal map χ^{s_j}. Since the m metrics on $\Sigma_j \setminus \cup_i N_i^{t_{i,j}}$ are uniformly equivalent to the hyperbolic metrics and are uniformly bounded in terms of each other in the neighborhood V^α, we see that $[(\chi^{s_j})^{-1}]^*\xi^a|_{\Sigma_j} \to [(\chi^s)^{-1}]^*\xi^a|_\Sigma$ converges in L^∞ with respect to the m^{α,Σ_j} metrics.

There is a small discrepancy in the overlap between these two regions, in that for the $[(\chi^{s_j})^{-1}]^*\xi^a|_{\Sigma_j}$, the coordinates in the plumbing locus are the standard ones we use above, up to a rotation $\theta = \theta(s_j)$. This discrepancy need not concern us, as θ varies continuously in s, as follows from [**78**].

Thus we have shown that the sections $\xi^a|_{\Sigma_j}$ converge to $\xi^a|_\Sigma$ in the sense of $L^{\infty,\mathrm{loc}}_m$. Now consider our original problem, in which $(\Sigma_j, U_j) \to (\Sigma, U)$ in the total space of the vector bundle. In terms of the frame $\{\xi^a\}$, write $U = u_a\xi^a$ and $U_j = u_{a,j}\xi^a$. Convergence in the total space then is equivalent to $\Sigma_j \to \Sigma$ and $u_{a,j} \to u_a$ for all a. Since the frame $\{\xi^a\}$ converges in $L^{\infty,\mathrm{loc}}_m$, we see $U_j \to U$ in the sense of $L^{\infty,\mathrm{loc}}_m$ as well.

To prove the converse, assume $\Sigma_j \to \Sigma$ and $U_j \to U$ in the $L^{\infty,\mathrm{loc}}_m$ sense. Write $U_j = u_{a,j}\,\xi^a|_{\Sigma_j}$ as above, and let $\mathbf{u_j} = (u_{j,a})_{a=1}^{5g-5} \in \mathbb{C}^{5g-5}$. It suffices to show that $\mathbf{u_j} \to \mathbf{u}$ in $\mathbb{C}^{5g-5}$. Consider two cases: First of all, assume $\mathbf{u_j}$ converges to some $\mathbf{v} \in \mathbb{C}^{5g-5}$, then $(\Sigma_i, U_i) \to (\Sigma_i, v_a\,\xi^a|_\Sigma)$ in the total space, and by the paragraphs above, we see $\mathbf{v} = \mathbf{u}$. In the second case, assume there is a sequence of real numbers $\lambda_j \to 0$ so that $\lambda_j \mathbf{u_j} \to \mathbf{v} \neq \mathbf{0} \in \mathbb{C}^{5g-5}$. In other words, $(\Sigma_j, \lambda_j U_j) \to (\Sigma, V)$ in the total space of the vector bundle, for $V = v_a\,\xi^a|_\Sigma \neq 0$. But since $(\Sigma_j, U_j) \to (\Sigma, U)$, we also have $(\Sigma_j, \lambda_j U_j) \to (\Sigma, 0)$ in the total space of the bundle. This contradiction rules out the second case. Since every subsequence of $\{\mathbf{u_j}\}$ is guaranteed to have a subsequence satisfying either the first or the second case, this proves the converse, and the proposition. q.e.d.

3. Hyperbolic affine spheres

3.1. Relationship to convex $\mathbb{RP}^2$ structures. For $\Omega \subset \mathbb{R}^n \subset \mathbb{RP}^n$ a properly convex domain, consider the cone $\mathcal{C}$ in $\mathbb{R}^{n+1}$ given by $\{t(x,1) : t \in \mathbb{R}^+, x \in \Omega\}$. Ω is the image of $\mathcal{C}$ under the projection $\pi\colon \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{RP}^n$. The *proper* convexity of Ω is equivalent to $\mathcal{C}$ being properly convex, in that it is convex and contains no lines. There is a unique (properly normalized) hyperbolic affine sphere $\mathcal{H}$ asymptotic to the boundary of $\mathcal{C}$ which is invariant under special linear automorphisms of $\mathcal{C}$ [**12**, **13**]. Then π restricts to a diffeomorphism from $\mathcal{H}$ to Ω, and projective automorphisms of Ω lift to special linear automorphisms of $\mathcal{C}$, which act on $\mathcal{H}$.

In order to define the hyperbolic affine sphere, we introduced the *affine normal*. To any strictly convex smooth hypersurface $\mathcal{H}$ in $\mathbb{R}^{n+1}$, the affine normal ξ is a smooth transverse vector field which is invariant under unimodular affine transformations of $\mathbb{R}^{n+1}$. The hyperbolic affine sphere (which we take to be normalized so the center is at the origin and the affine mean curvature is -1) can be defined as a convex hypersurface $\mathcal{H}$ so that at all points, the affine normal is equal to the position vector. See e.g. [**10**, **11**, **13**, **44**, **50**, **59**]. A hyperbolic affine sphere is always equivalent to one normalized as above by an affine motion in $\mathbb{R}^{n+1}$. For the rest of this work, we will always assume all hyperbolic affine spheres are so normalized.

The first natural structure equation on $\mathcal{H}$ is the following formula of Gauss type:

$$D_X Y = \nabla_X Y + h(X,Y) f, \tag{9}$$

where D is the flat connection on $\mathbb{R}^{n+1}$, X and Y are tangent vector fields on $\mathcal{H}$, f is the position vector of points on $\mathcal{H}$ (which is transverse to the tangent space). Then $D_X Y$ is split into $\nabla_X Y$, the part in the tangent space, and $h(X,Y)f$, the part in the span of f. ∇ is a projectively-flat torsion-free connection on $\mathcal{H}$, while $h(X,Y)$ is a positive-definite symmetric tensor called the *Blaschke metric* or the *affine metric*. Another important local invariant is the *cubic tensor*, or *Pick form*, which is the difference of the Levi-Civita connection of h and the connection ∇. The cubic tensor measures how far a hypersurface is from a hyperquadric, as a general theorem of Maschke, Pick, and Berwald implies

Theorem 3.1.1. *$\mathcal{H}$ is a hyperboloid if and only if its cubic tensor vanishes identically.*

(This is a special case of the more general theorem that any nondegenerate smooth hypersurface is a hyperquadric if and only if its cubic tensor, when defined with respect to the affine normal ξ, vanishes identically.)

For a hyperbolic affine sphere $\mathcal{H}$, the completeness of the Blaschke metric is equivalent to $\mathcal{H}$ being properly embedded. In fact, we have the following theorem of Cheng-Yau [**12**, **13**] and Calabi-Nirenberg (unpublished), with clarifications by Gigena [**24**], Sasaki [**63**], and A.M. Li [**42**, **43**].

Theorem 3.1.2. *For Ω a properly convex domain in $\mathbb{RP}^n$, consider the cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ over Ω. Then there is a unique properly embedded hyperbolic affine sphere $\mathcal{H} \subset \mathcal{C}$ which is centered at the origin, has affine mean curvature -1, and which is asymptotic to $\partial\mathcal{C}$. $\mathcal{H}$ is invariant under volume-preserving linear automorphisms of $\mathcal{C}$, and $\mathcal{H}$ is diffeomorphic to Ω under projection. The Blaschke metric on $\mathcal{H}$ is complete.*

Conversely, let $\mathcal{H}$ be an immersed hyperbolic affine sphere normalized to have center 0 and affine mean curvature -1. If the Blaschke metric on $\mathcal{H}$ is complete, then $\mathcal{H}$ is properly embedded in a proper convex cone $\mathcal{C}$ centered at the origin and is asymptotic to the boundary $\partial\mathcal{C}$.

If $M = \Gamma\backslash\Omega$ is a properly convex $\mathbb{RP}^n$ manifold, then we may lift the representation Γ to $\mathbf{PGL}(n+1,\mathbb{R})$ to volume-preserving linear actions $\tilde\Gamma$ on the cone $\mathcal{C}$ over Ω. By the invariance of $\mathcal{H}$, we find $\tilde\Gamma$ acts on $\mathcal{H}$, and M is naturally diffeomorphic to $\tilde\Gamma\backslash\mathcal{H}$. The invariant tensors on $\mathcal{H}$ (the Blaschke metric and the cubic form) descend to M.

Given a properly convex cone $\mathcal{C} \subset \mathbb{R}^{n+1}$, the dual cone $\mathcal{C}^*$ is the cone in the dual vector space $\mathbb{R}_{n+1}$ to $\mathbb{R}^{n+1}$ given by all $\ell \in \mathbb{R}_{n+1}$ so that $\ell(x) > 0$ for all $x \in \mathcal{C}$. Upon projecting to projective space, if $\Omega = \pi(\mathcal{C}) \subset \mathbb{RP}^n$, the we also have a dual convex projective domain $\Omega^* \subset \mathbb{RP}_n$. From this formulation, we remark

Lemma 3.1.3. *If $\Omega_1 \subset \Omega_2$ are properly convex domains in $\mathbb{RP}^n$, then $\Omega_1^* \supset \Omega_2^*$.*

There is a related duality result on hyperbolic affine spheres due to Calabi (see [**23**], and [**48**] for an exposition). For $\mathcal{H}$ a hyperbolic affine sphere, consider the conormal map $\mathcal{H} \to \mathbb{R}_{n+1}$ given by

$$x \mapsto \ell, \quad \text{where} \quad \ell(x) = 1 \quad \text{and} \quad \ell(T_x H) = 0.$$

Theorem 3.1.4. *Given a properly convex cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ with corresponding hyperbolic affine sphere $\mathcal{H} \subset \mathcal{C}$, then the conormal map maps $\mathcal{H}$ diffeomorphically onto the unique hyperbolic affine sphere $\mathcal{H}^*$ corresponding to the dual cone $\mathcal{C}^* \subset \mathbb{R}_{n+1}$. The conormal map is an isometry with respect to the Blaschke metrics on $\mathcal{H}$ and $\mathcal{H}^*$ and takes the cubic form $C \mapsto -C$.*

We will also use the relationship between the conormal map and the *Legendre transform*. If $v : \Omega \to \mathbb{R}$ is a smooth convex function on a convex domain $\Omega \subset \mathbb{R}^n$ with coordinates x^i, the we define the Legendre transform function p by

$$p + v = x^i \, \frac{\partial v}{\partial x^i}.$$

The domain of the function p is considered primarily to be the image of the gradient $(dv)(\Omega)$, and so p is a function of the variables $\frac{\partial v}{\partial x^i}$. The Legendre transform is an involution on the space of convex functions. If $\mathcal{H}$ is a hypersurface given by a radial graph of $-\frac{1}{v}$ for a convex function v,

$$\mathcal{H} = \left\{ -\frac{1}{v(x)}(x^1, \ldots, x^n, 1) : x \in \Omega \right\},$$

Then the image of the conormal map of $\mathcal{H}$ is given by

$$\left\{ \left(-\frac{\partial v}{\partial x^1}, \cdots, -\frac{\partial v}{\partial x^n}, p \right) \right\}. \tag{10}$$

Therefore, the conormal map essentially (up to a few minus signs) interchanges the radial graph of $-\frac{1}{v}$ with the Cartesian graph of the Legendre transform p.

We also mention here the relationship between hyperbolic affine spheres and a real Monge-Ampère equation, which is due to Calabi [**11**]. The formulation here also depends on results in Gigena [**24**].

Theorem 3.1.5. *Given a properly convex domain $\Omega \subset \mathbb{R}^n \subset \mathbb{RP}^n$, the hyperbolic affine sphere asymptotic to the boundary of the cone over Ω is given by the radial graph of $-\frac{1}{v}$*

$$\left\{ -\frac{1}{v(x)}(x, 1) : x \in \Omega \right\},$$

for v the unique convex solution of the Dirichlet problem v continuous on $\bar\Omega$, $v = 0$ on $\partial\Omega$, and

$$\det(v_{ij}) = \left(-\frac{1}{v}\right)^{n+2} \tag{11}$$

on Ω, for v_{ij} the Hessian matrix of v. The Blaschke metric is $-\frac{1}{v} v_{ij}\, dx^i\, dx^j$.

Loewner-Nirenberg first solved this equation for convex domains with regular boundary in dimension two [**45**]. Cheng-Yau solved this equation in the general case [**12**].

3.2. Benoist-Hulin's convergence of invariant tensors. For the reader's convenience we provide a detailed proof of Benoist-Hulin's theorem.

Theorem 3.2.1. [**6**] *Let Ω_j, Ω_∞ be bounded convex domains in $\mathbb{R}^n \subset \mathbb{RP}^n$. Assume $\Omega_j \to \Omega_\infty$ converges in the Hausdorff topology with respect to the Fubini-Study metric on $\mathbb{RP}^n$. Then the solutions v_j to the Dirichlet problem (11) on Ω_j converge in C^∞_{loc} to the solutions v_∞ on Ω_∞.*

Since the projectively-invariant tensors the Blaschke metric and cubic tensor are formed from v and its derivatives, we have the following result of Benoist-Hulin:

Theorem 3.2.2. *Under the hypotheses of the theorem, the Blaschke metrics and cubic tensors converge in C^∞_{loc}.*

Proof of Theorem 3.2.1. The proof here of Benoist-Hulin's theorem provides more details than in [**6**]. For C^0_{loc} estimates, we use the maximum principle. Pick an inhomogeneous affine coordinate chart $\mathbb{R}^2 \subset \mathbb{RP}^2$ so that $0 \in \Omega_\infty$ and Ω_∞ is bounded in $\mathbb{R}^2$. This implies there are $\epsilon_j \to 0$ so that

$$(1+\epsilon_j)\Omega_j \supset \Omega_\infty \supset (1-\epsilon_j)\Omega_j.$$

For v_j the solutions to the Dirichlet problem for (11) on Ω_j, the corresponding solution on $t\Omega_j$ is $t^{\frac{n}{n+1}} v_j(t^{-1}x)$, and the maximum principle implies that if $\mathcal{O} \subset \mathcal{U}$, then $v_{\mathcal{O}} \geq v_{\mathcal{U}}$. In particular, this shows

$$(1+\epsilon_j)^{\frac{n}{n+1}} v_j(x/(1+\epsilon_j)) \leq v_\infty(x) \leq (1-\epsilon_j)^{\frac{n}{n+1}} v_j(x/(1-\epsilon_j)),$$

which in turns shows $v_j \to v_\infty$ in C^0_{loc} on Ω_∞. Define

$$v_j^+(x) = (1+\epsilon_j)^{\frac{n}{n+1}} v_j(x/(1+\epsilon_j)), \quad v_j^-(x) = (1-\epsilon_j)^{\frac{n}{n+1}} v_j(x/(1-\epsilon_j)).$$

Then $v_j^- \to v_\infty$ in C^0_{loc}, $v_j^+ \to v_\infty$ in C^0_{loc}, and $v_j^+(x) \geq v_j(x) \geq v_j^-(x)$.

The C^1_{loc} estimates depend only on convexity. Let T be a large triangle in $\mathbb{R}^2$ which contains Ω_∞ and all the Ω_k for k large. Then the solution v_T to the Monge-Ampère equation on T has a minimum value $-M$, and the maximum principle shows that the solutions v_Ω and v_{Ω_k} also must satisfy $v \geq -M$. Now for such a v, let $g = v(y+tw)$, where $t \in \mathbb{R}$, y is a boundary point, and w is a unit vector pointing into the domain. Then g is continuous on an interval $[0,R]$, $g(0) = g(R) = 0$, and g is smooth and strictly convex on $(0,R)$. In particular, g' is increasing on $(0,R)$. For $t \in (0,R)$,

$$g'(t) \geq \frac{g(t)-g(0)}{t-0} = \frac{g(t)}{t} \geq -\frac{M}{t}.$$

Together with the estimate along the same ray traversed in the opposite direction, this shows $|g'|$ is uniformly bounded on any compact set, with the bound depending on the diameter of the domain and the distance to the boundary.

For the interior C^2 estimates, we have the following standard result following Pogorelov. The following theorem we quote is a direct application of Theorem 17.19 in [**25**]. We note that we must restrict to a sub-level domain $\{v < -\epsilon\}$, as estimates on the function $v \mapsto \left(-\frac{1}{v}\right)^{n+2}$ and its first two derivatives are needed to apply Theorem 17.19.

Theorem 3.2.3. *Consider the solution v to the Dirichlet problem (11) on a bounded convex domain $\Omega \subset \mathbb{R}^n$. Let $\epsilon > 0$, and let $\mathcal{O} = \{v < -\epsilon\}$. Then there is a constant C depending on n, ϵ, $\|v\|_{C^1(\mathcal{O})}$ and the diameter of $\mathcal{O}$ so that if $\mathcal{O}' \subset\subset \mathcal{O}$, then*

$$\sup_{\mathcal{O}'} |D^2 v| \leq \frac{C}{\operatorname{dist}(\mathcal{O}', \partial\mathcal{O})}.$$

By the arguments above, we have $v_k^+ \to v$, $v_k^- \to v$, and $v_k^+ \geq v \geq v_k^-$. On each compact set, convexity shows that the convergence is uniform (this follows from the C^1 estimates above and Ascoli-Arzela). Since $v_k^+ \geq v_k \geq v_k^-$, we have that $v_k \to v$ uniformly on compact subsets of Ω_∞. Now let K be a compact subset of Ω_∞. For large k, $K \subset \Omega_k$ as well. By continuity, $\max_K v = -3\epsilon$ for some $\epsilon > 0$. The set $\{v \leq -2\epsilon\}$ is also compact, and by uniform convergence, we can see that for large enough k, we have

$$\{v_k < -\epsilon\} \supset \{v < -2\epsilon\} \supset \{v \leq -3\epsilon\} \supset K.$$

So if we define $\mathcal{O}_k = \{v_k < -\epsilon\}$,

$$\operatorname{dist}(K, \partial\mathcal{O}_k) > \operatorname{dist}(K, \{v = -2\epsilon\}) > 0.$$

Moreover, the diameter of $\mathcal{O}_k$ is bounded by that of Ω_k, which is uniformly bounded. Thus we have estimates for Theorem 3.2.3 which are independent of k, and on K, we have uniform C^2 estimates on v_k.

Finally, we use the Evans-Krylov estimates to find interior $C^{2,\alpha}$ estimates. See Theorem 17.14 in [**25**]. In particular, on any compact subset K of Ω, there is an $\alpha \in (0,1)$ so that the $C^{2,\alpha}$ estimates on a slightly smaller compact subset K' depends only on the distance $\operatorname{dist}(K', \partial K)$, the C^2 estimates of v on K, and bounds on the eigenvalues of the Hessian matrix of v. These estimates are similar to but easier to apply than the Pogorelov estimates above. The main new ingredient is to bound the eigenvalues of the Hessian matrix of v_k. The largest eigenvalue is bounded by Pogorelov's bounds on the second derivatives, while the smallest eigenvalue is bounded away from 0 by using the Monge-Ampère equation $\det(v_{k,ij}) = (-v_k)^{-n-2}$ and the bounds away from 0 for v_k.

We have shown so far that on every compact $K \subset \Omega_\infty$, there are uniform $C^{2,\alpha}$ bounds on v and v_k for k large, and also $v_k \to v$ uniformly. This means that the Ascoli-Arzela Theorem applies to show that (subsequentially at least), $v_k \to v$ in C^2 on K. But every subsequence of $\{v_k\}$ then has a subsequence converging to v in C^2, and so we see $v_k \to v$ in C^2 on K.

Higher-order interior estimates and convergence are standard once $C^{2,\alpha}$ estimates are in place, and so $v_k \to v$ in C^∞_{loc} on Ω_∞. q.e.d.

Benoist-Hulin's C^∞_{loc} convergence of affine invariants is quite strong. However, the conformal structure at the end of a surface is not local in this sense, and so we will need to expend more effort to compute the conformal structures at the ends.

3.3. An estimate on Blaschke metrics. We begin this subsection with a quantitative version of the following theorem of Cheng-Yau [**13**] (as clarified by Li [**42**]): A hyperbolic affine sphere $\mathcal{H}$ with complete Blaschke metric is properly embedded in $\mathbb{R}^{n+1}$ and is asymptotic to the boundary of the convex cone given by the convex hull of $\mathcal{H}$ and its center. We consider a quantitative version involving geodesic balls of large radius.

Proposition 3.3.1. *Let $\mathcal{H}$ be a hyperbolic affine sphere given by the radial graph of $-\frac{1}{v}$, where v solves the Dirichlet problem (11) over a convex domain Ω. Let v be normalized so that $v(0) = -1$ and $dv(0) = 0$. Assume there are positive constants γ, δ, ϵ, and consider the Euclidean ball $\{x : |x| < \epsilon\}$ in Ω. Assume the Blaschke arc-length $\ell(0,x)$ of radial paths $\{tx : t \in [0,1]\}$ satisfies*

$$\ell(0,x) > \delta \quad \textit{for} \quad |x| = \epsilon. \tag{12}$$

Also assume

$$v(x) > -1+\gamma, \quad \frac{1}{\epsilon} x^i \frac{\partial v}{\partial x^i}(x) > \gamma \quad \textit{for} \quad |x| = \epsilon. \tag{13}$$

Then there are positive constants $C = C(n)$ and $A = A(C,\gamma,\delta)$ so that if $B^h_0(Q)$ is the geodesic ball centered at $x = 0$ of radius Q, then

$$\{x : v(x) < -e^{(A-Q)/C}\} \subset B^h_0(Q).$$

Proof. Consider a hyperbolic affine sphere $\mathcal{H}$ normalized with its center at that origin, affine mean curvature -1. Moreover, assume

$(0,\dots,0,1) \in \mathcal{H}$ and that the tangent plane at that point is horizontal. Consider the dual affine sphere $\mathcal{H}^*$ can be written as the graph $\{(y,p(y))\}$. $\mathcal{K} = \{(-y,p(y))\}$, as the image of $\mathcal{H}^*$ under a volume-preserving linear map, is also a hyperbolic affine sphere.

We follow a suggestion in [**44**], Remark 2.7.2.6(ii). The height function p on $\mathcal{K}$ is a positive eigenfunction of the Laplacian with respect to the Blaschke metric h, which is complete and has Ricci curvature uniformly bounded below. A gradient estimate of Yau [**64**, Theorem I.3.2] then applies to show that there is a uniform constant C depending only on the dimension so that

$$\|d(\log p)\|_h \le C \tag{14}$$

Consider v the Legendre transform of p. Theorem 3.1.4 and (10) show that $\mathcal{H} = \{-\frac{1}{v(x)}(-x,1) : x \in \Omega\}$ is essentially the radial graph of $-\frac{1}{v}$. Recall the Legendre transform is given by

$$p + v = x^i y_i, \qquad y_i = \frac{\partial v}{\partial x^i}, \quad x^i = \frac{\partial p}{\partial y_i}. \tag{15}$$

We primarily consider $p = p(y)$ and $v = v(x)$. Choose coordinates on $\mathbb{R}^{n+1}$ so that $v(0) = -1$ and $dv(0) = 0$. Since v is convex, it has its minimum at $x = 0$. Differentiating (15) shows that

$$\frac{\partial p}{\partial x^i} = x^j \, \frac{\partial^2 v}{\partial x^i \partial x^j}. \tag{16}$$

We follow the proof of Theorem 2.7.1.9 in [**44**]. Use the expression for the Blaschke metric in Theorem 3.1.5 above to compute for $\bar{x} \in \Omega$ the Blaschke length ℓ of the path $\mathcal{P} = \{t\bar{x} : 0 \le t \le 1\}$ to be

$$\ell(0,\bar{x}) = \int_0^1 \left(-\frac{1}{v} \, \frac{\partial^2 v}{\partial x^i \partial x^j} \bar{x}^i \bar{x}^j \right)^{\frac{1}{2}} \, dt.$$

Assume (12) and (13) and use (14), (15) and (16). Let v^{ij} denote the inverse matrix of $\frac{\partial^2 v}{\partial x^i \partial x^j}$. Compute

$$\begin{aligned}
\ell(0,\bar{x}) &\le \delta + \int_{\epsilon/|\bar{x}|}^{1} \left(-\frac{1}{v} \frac{\partial^2 v}{\partial x^i \partial x^j} \bar{x}^i \bar{x}^j \right)^{\frac{1}{2}} dt \\
&= \delta + \int_{\epsilon/|\bar{x}|}^{1} \left(-\frac{1}{vt^2} \frac{\partial^2 v}{\partial x^i \partial x^j} (t\bar{x}^i)(t\bar{x}^j) \right)^{\frac{1}{2}} dt \\
&= \delta + \int_{\epsilon/|\bar{x}|}^{1} \left(-\frac{1}{vt^2} v^{ij} \frac{\partial p}{\partial x^i} \frac{\partial p}{\partial x^j} \right)^{\frac{1}{2}} dt \\
&= \delta + \int_{\epsilon/|\bar{x}|}^{1} -\frac{1}{vt} \|dp\|_h \, dt \\
&\le \delta + \int_{\epsilon/|\bar{x}|}^{1} -\frac{C}{vt} p \, dt \\
&= \delta + \int_{\epsilon/|\bar{x}|}^{1} -\frac{C}{vt} \left(t\bar{x}^j \frac{\partial v}{\partial x^j} - v \right) dt \\
&= \delta - C \int_{\epsilon/|\bar{x}|}^{1} \frac{d}{dt} \log |v(\mathcal{P}(t))| \, dt + \int_{\epsilon/|\bar{x}|}^{1} \frac{C}{t} \, dt \\
&= \delta + C \left[-\log |v(\bar{x})| + \log \left| v\left(\frac{\epsilon \bar{x}}{|\bar{x}|} \right) \right| - \log \left| \frac{\epsilon}{|\bar{x}|} \right| \right] \\
&\le \delta - C \log |v(\bar{x})| + C \log(1-\gamma) - C \log \epsilon + C \log |\bar{x}|.
\end{aligned}$$

Now we claim that $|\bar{x}| < \frac{\epsilon}{\gamma}$. To prove this claim, we consider the convex function $g(t) = v(t\bar{x})$ and an appropriate linear secant-line function f to g. By (13) and $\Omega = \{v < 0\}$, g satisfies $g(0) = -1$, $g(\epsilon|\bar{x}|^{-1}) > -1 + \gamma$, and $g(T) = 0$ for some $T > 1$. Now consider the linear function $f(t)$ so that $f(0) = -1$ and $f(\epsilon|\bar{x}|^{-1}) = g(\epsilon|\bar{x}|^{-1}) > -1 + \gamma$. The convexity of g implies that $f(t) \le g(t)$ for $t \ge \epsilon|\bar{x}|^{-1}$. In particular $f(\tau) = 0$ for $\tau = \epsilon|\bar{x}|^{-1}/[g(\epsilon|\bar{x}|^{-1}) + 1]$, and $f(t) \le g(t)$ implies

$$\frac{\epsilon|\bar{x}|^{-1}}{\gamma} > \frac{\epsilon|\bar{x}|^{-1}}{g(\epsilon|\bar{x}|^{-1}) + 1} = \tau \ge T > 1.$$

This proves the claim.

Therefore, there is a constant $A = A(C, \gamma, \delta)$ so that

$$d_h(0,\bar{x}) \le \ell(0,\bar{x}) \le A - C \log |v(\bar{x})|$$

for d_h the Blaschke distance. q.e.d.

Now we use Proposition 3.3.1 to show that sequences of points in convex domains must be separated from each other if their Blaschke distance approaches ∞. In fact, the set of sequences of points x_j in Ω_j which converge in the Benzécri sense, in that there exist $\rho_j \in \mathbf{SL}(3,\mathbb{R})$

so that $\rho_j(\Omega_j, x_j) \to (\mathcal{O}, x)$ in the Hausdorff sense, may be partitioned into equivalence classes according to whether their Blaschke distances remain bounded.

Proposition 3.3.2. *Let $\Omega_j \to \mathcal{O}$ be a convergent sequence of properly convex domains in $\mathbb{RP}^n$ with respect to the Hausdorff topology. Assume $\rho_j\Omega_j \to \mathcal{U}$ for $\rho_j \in \mathbf{SL}(n+1,\mathbb{R})$. Assume $q_j \to q$ for $q_j \in \Omega_j$ and $q \in \mathcal{O}$, and $r_j \to r$ for $r_j \in \rho_j\Omega_j$ and $r \in \mathcal{U}$. Assume that the Blaschke distance $d_{\Omega_j}(q_j, \rho_j^{-1}r_j) \to \infty$. Then there does not exist a sequence $z_j \in \Omega_j$ so that $z_j \to z \in \mathcal{O}$ and $\rho_j z_j \to w \in \mathcal{U}$.*

Proof. Choose coordinates on $\mathcal{O}$ so that $q = 0$ and so that the hyperbolic affine sphere $\mathcal{H}_{\mathcal{O}}$ is given by the radial graph of $-\frac{1}{v}$. Also assume $v(0) = -1$ and $dv(0) = 0$. In these coordinates, Theorem 3.2.1 shows that the corresponding functions v_j on Ω_j converge to v in C^∞_{loc}. In particular, we can change coordinates on Ω_j to assume $q_j = 0$, $v_j(0) = -1$, and $dv_j(0) = 0$, while still maintaining $\Omega_j \to \mathcal{O}$. The C^∞_{loc} convergence of $v_j \to v$ implies that there are positive constants γ, δ, ϵ independent of j so that the hypotheses of Proposition 3.3.1 are satisfied. If $d_{\Omega_j}(q_j, \rho_j^{-1}r_j) \ge Q_j$, then $v_j(\rho_j^{-1}r_j) \ge -e^{A-Q_j}$. Thus $v_j(\rho^{-1}r_j) \to 0$ and $\rho_j^{-1}r_j$ has no limit in $\mathcal{O}$.

We prove the result by contradiction. Assume there does exist such a sequence z_j. Note that z and q are of finite Blaschke distance from each other in $\mathcal{O}$, as are r and w in $\mathcal{U}$. Thus Theorem 3.2.2 implies there is a constant C so that for all j large enough, the Blaschke distances $d_{\Omega_j}(z_j, q_j) \le C$ and $d_{\Omega_j}(z_j, \rho_j^{-1}r_j) = d_{\rho_j\Omega_j}(\rho_j z_j, r_j) \le C$. Then the triangle inequality implies that $0 = d_{\Omega_j}(z_j, z_j) \ge d_{\Omega_j}(q_j, \rho_j^{-1}r_j) - d_{\Omega_j}(z_j, q_j) - d_{\Omega_j}(\rho_j^{-1}r_j, z_j) \to \infty$, which provides a contradiction. q.e.d.

Corollary 3.3.3. *Proposition* 3.3.2 *holds with the Hilbert metric in place of the Blaschke metric.*

Proof. Benoist-Hulin [**6**, proof of Prop. 2.6] show that for each dimension n, there is a positive constant C_n so that for all convex bounded domains $\Omega \subset \mathbb{R}^n$ and all $x, y \in \Omega$, $C_n^{-1} d^H_\Omega(x,y) \le d_\Omega(x,y) \le C_n d^H_\Omega(x,y)$, where d^H_Ω is the Hilbert distance. q.e.d.

3.4. Wang's developing map. For a two-dimensional hyperbolic affine sphere $\mathcal{H}$, one may consider the conformal structure with respect to the Blaschke metric to give $\mathcal{H}$ the structure of a simply-connected open Riemann surface. As the geometry is derived from the elliptic Monge-Ampère equation (11), it should not be surprising that holomorphic data on the Riemann surface comes into play. In particular, with respect to a local conformal coordinate z, the cubic form, upon lowering an index with the metric, is of the form $C = U\,dz^3 + \bar{U} d\bar{z}^3$, for $U\,dz^3$ a holomorphic cubic differential.

C.P. Wang worked out the developing map for hyperbolic affine spheres in $\mathbb{R}^3$ [**71**]. (Much earlier, Ţiţeica analyzed a slightly different case of non-convex proper affine spheres [**68**, **69**].) Below, we present a synopsis of Wang's theory, as presented in [**50**].

Let $\mathcal{D}$ be a simply-connected domain in $\mathbb{C}$, and let $f : \mathcal{D} \to \mathbb{R}^3$ represent an immersed surface so that f is conformal with respect to the Blaschke metric. Let z be a conformal coordinate on $\mathcal{D}$. Let $\mathcal{H} = f(\mathcal{D})$ be the immersed surface. Then $f_z, f_{\bar z}$ span the complexified tangent space $T_f\mathcal{H} \otimes_{\mathbb{R}} \mathbb{C}$, and thus $\{f, f_z, f_{\bar z}\}$ is a frame of $\mathbb{R}^3 \otimes_{\mathbb{R}} \mathbb{C}$ at each point of $\mathcal{H}$.

Consider $f, f_z, f_{\bar z}$ as column vectors, and form the frame matrix

$$F = (f, f_z, f_{\bar z}).$$

Let $e^\psi |dz|^2$ and U be a conformal metric and cubic differential respectively on $\mathcal{D}$. Then the structure equation (9) is equivalent to the following first-order system

$$F_z = F \begin{pmatrix} 0 & 0 & \frac12 e^\psi \\ 1 & \psi_z & 0 \\ 0 & Ue^{-\psi} & 0 \end{pmatrix}, \qquad F_{\bar z} = F \begin{pmatrix} 0 & \frac12 e^\psi & 0 \\ 0 & 0 & \bar U e^{-\psi} \\ 1 & 0 & \psi_{\bar z} \end{pmatrix}. \tag{17}$$

This system of equations is integrable if and only if the following two conditions hold:

$$\psi_{z\bar z} + |U|^2 e^{-2\psi} - \tfrac12 e^\psi = 0, \tag{18}$$

$$U_{\bar z} = 0. \tag{19}$$

In this case, if at a point $z_0 \in \mathcal{D}$ initial conditions

$$f(z_0) \in \mathbb{R}^3, \qquad f_{\bar z}(z_0) = \overline{f_z(z_0)}, \qquad \det F(z_0) = \tfrac12 i e^{\psi(z_0)} \tag{20}$$

are satisfied, then the frame F can be uniquely defined on all of $\mathcal{D}$ and f is an immersion of a hyperbolic affine sphere in $\mathbb{R}^3$ with Blaschke metric e^ψ and cubic form (with index lowered by the metric) $U\,dz^3 + \bar U d\bar z^3$. Note the integrability conditions can be thought of as the flatness condition for the connection D in (9). In particular, we consider D as a connection on the vector bundle $E = 1 \oplus T_{\mathcal{D}}$, where 1 represents the trivial line bundle and $T_{\mathcal{D}}$ is the tangent bundle, with Christoffel symbols with respect to the frame F given by (17). The holonomy of the corresponding $\mathbb{RP}^2$ structure around a loop on a quotient surface is given by the inverse of the parallel transport of D around the same loop. See e.g. [**49**].

The map $f : \mathcal{D} \to \mathbb{R}^3$ is the *developing map* for the affine sphere, and $[f]$ is a developing map for the corresponding $\mathbb{RP}^2$ structure. We will use more details of this developing map below: Given appropriate initial conditions as above, the equations (17) become a linear system of ODEs along any path from z_0 in $\mathcal{D}$ (this is again related to parallel transport of the flat connection D on E; see e.g. [**49**]). The flatness of the connection D shows that the solution to the ODE initial value problem is independent of paths in a given homotopy class.

The initial value problem is particularly useful in the case the Blaschke metric $e^\psi|dz|^2$ is complete, as then Theorem 3.1.2 above shows the developed image $f(\mathcal{D})$ is a hyperbolic affine sphere asymptotic to the boundary of a convex cone $\mathcal{C} \subset \mathbb{R}^3$ and diffeomorphic to a properly convex domain $\Omega \subset \mathbb{RP}^2$.

We also consider a Riemann surface Σ with universal cover $\mathcal{D}$. Assume Σ is a Riemann surface equipped with a holomorphic cubic differential U and background conformal metric g. Then if $e^\psi|dz|^2 = e^u g$, the elliptic equation (18) becomes

$$\Delta u + \tfrac{1}{2}e^{-2u}\|U\|^2 - 2e^u - 2\kappa = 0, \tag{21}$$

where Δ is the Laplacian, $\|\cdot\|$ is the induced norm on cubic differentials, and κ is the Gauss curvature, all with respect to g.

Remark. Equation (21) differs by a normalization factor from the corresponding equations in [**46**, **50**]. The discrepancy comes from the fact that for the metric $|dz|^2$, the norm-squared of dz is 2, not 1, as was incorrectly assumed earlier. This change does not affect the results in [**46**, **50**], as the change in normalization simply amounts to scaling the cubic differential in solving (21). I would like to thank Ian McIntosh for pointing this out to me.

The following theorem is the basic correspondence between convex $\mathbb{RP}^2$ structures on a surface and solutions u to (21) on a Riemann surface Σ equipped with a cubic differential U. As the Blaschke metric and cubic form on the affine sphere $\mathcal{H}$ are invariant under the action of $\mathbf{SL}(3,\mathbb{R})$, so too are the function u and cubic differential U.

Theorem 3.4.1. *Let Σ be a Riemann surface equipped with a holomorphic cubic differential U and a conformal background metric g. Let u be a solution to (21) so that $e^u g$ is complete. For a basepoint $z_0 \in \tilde\Sigma$ and an initial frame $F(z_0)$ satisfying (20), we have a complete hyperbolic affine sphere $f(\tilde\Sigma)$ asymptotic to the boundary of a properly convex cone in $\mathbb{R}^3$. Different choices of z_0 and $F(z_0)$ lead to moving $f(\tilde\Sigma)$ by a motion of an element of $\mathbf{SL}(3,\mathbb{R})$ acting on $\mathbb{R}^3$.*

Upon projection to $\mathbb{RP}^2$, the universal cover $\tilde\Sigma$ is identified with a convex domain Ω. Holomorphic deck transformations of $\tilde\Sigma$ correspond to orientation-preserving projective automorphisms of Ω. In this way, the triple $(\Sigma, U, e^u g)$ corresponds to a convex $\mathbb{RP}^2$ structure on Σ.

Remark. In the case of Σ closed, this theorem is due independently to Labourie and the author [**37**, **38**, **46**]. See also [**71**]. In this case, existence of solutions to (21) is also proved and uniqueness is a straightforward application of the maximum principle, and thus a properly convex

$\mathbb{RP}^2$ structure on a closed surface S of genus at least two is equivalent to a pair (Σ, U) of a conformal structure and holomorphic cubic differential.

3.5. Ţiţeica's example. Consider the first octant in $\mathbb{R}^3$ as a convex cone. The hyperbolic affine sphere associated to this cone was discovered by Ţiţeica [**69**] (and generalized to any dimension by Calabi [**11**]). As we will use this example below, we summarize the basics of its construction. See e.g. [**50**] or [**22**] for justification.

The hyperbolic affine sphere $\mathcal{H}$ is equal to the set $\{(x^1, x^2, x^3) : x^i > 0, x^1x^2x^3 = 3^{-\frac{3}{2}}\}$. With respect to the induced Blaschke metric, $\mathcal{H}$ is conformally equivalent to $\mathbb{C}$. If z is a complex coordinate on $\mathbb{C}$, the Blaschke metric is given by $h = 2\,|dz|^2$ and the cubic differential is $U = 2\,dz^3$. If $z = \sigma + i\tau$, an embedding f of $\mathcal{H}$ is given by

$$f(z) = \tfrac{1}{\sqrt{3}}\left(e^{2\sigma}, e^{-\sigma+\sqrt{3}\tau}, e^{-\sigma-\sqrt{3}\tau}\right).$$

4. Regular cubic differentials to regular convex $\mathbb{RP}^2$ structures

4.1. The regular limits. We recall our earlier work in [**50**]. For every pair (Σ, U) for Σ a noded Riemann surface and U a regular cubic differential, a regular convex $\mathbb{RP}^2$ structure is constructed on $\Sigma^{\rm reg}$. In particular, if we view $\Sigma^{\rm reg}$ as a punctured Riemann surface, then at each puncture, the residue of the cubic differential determines the local $\mathbb{RP}^2$ geometry of the end. In particular, we have the following

Theorem 4.1.1. *Let $\Sigma = \bar\Sigma - \{p_1, \dots, p_n\}$ be a Riemann surface of finite hyperbolic type, and let U be a cubic differential on Σ with poles of order at most 3 and residue R_i at each puncture p_i. Then there is a background metric g on Σ and a solution u to* (21) *so that $e^u g$ is complete. Then for the corresponding convex $\mathbb{RP}^2$ structure on Σ provided by Theorem* 3.4.1*, the $\mathbb{RP}^2$ holonomy and developing map of each end is determined in the following way:*

For a residue $R \in \mathbb{C}$, consider the roots $\lambda_1, \lambda_2, \lambda_3$ of the cubic equation

$$\lambda^3 - 3(2^{-\frac{2}{3}})|R|^{\frac{2}{3}}\lambda - \operatorname{Im} R = 0.$$

Then the eigenvalues of the holonomy of the $\mathbb{RP}^2$ structure along an oriented loop around p_i are given by $\alpha^i = \exp(2\pi\lambda_i)$. When there are repeated eigenvalues, the Jordan blocks are all maximal. In the cases where the eigenvalues are distinct (hyperbolic holonomy), the bulge is $\pm\infty$, with the sign coinciding with the sign of $\operatorname{Re} R$.

More specifically, there are four cases. First of all if $R = 0$, then all $\alpha_i = 1$ and the $\mathbb{RP}^2$ holonomy is parabolic, conjugate to

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $\operatorname{Re} R = 0$ but $R \neq 0$, then there are two positive repeated eigenvalues $\alpha_1 = \alpha_2$, and the holonomy is quasi-hyperbolic, conjugate to

$$\begin{pmatrix} \alpha_1 & 1 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \qquad \alpha_1^2\alpha_3 = 1.$$

If $\operatorname{Re} R \neq 0$, then the eigenvalues $\alpha_1, \alpha_2, \alpha_3$ are positive and distinct, and so the holonomy is hyperbolic. The holonomy matrix is conjugate to

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \qquad \alpha_1\alpha_2\alpha_3 = 1.$$

The bulge is $\pm\infty$, with the same sign as $\operatorname{Re} R$.

Remark. In [**50**], the bulge parameter is called the vertical twist parameter.

Remark. This geometric correspondence can also be approached by Simpson's theory of Higgs bundles with regular singularities near punctures [**65**]. In particular, Labourie [**38**] proves that a cubic differential and corresponding Blaschke metric on a Riemann surface can be used to construct a Higgs bundle of a type considered by Hitchin [**32**]. See also Baraglia [**3**]. In the case of cubic differentials of poles of order 3, these Higgs bundles are regular near the pole locus in the sense of Simpson.

Remark. On a noded Riemann surface equipped with a regular cubic differential, the $\mathbb{RP}^2$ structures of the ends pair up to form regular separated necks. Consider each node as two punctures glued together. Then a regular cubic differential near the node has residues around the punctures which sum to zero. Then we may apply the dictionary in Theorem 4.1.1 to show the $\mathbb{RP}^2$ structures near each puncture satisfy the conditions for a regular separated neck as in Subsection 2.6 above. In particular, if the residue changes $R \mapsto -R$, then the eigenvalues of the holonomy change by $\{\alpha_i\} \mapsto \{\alpha_i^{-1}\}$. In the cases above, if $R = 0$ at one puncture, the other puncture also has residue 0, and both holonomies are parabolic. If $\operatorname{Re} R = 0$ but $R \neq 0$, then both holonomies are quasi-hyperbolic with inverse holonomy type. And finally if $\operatorname{Re} R \neq 0$, then the holonomies are hyperbolic and inverses of each other, and the bulge $\pm\infty \mapsto \mp\infty$ under $R \mapsto -R$.

In the context of the present work, we may summarize the main results of [**50**] in

Theorem 4.1.2. *Let (Σ, U) be a pair of a compact noded Riemann surface Σ and a cubic differential U. Then there is a corresponding regular $\mathbb{RP}^2$ structure on Σ^{reg} the type of whose regular separated necks is determined by the residue of U at each node. This defines an injective map $\Phi\colon \mathcal{V}_g \to \mathcal{R}_S^{\mathrm{aug}}$ which takes (Σ, U) to the corresponding regular $\mathbb{RP}^2$ structure. Moreover, the local invariants of the regular separated necks (the holonomy and bulge parameters of $\pm\infty$) depend continuously on (Σ, U) with the topology described above, as long as the residues of U are not 0.*

Remark. In the case one of the residues of U is 0, we also prove in [**50**] that the local invariants of the regular separated necks converge along many paths in the total space $\mathcal{V}_g$ of all (Σ, U).

Our present work improves this result in many ways: A natural topology is described on the space of regular convex $\mathbb{RP}^2$ structures, for which Φ is shown to be a homeomorphism. In the case of residue 0, this is due to Benoist-Hulin [**6**], and recently the surjectivity of Φ is due to Nie [**58**]. We also construct Φ^{-1} to prove Φ is onto. Under the topology on the space of convex $\mathbb{RP}^2$ structures, the local invariants addressed in [**50**] also vary continuously, and so we have a better understanding of the geometry of the regular convex $\mathbb{RP}^2$ structures. We can also remove the technical hypothesis on paths needed in [**50**]. The continuity of Φ and Φ^{-1} is new.

4.2. Uniform estimates. One of the main steps to construct the $\mathbb{RP}^2$ structures in [**50**] is to find sub- and super-solutions to (21) which are quite precise near the punctures. These sub- and super-solutions work well in most degenerating families of (Σ, U), except for those in which the residue at a node is 0. In this paper, we take a different tack: We find sub- and super-solutions which are not particularly precise but which have the virtue of being uniformly bounded in the universal curve away from singularities. This allows us to take limits of families of solutions without the restrictions above. Then we use a uniqueness theorem of Dumas-Wolf [**22**] to show that the limiting Blaschke metric is the one constructed in [**50**]. We record Dumas-Wolf's result here:

Theorem 4.2.1. *Let Σ be a Riemann surface which may or may not be compact, and let U be a holomorphic cubic differential on Σ. Let g be a conformal background metric on Σ. Then there is at most one solution u to (*21*) so that $e^u g$ is complete.*

We remark the proof of this theorem closely follows Wan [**70**], who studies similar equations for quadratic differentials. The theorems in [**22**, **70**] are phrased in terms of differential equations on domains in $\mathbb{C}$. The theorem as we state it here follows from passing to the universal cover of Σ.

Recall the finite cover of $\overline{\mathcal{M}}_g$ by $\{V^\alpha\}$ from Subsection 2.9 above, and consider the universal curve $\pi\colon \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$. Let $K^\alpha = \pi^{-1}V^\alpha$, and let $K^{\alpha,\mathrm{reg}}$ denote K^α with the nodes removed. Recall each V^α consists of an (s,t) neighborhood of a noded Riemann surface Σ, where the s parameters represent Beltrami differentials $\nu(s)$ which are supported away from the nodes and which preserve hyperbolic cusp coordinates, and the t parameters open the nodes by taking $\{zw=0\}$ to $\{zw=t\}$.

Theorem 4.2.2. *Let $(\Sigma_j, U_j) \to (\Sigma_\infty, U_\infty)$ in the total space of regular cubic differentials. Then the corresponding Blaschke metrics h_j converge in C^∞ in the following sense: We may assume the elements of the sequence all lie in one $V^\alpha \subset \overline{\mathcal{M}}_g$. Then the Blaschke metrics h_j converge in the same manner that the cubic differentials U_i do:*

In particular, there is a fixed noded Σ so that $\Sigma_j = \Sigma_{s_j,t_j}$, for $j \in \{1,2,\dots,\infty\}$, with respect to the hyperbolic-metric plumbing coordinates. On the thick part of each Riemann surface, the Blaschke metrics converge upon being pulled back by the quasi-conformal diffeomorphisms induced by the Beltrami coefficients so that $[(\chi^{s_j})^{-1}]^* h_i \to [(\chi^{s_\infty})^{-1}]^* h_\infty$ in C^∞. On the cusp and collar neighborhoods, the Blaschke metrics converge in C^∞_{loc} with respect to the cusp coordinates z and w.

Proof. We use the method of sub- and super-solutions. Consider the hyperbolic metric k_j on Σ_j as a background. In this case, the equation for the conformal factor (21) becomes

$$L_j(u) \equiv \Delta_j u + \tfrac{1}{2} e^{-2u} \|U_j\|_j^2 - 2e^u + 2 = 0. \tag{22}$$

Note that $L_j(0) \ge 0$ always, and we use the hyperbolic metric as a sub-solution for our equations. In order to find a family of super-solutions S_j, we need to ensure

$$S_j \ge 0, \qquad L_j(S_j) \le 0.$$

Then we will always be able to find a solution u_j satisfying $0 \le u_j \le S_j$ everywhere. Note the method of sub- and super-solutions works on non-compact Riemann surfaces (see e.g. [**70**]), and it is not necessary to have an L^∞ bound on the difference $S_j - 0$ of the super- and sub-solutions.

To construct a family of super-solutions, recall that with respect to the metrics m_j defined by (6) and (7) on Σ_j^{reg}, the convergence of (Σ_j, U_j) implies there is a uniform constant C so that $\|U_j\|_{m_j} \le C$. Write $m_j = e^{\phi_j} k_j$. Then for a constant B,

$$L_j(\phi_j + B) = e^{\phi_j}\left(\tfrac{1}{2}\|U_j\|_{m_j}^2 e^{-2B} - 2e^B - 2\kappa_{m_j}\right).$$

Moreover, (4) implies the grafting metric g_j and the hyperbolic metric k_j are uniformly comparable. By the construction of m_j above in (6) and (7), ϕ_j is uniformly bounded on the region of interpolation, $\frac{m_j}{g_j} \ge 1$ where m_j is flat, and κ_{m_j} is uniformly bounded. In particular, ϕ_j has a uniform lower bound, and so for B large enough, $L_j(\phi_j + B) < 0$ and $\phi_j + B \geq 0$. Therefore, $S_j = \phi_j + B$ is a super-solution.

S_j is a smooth function on each Σ_j^{reg}. Note that the S_j can be chosen to vary continuously as Σ_j changes for t_j small (within our coordinate neighborhood $V^\alpha \subset \overline{\mathcal{M}}_g$), but within each N^{t_i} neighborhood, S_j varies discontinuously for values of $|t_j|$ large enough. However, there are still uniform bounds. This follows since for $|t_j|$ bounded away from zero, the g_j metrics on the hyperbolic collars are uniformly equivalent (depending on the bound on $|t_j|$) to the flat metrics we glue in to form m_j. In other words, for $|t_j| > P > 0$, there is a uniform positive constant $C = C(P)$ so that $Cg_j \leq m_j \leq C^{-1}g_j$.

With the sub-solution 0 and super-solution S_j in place, there is a solution u_j to (22) satisfying $0 \leq u_j \leq S_j$. This implies the Blaschke metric $h_j = e^{u_j}k_j$ is complete, since the hyperbolic metric k_j is complete.

Now consider the sequence of Blaschke metrics $h_j = e^{u_j}k_j$ on Σ_j^{reg}. It is a theorem of Bers that the hyperbolic metrics k_j vary smoothly on compact subsets of $\overline{\mathcal{C}}_g^{\mathrm{reg}} \supset K^{\alpha,\mathrm{reg}}$ [**9**] (this also follows from e.g. Wolf-Wolpert [**73**, p. 1090]). The uniform local bounds on S_j on $K^{\alpha,\mathrm{reg}}$, together with interior elliptic estimates, imply that the u_j have locally uniform $C^{2,\beta}$ estimates on each $\Sigma_j^{\mathrm{reg}} \subset K^{\alpha,\mathrm{reg}}$. (See e.g. [**50**] for the elliptic regularity argument.) This implies by Ascoli-Arzela that there is a limit (up to a subsequence) in C^2_{loc}: $u_{j_k} \to w$ on $\Sigma_\infty^{\mathrm{reg}}$. Since each $u_{j_k} \geq 0$, $w \geq 0$ as well, and $e^w k_\infty$ is complete. Since the convergence is C^2_{loc}, w satisfies (22) and so $e^w k_\infty$ is a complete Blaschke metric on Σ_∞. But Dumas-Wolf's uniqueness Theorem 4.2.1 above then shows that $w = u_\infty$. Moreover, the same argument shows that every subsequence of u_j has a subsequence which converges to u_∞ in C^2_{loc}. This shows $u_j \to u_\infty$ in C^2_{loc} (and in C^∞_{loc} by elliptic regularity). q.e.d.

Since 0 is a sub-solution to (22), we have the following

Proposition 4.2.3. *Let Σ be a Riemann surface equipped with a complete conformal hyperbolic metric k. Let U be a cubic differential over Σ, and let $h = e^u k$ be a complete Blaschke metric for which u satisfies (22). Then $h \geq k$ on Σ.*

Proof. This follows from the proof of Theorem 4.2.2 for the pairs (Σ, U) we consider in this paper, in which Σ can be conformally compactified by adding a finite number of points at which U has poles of order at most 3. More generally, we can modify the proof of Theorem 6 in Wan [**70**], and use a global pointwise bound on the norm of U with respect to the Blaschke metric due to Calabi [**11**]. Since we do not need the general result presently, we leave the details to the reader. q.e.d.

4.3. Convergence of the holonomy. In the this subsection and the next we prove

Theorem 4.3.1. *The map* $\Phi\colon \mathcal{V}_g \to \mathcal{R}_S^{\rm aug}$ *defined in Theorem* 4.1.2 *is continuous.*

Proof. This theorem follows from Theorems 4.3.2 and 4.4.1 below. q.e.d.

On a pair of noded Riemann surface and regular cubic differential (Σ_j, U_j), the holonomy is determined by a tuple of representations $\mathrm{hol}_{j,k}\colon \pi_1(\Sigma_{j,k}, p_{j,k}) \to \mathbf{SL}(3,\mathbb{R})$, where the $\Sigma_{j,k}$ are the connected components of $\Sigma_j^{\rm reg}$. We consider in the next theorem the case of limits $(\Sigma_j, U_j) \to (\Sigma_\infty, U_\infty)$. In the case in which one or more nodes form as $\Sigma_j \to \Sigma_\infty$, we will need to consider as many sequence of basepoints $p_{j,k} \to p_{\infty,k}$, as there are connected component of $\Sigma_\infty^{\rm reg}$. This may be strictly larger than the number of connected components of $\Sigma_j^{\rm reg}$, and is in accordance with the topology on augmented Goldman space defined in Subsection 2.7 above.

Theorem 4.3.2. *Let* (Σ_j, U_j) *be a sequence of pairs of (possibly) noded Riemann surfaces and regular cubic differentials which is convergent to* $(\Sigma_\infty, U_\infty)$ *with respect to the topology on* $\mathcal{V}_g$ *defined in Subsection* 2.9. *Consider a family of parametrized smooth loops* $\mathcal{L}_j$ *based at points* p_j *in* $\Sigma_j^{\rm reg}$ *which converge uniformly to* $(\mathcal{L}_\infty, p_\infty)$ *on* $\Sigma_\infty^{\rm reg}$. *Given a choice of a continuous family of initial frames at the basepoints, consider the* $\mathbb{RP}^2$ *holonomy* $\mathrm{hol}_j \in \mathbf{SL}(3,\mathbb{R})$. *Then* $\mathrm{hol}_j \to \mathrm{hol}_\infty$.

Proof. The idea of the proof is that since all the loops avoid the nodes, there is an $\epsilon > 0$ so that we remain in the thick part Thick_ϵ of the universal curve $\overline{\mathcal{C}}_g$, where we have uniform estimates and thus convergence of the cubic differentials U_j, the conformal factors u_j and the hyperbolic metrics. The convergence of the holonomy will follow from the theory of linear ODEs with parameters.

We may assume all Σ_j for j large enough are represented in a single neighborhood $V^\alpha \subset \overline{\mathcal{M}}_g$. Then for each Beltrami differential ν in the definition of the hyperbolic metric plumbing coordinates, consider the diffeomorphism $\tilde\chi^\nu$ of $\mathcal{D}$ which fixes three points (such as $1, i, -1$) on the boundary $\partial\mathcal{D}$ and is a lift of the quasi-conformal diffeomorphism χ^ν of the connected component of $\Sigma_j^{\rm reg}$ containing p_j, as in Subsection 2.9 above. In this way, we can choose lifts $\tilde p_j \in \mathcal{D}$ of p_j so that $\tilde p_j \to \tilde p_\infty$. Also consider lifts $\tilde{\mathcal{L}}_j$ based at $\tilde p_j$ of the loop $\mathcal{L}_j$. Let ι_j denote the deck transformation corresponding to $\mathcal{L}_j$. Then $\tilde{\mathcal{L}}_j$ has endpoints $\tilde p_j$ and $\iota_j \tilde p_j$.

Let $\hat{\mathcal{L}}_j : [0,1] \to \mathcal{D}$ be the hyperbolic-geodesic constant-speed path from $\tilde p_j$ to $\iota_j \tilde p_j$. $\hat{\mathcal{L}}_j$ and $\tilde{\mathcal{L}}_j$ are homotopic, but $\hat{\mathcal{L}}_j$ enjoys better convergence properties: $\hat{\mathcal{L}}_j \to \hat{\mathcal{L}}_\infty$ in C^∞, while $\tilde{\mathcal{L}}_j \to \tilde{\mathcal{L}}_\infty$ only uniformly. Upon projecting to $\Sigma_j^{\rm reg}$, the image of $\hat{\mathcal{L}}_j$ is a hyperbolic geodesic typically with a corner at the basepoint p_j.

On a Riemann surface Σ with local coordinate z, consider any smooth path $z(t)$. Then (17) becomes a linear system of ODEs via $\frac{dF}{dt} = F_z\frac{dz}{dt} + F_{\bar z}\frac{d\bar z}{dt}$ for the frame $F = F^z = (f, f_z, f_{\bar z})$. Moreover, (17) gives the Christoffel symbols for a connection on the rank-3 vector bundle $E = (1 \oplus T\Sigma) \otimes \mathbb{C}$ with frame F. Around each loop, the inverse of the parallel transport map of this connection is the $\mathbb{RP}^2$ holonomy. See [**28**, **49**] for details.

To compute the $\mathbb{RP}^2$ holonomy along $\mathcal{L}_j$, first lift to the loop $\tilde{\mathcal{L}}_j$ as above and consider instead the hyperbolic geodesic path $\hat{\mathcal{L}}_j$. We first solve an initial-value problem for (17) along $\hat{\mathcal{L}}_j$, to find the frame F^z evolves to be $F^z M_j$ along the path. In other words, M_j is the solution to an initial value problem for the linear system of ODEs along $\hat{\mathcal{L}}_j$. But upon projecting $\mathcal{D} \to \Sigma_j$, the pullback of the frame at $p_j \in \Sigma$ to $\iota_j(\tilde p_j)$ is then $(f, f_w, f_{\bar w})$ for $w = \iota_j(z)$. Then this computes the $\mathbb{RP}^2$ holonomy to be

$$D(1, \frac{d\iota_j}{dz}, \frac{d\bar{\iota_j}}{d\bar z})|_{\tilde p_j} \cdot M_j.$$

Since each loop $\mathcal{L}_j$ avoids the nodes in Σ_j shows that there is an $\epsilon > 0$ so that $\mathcal{L}_j$ lies in Thick_ϵ inside the noded Riemann surface Σ_j. Since $(\Sigma_j, U_j) \to (\Sigma_\infty, U_\infty)$, we have $U_j \to U_\infty$ uniformly on the paths $\hat{\mathcal{L}}_j \to \hat{\mathcal{L}}_\infty$. Theorem 4.2.2 then shows the conformal factors u_j for each Blaschke metric, and their first derivatives, also converge uniformly as $\hat{\mathcal{L}}_j \to \hat{\mathcal{L}}_\infty$. Recall the Blaschke metric $h_j = e^{u_j} k_j$ for k_j the hyperbolic metric. On $\mathcal{D}$, the union of the $\hat{\mathcal{L}}_j$ for $j = 1, 2, \dots, \infty$ is compact, and so there is a $\delta > 0$ so that they all lie in $\{z : |z| < 1 - \delta\} \subset \mathcal{D}$. This shows the hyperbolic metric and $|dz|^2$ are uniformly bounded in terms of each other on the $\hat{\mathcal{L}}_j$, and so for $h_j = e^{\psi_j}|dz|^2$, ψ_j and its first derivatives converge uniformly as $j \to \infty$. As these cover all the terms in the coefficients in (17), the theory of linear systems of ODEs with parameters shows that the solutions M_j to the appropriate initial-value problems converge as $j \to \infty$.

So it remains to analyze how $D(1, \frac{d\iota_j}{dz}, \frac{d\bar{\iota_j}}{d\bar z})|_{\tilde p_j}$ varies as $j \to \infty$. Each ι_j is a deck transformation in the Fuchsian group whose quotient is Σ_j^{reg}. These Fuchsian groups, when normalized at $\partial\mathcal{D}$ as above, vary continuously on compact subsets of $\mathcal{D}$ as the hyperbolic plumbing coordinates (s, t) vary, and so the change-of-frame factor $D(1, \frac{d\iota_j}{dz}, \frac{d\bar{\iota_j}}{d\bar z})|_{\tilde p_j}$ converges as $j \to \infty$. q.e.d.

4.4. Convergence of the developing map. The convergence of the developing map as $(\Sigma_j, U_j) \to (\Sigma_\infty, U_\infty)$ is a little trickier, as it is necessary to consider paths that go to the boundary of the universal cover, and thus the standard theory of linear systems of ODE's with parameters does not directly apply. Our proof is similar to Dumas-Wolf ([**22**], the proof of Theorem 8.1).

Theorem 4.4.1. *Let $(\Sigma_j, U_j) \to (\Sigma_\infty, U_\infty)$ be a convergent family of pairs of (possibly) noded Riemann surfaces and regular cubic differentials. Let $p_j \in \Sigma_j^{\rm reg}$ and let $p_j \to p_\infty \in \Sigma_\infty^{\rm reg}$. Consider the connected component of $\Sigma_j^{\rm reg}$ containing p_j, and take a universal cover of this component to be the unit disk $\mathcal{D}$, with a lift of p_j placed at 0. Let $\Omega_j \subset \mathbb{RP}^2$ be the convex domain determined by projecting the complete affine sphere determined by the initial value problem (17) with a fixed initial frame from $\mathbb{R}^3$ to $\mathbb{RP}^2$. Then $\Omega_j \to \Omega_\infty$ in the Hausdorff sense.*

Proof. The idea of the proof is to use the theory of linear ODE systems with parameters as in Theorem 4.3.2 above. But there is an important difference in that our paths do not necessarily remain in Thick_ϵ for any $\epsilon > 0$, and so using the hyperbolic metric as a background is inadequate. Instead, we graft in flat annuli into the thin parts of Σ_j, similarly to Figure 2 in Subsection 2.9 above. Assume Σ_∞ and Σ_j for j large are all represented in a coordinate chart V^α of $\overline{\mathcal{M}}_g$. We will define continuous conformal metrics n_j on $\Sigma_j^{\rm reg}$ which are related to the metrics $m^{\alpha,s,t}$ introduced above. In particular, for Σ_j represented by the coordinates (s,t) on V^α, define n_j to be the hyperbolic metric induced by the Beltrami coefficient $\nu(s)$ on $\Sigma_j \setminus \cup_i N_i^{t_i,j}$. On each cusp/collar neighborhood $N_i^{t_i,j} \cap \Sigma_j^{\rm reg}$, define n_j to be the flat cylindrical metric continuously extending the hyperbolic metric. The n_j metrics are uniformly equivalent to the the $m_j = m^{\alpha,s,t}$ metrics by the constructions in Subsection 2.9 above. From the point of view of the hyperbolic geometry on the thick parts of Σ_j, the singular locus of the n_j is represented by images of horocycles projected from the Poincaré disk. The n_j metrics are convenient in that every path between two endpoints in $\Sigma_j^{\rm reg}$ is homotopic to a unique n_j-geodesic path between the endpoints. Moreover, $(n_j, p_\infty) \to (n_\infty, p_\infty)$ in the sense of geometric limits of pointed Riemannian manifolds; in other words, the metrics n_j converge to n_∞ in any geodesic ball centered at p_∞.

We need to prove that for each $\epsilon > 0$, there is a J so that for all $j \geq J$, $\Omega_n \subset N_\epsilon(\Omega_\infty)$ and $\Omega_\infty \subset N_\epsilon(\Omega_j)$, where N_ϵ is an ϵ-neighborhood with the respect to the Fubini-Study metric on $\mathbb{RP}^2$.

For simplicity, assume that $p_\infty \in \Sigma_\infty^{\rm reg}$ is in the thick part of Σ_∞, in the interior of the region where n_∞ is hyperbolic. Then for j large, the same is true for the n_j, and neighborhoods of lifts $\tilde p_\infty$ and $\tilde p_j$ may be naturally identified with neighborhoods in the Poincaré disk $\mathcal{D}$ by the quasi-conformal diffeomorphisms χ^{s_j}. In particular, we assume that $0 \in \mathcal{D}$ represents the lift $\tilde p_\infty$.

Let $F_j(z)$ denote the frame for $z \in \mathcal{D}$ corresponding to (Σ_j, U_j) as above. Then the component $f_j(z)$ is the parametrization of the hyperbolic affine sphere in $\mathbb{R}^3$, and $[f_j(z)]$ is the projection to $\mathbb{RP}^2$. For $R > 0$, consider the closed n_j-geodesic disk $\overline{B_j(R)}$ in the universal cover of the connected component of $\Sigma_j^{\rm reg}$ centered at the origin. Since n_∞ is complete, we may choose R large enough so that $\Omega\subset N_{\epsilon/2}([f_\infty](\overline{B_\infty(R)}))$.

Now for $z\in\overline{B_j(R)}$, $F_j(z)$ can be determined by a linear system of ODE's given by integrating the frame along an n_j-geodesic from 0 to z. The n_j-exponential maps $\exp_{n_j}$ centered at the origin identify all the $\overline{B_j(R)}$ with $\overline{B(R)}$, the closed disk of radius R in the tangent space $T_0\mathcal{D}$. As above in the proof of Theorem 4.3.2, the coefficients of these ODE systems on the compact set $\overline{B(R)}$ converge uniformly as $j\to\infty$. Also, $n_j\to n_\infty$ in $\overline{B(R)}$. Therefore, the theory of linear ODE systems with parameters shows $F_j\circ\exp_{n_j}\to F_\infty\circ\exp_{n_\infty}$ uniformly on $\overline{B(R)}$. Denote $\tilde f_j=[f_j\circ\exp_{n_j}]$. So there is a J so that for $j\ge J$,

$$\tilde f_j(\overline{B(R)})\subset N_{\epsilon/2}(\tilde f_\infty(\overline{B(R)}))\subset N_{\epsilon/2}(\tilde f_\infty(T_0\mathcal{D}))=N_{\epsilon/2}(\Omega_j),$$

and thus $\Omega_\infty\subset N_\epsilon(\Omega_j)$.

To prove the opposite inclusion, we consider the dual hyperbolic affine sphere, which has the same metric $e^{u_j}k_j$ on Σ_j and the opposite cubic differential $-U_j$, by Theorem 3.1.4 above. Now we can lift the data to $T_0\mathcal{D}$ as above, and consider an appropriate initial frame to form the dual hyperbolic affine sphere and projective dual convex domain. Then the previous case implies there is a J so that if $j\ge J$, then

$$\Omega^*\subset N_\epsilon(\Omega_j^*).$$

But then Lemma 3.1.3 implies that

$$\Omega\supset(N_\epsilon(\Omega_j^*))^*\supset N_{\epsilon'}(\Omega_j),$$

where $\epsilon'\to0$ if and only if $\epsilon\to0$. This follows from the continuity under the Hausdorff distance of the projective duality of uniformly bounded convex domains which contain a fixed ball. This in turn implies (for convex domains) that there is an ϵ'' which approaches 0 if and only if ϵ does so that

$$\Omega_j\subset N_{\epsilon''}(\Omega_\infty)$$

for $j\ge J$. This is enough to prove the theorem. q.e.d.

Remark. As should be clear from the proofs, Theorems 4.3.2 and 4.4.1 also apply to the case of the vector bundle of regular cubic differentials over $\mathcal{Q}_c$, where c is the set of loops in S pinched to nodes in Σ_∞.

5. Regular convex $\mathbb{RP}^2$ structures to regular cubic differentials

5.1. The singular limit cases. In this subsection, we show the regular convex $\mathbb{RP}^2$ structures each correspond to a pair (Σ,U) of a noded Riemann surface Σ and regular cubic differential U on Σ. It suffices to consider each connected component of $\Sigma^{\rm reg}$ separately. Consider a connected oriented properly convex $\mathbb{RP}^2$ surface each of whose ends is regular. Then use the hyperbolic affine sphere to construct a Riemann surface of finite type and regular cubic differential so that the $\mathbb{RP}^2$ geometry of each end corresponds to the residue of the cubic differential as in Theorem 4.1.1 above. The results in this subsection are also recently due to Nie [**58**], using similar techniques. We include our version, as we find the material both short and instructive. We summarize these results in

Theorem 5.1.1. *Let S be a closed oriented surface of genus $g \ge 2$. Then the map $\Phi\colon \mathcal{V}_g \to \mathcal{R}_S^{\rm aug}$ defined in Theorem* 4.1.2 *is surjective.*

This theorem follows from Theorem 5.1.6 below and the definition of a regular separated neck in Subsection 2.6 above.

Consider a single end $\mathcal{E}$ of X. We proceed by considering the four cases of regular ends separately.

Theorem 5.1.2. [**6**] *Let $\mathcal{E}$ be an end of parabolic type. Then with respect to the Blaschke metric, $\mathcal{E}$ can be conformally compactified by adding a single point. The cubic differential U has at worst a pole of order 2 at this point, and so the residue is 0.*

This case is settled by Benoist-Hulin [**6**], who prove that the conformal structure at the end can be compactified by adding a single point, and that the corresponding cubic differential U has a pole of order at most 2. In our language, this corresponds to the residue's being 0. To be more specific, Benoist-Hulin consider convex $\mathbb{RP}^2$ surfaces with finite area with respect to the Hilbert metric, which Marquis [**53**] has proved are equivalent to having a finite number of ends each with parabolic holonomy. Thus [**6**] is concerned with convex $\mathbb{RP}^2$ surface *all* of whose ends are parabolic. But the techniques used to analyze each end are essentially local, and apply to each end separately, and indeed they prove that each such end has finite conformal type and has cubic differential with residue 0.

Proposition 5.1.3. *Let $\mathcal{E}$ be a regular end of quasi-hyperbolic type, or of hyperbolic type with bulge $\pm\infty$. Then there is a family of loops L_s around $\mathcal{E}$ which depend on a parameter $s \to 0^+$ so that*

- *L_s uniformly approaches the end as $s \to 0^+$. More precisely, represent $\mathcal{E}$ as homeomorphic to a closed half-cylinder $[0,\infty) \times S^1$. Then for every compact $K \subset \mathcal{E}$, there is an $\epsilon > 0$ so that if $s < \epsilon$, $L_s \cap K = \emptyset$.*
- *There is a family of elements $M_s \in \mathbf{SL}(3,\mathbb{R})$ so that $M_s\,\Omega \to T$ in the Hausdorff topology as $s \to 0^+$ and $M_s\,L_s$ lies in a compact subset of T for s small enough. Here T is a triangle in $\mathbb{RP}^2$.*

Proof. The proof is broken into 3 cases.

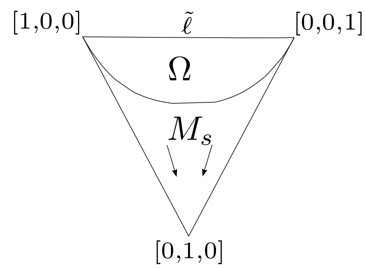

Figure 3

First, we consider the case in which $\mathcal{E}$ is of hyperbolic type with bulge $-\infty$. Choose a based loop $\mathcal{L}$ in X freely homotopic to a loop around $\mathcal{E}$, and coordinates on $\mathbb{RP}^2$ so that the $\mathbf{SL}(3,\mathbb{R})$ holonomy along a lift $\tilde{\mathcal{L}}$ of $\mathcal{L}$ is represented by $H = D(\lambda,\mu,\nu)$ so that $\lambda > \mu > \nu > 0$ and $\lambda\mu\nu = 1$. Let T denote the principal triangle given by the projection of the first octant in $\mathbb{R}^3$ to $\mathbb{RP}^2$. Note the vertices of T are the fixed points of H. Since the bulge is $-\infty$, Ω, the image of the developing map of X, is contained in T and the boundary of Ω contains the principal geodesic $\tilde{\ell} = \{[t,0,1-t] : 0 \le t \le 1\}$. For $p = [1,s,1]$ as $s \to 0^+$, consider the lift of a loop $L_s = \{H^t\,p : t \in [0,1)\}$. Let $M_s = D(s^{\frac{1}{3}}, s^{-\frac{2}{3}}, s^{\frac{1}{3}})$ so that M_s acts on the hyperbolic affine sphere $\mathcal{H}$ by sending p to $[1,1,1]$. For $t \in [0,1)$, we have $M_s\,H^t\,p = H^t\,[1,1,1]$. Thus the limit of of $M_s\,L_s$ lies in a bounded neighborhood of $[1,1,1]$ in T.

Since $\partial\Omega$ contains the principal geodesic $\tilde{\ell}$, $\Omega \subset T$, and Ω is convex, $M_s\,\Omega \to T$ in the Hausdorff topology as $s \to 0$. One can see this by noting $\tilde{\ell}$ is fixed by M_s, and all other points in $\bar{\Omega}$ approach $[0,1,0]$ as $s \to 0^+$. The interior of the convex hull of $\tilde{\ell}$ and $[0,1,0]$ is T. In fact, $M_s\,\Omega$ increases to T as $s \to 0^+$. See Figure 3.

The second case is of hyperbolic holonomy H with bulge $+\infty$. In this case, we choose coordinates so that the convex domain Ω contains T, $\tilde{\ell}$, and a proper nontrivial subset of $\bar{T}$. Consider the point $p = [s,1,s] \in T$ as $s \to 0^+$. Consider the map $M_s = D(s^{-\frac{1}{3}}, s^{\frac{2}{3}}, s^{-\frac{1}{3}})$, which sends p to $[1,1,1]$. As $s \to 0$, the action of M_s is essentially a bulge parameter approaching $-\infty$. Since $\Omega \cap \bar{T}$ is bounded away from $[0,1,0]$, we see that $M_s\,\Omega \to T$ as $s \to 0$.

Moreover, as $s \to 0$, $p \to [0,1,0] \in \partial\Omega$, and the points in the lift of the loop $L_s = \{H^t\,p : t \in [0,1)\}$ approach $H^t\,[0,1,0] = [0,1,0]$. Thus the family of loops do approach the end as $s \to 0$. Also, $M_sH^tp \to H^t\,[1,1,1]$. Since $t \in [0,1)$, $\lim_{s\to 0^+} M_s\,L_s$ lies in a compact subset T. See Figure 4.

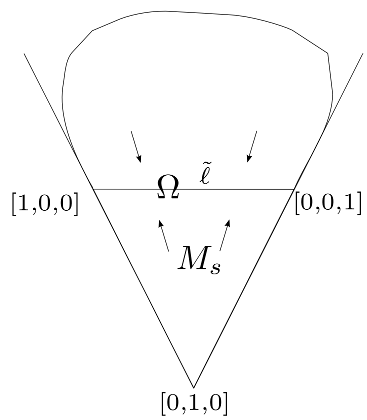

Figure 4

The remaining case is that of quasi-hyperbolic holonomy. It is a bit different, in that the dynamics do not involve a principal triangle. Nevertheless, we analyze this case in terms of T as well. We may assume the holonomy matrix $H = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}$, with λ, μ positive and $\lambda^2\mu = 1$. Assume without loss of generality that $\lambda > \mu$ (otherwise, we could analyze H^{-1} similarly). Then the fixed points of H are the attracting fixed point $[1, 0, 0]$ and the repelling fixed point $[0, 0, 1]$. Consider the geodesic $\tilde{\ell} = \{[t, 0, 1 - t] : t \in [0, 1]\}$. The proper convexity of Ω and a simple analysis of the dynamics of H imply that we can choose coordinates so that $\partial\Omega \cap \{[x, y, z] : y = 0\} = \tilde{\ell}$ and $\Omega \subset \{[x, y, z] : y, z > 0\}$.

As above, we consider $p = [1, s, 1] \in \Omega$ as $s \to 0^+$, and note $\lim_{s\to0} p = [1, 0, 1] \in \partial\Omega$. Then the map $M_s = D(s^{\frac{1}{3}}, s^{-\frac{2}{3}}, s^{\frac{1}{3}})$ takes p to $[1, 1, 1]$. Moreover, as $s \to 0^+$, $M_s\,\Omega \to T$ in the Hausdorff sense. This can be seen because $\tilde{\ell}$ is fixed under the action of M_s, and for every $q = [x, y, z] \in \mathbb{RP}^2$ with $y \neq 0$, the orbit $M_s\,q$ is a straight line approaching $[0, 1, 0]$ as $s \to 0$. This implies that any point $q \in \Omega$ which is ϵ-close to $\tilde{\ell}$ remains ϵ-close to T under the action of M_s as $s \to 0^+$. On the other hand, there is a $\sigma > 0$ so that if $0 < s < \sigma$, and $r \in \Omega$ is not ϵ-close to $\tilde{\ell}$, then $M_s\,r$ is ϵ-close to $[0, 1, 0]$. Thus all points of $M_s\,\Omega$ are within ϵ of T for s small enough.

Consider a family of loops $L_s = \{H^t\,p : t \in [0, 1)\}$ which uniformly approaches $\partial\Omega$ as $s \to 0^+$. Compute

$$M_s\,H^t\,p = [\lambda^t + st\lambda^{t-1}, \lambda^t, \mu^t] \to [\lambda^t, \lambda^t, \mu^t]$$

as $s \to 0$. This shows that the closure of $\lim_{s\to0} M_s\,L_s$ is compactly contained in T. See Figure 5.

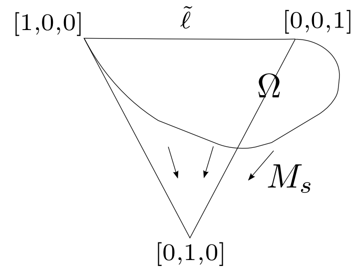

Figure 5

q.e.d.

Proposition 5.1.4. *Let X be a convex $\mathbb{RP}^2$ surface. Let $\mathcal{E}$ be a regular end of X of quasi-hyperbolic type, or of hyperbolic type with bulge $\pm\infty$. For the cubic differential U and Blaschke metric h, the norm squared $\|U\|_h^2$ approaches the constant 4 uniformly at the end $\mathcal{E}$. Moreover, the metric $|U|^{\frac{2}{3}}$ is a flat Riemannian metric which is complete and has bounded circumference on $\mathcal{E}$. The induced conformal structure of the end can be compactified by adding a single point.*

Proof. Under the previous proposition, we know that $M_s\,\Omega \to T$ in the Hausdorff topology. Then Theorem 3.2.2 applies to show the Blaschke metric and cubic tensor on $M_s\,\Omega$ converge to those on T in $C^\infty_{\rm loc}$. We know by Subsection 3.5 above that the norm squared $\|U\|_h^2 = 4$ on T, and so on each compact subset of T, the same quantities on $M_s\,\Omega$ satisfy $\lim_{s\to 0}\|U\|_h^2 = 4$. By our construction of M_s, such a compact set $K \subset T$ pulls back to $M_s^{-1}\,K$, which approaches a lift of the end $\mathcal{E}$ in Ω. This shows $\|U\|_h^2 \to 4$ uniformly approaching the end $\mathcal{E}$ on X.

This implies that there are no zeros of U in a neighborhood of $\mathcal{E}$. Also, the Blaschke metric h is complete by Theorem 3.1.2. Then $\|U\|_h^2 = 8|U|^2h^{-3} \to 4$, which shows that $|U|^{\frac{2}{3}}$ is complete at $\mathcal{E}$. Away from the zeros of U, $|U|^{\frac{2}{3}}$ is a flat metric, and thus $|U|^{\frac{2}{3}}$ is a flat metric on $\mathcal{E}$ which is complete at the end.

Since the loops L_s from the previous proposition converge to the end and the $M_s\,L_s$ lie in a compact subset of T for s near 0, the length with respect to the flat metric $|U|^{\frac{2}{3}}$ (or the Blaschke metric h) of L_s remains bounded as $s \to 0$. This shows the $|U|^{\frac{2}{3}}$-circumference of the $\mathcal{E}$ is bounded. Since the metric is flat and complete, the circumference must be constant. In fact, by considering the Euclidean developing map of $(\mathcal{E}, |U|^{\frac{2}{3}})$, we find it must be a flat half-cylinder. This flat half-cylinder can be conformally compactified by adding one point, as one can choose a conformal coordinate z so that

$$|U|^{\frac{2}{3}} = C\,|z|^{-2}\,|dz|^2 \tag{23}$$

for a constant C and $0 < |z| \le 1$. q.e.d.

Proposition 5.1.5. *Let $\mathcal{E}$ be a quasi-hyperbolic end, or hyperbolic end with bulge $\pm\infty$ on a convex $\mathbb{RP}^2$ surface. At the puncture on the Riemann surface induced by the Blaschke metric h, the cubic differential U has a pole of order exactly three.*

Proof. Dumas-Wolf show that the completeness of $|U|^{\frac{2}{3}}$ implies U cannot have an essential singularity at $z = 0$ [**22**, Lemma 7.6] (see also Osserman [**60**]). Moreover, the completeness of $|U|^{\frac{2}{3}}$ implies U has a pole of order at least 3. The finite circumference of a loop around the end with respect to $|U|^{\frac{2}{3}}$ then shows that the pole order of U is at most 3, by the choice of coordinates in (23) above. q.e.d.

Theorem 5.1.6. *Let X be a connected oriented properly convex $\mathbb{RP}^2$ surface of genus g and n ends, so that $2g + n > 2$. Assume the $\mathbb{RP}^2$ structure of each end is regular. Then the conformal structure Σ induced by the Blascke metric on X is of finite type, and the induced cubic differential U has poles of order at most 3 at each puncture of Σ. The residue of U at each puncture corresponds to the $\mathbb{RP}^2$ structure of the end as in Theorem* 4.1.1 *above.*

Proof. The case of parabolic holonomy was settled by Benoist-Hulin's Theorem 5.1.2 above.

Denote the conformal structure by Σ. For the other cases, we have shown that they each lead to a regular cubic differential U of pole order 3. We proved in [**50**] that given such a pair (Σ, U), we can construct from a background metric the complete Blaschke metric $\tilde{h}$, and also integrate the equations to determine an $\mathbb{RP}^2$ structure $\tilde{X}$ of the surface. On $\tilde{X}$, the residue of the corresponding cubic differential determines the holonomy and bulge parameters of the end as in Theorem 4.1.1. Theorem 4.2.1 shows $\tilde{h} = h$ and, as the $\mathbb{RP}^2$ structure is determined by (Σ, U, h), the ends of the $\mathbb{RP}^2$ structure X conform to Theorem 4.1.1. q.e.d.

5.2. Convergence in families. We introduce some terminology of convergence of pointed convex domains and Benzécri's Theorem. A pair (Ω, x) with Ω a properly convex domain in $\mathbb{RP}^n$ and $x \in \Omega$ is called a *pointed convex domain*. A sequence (Ω_j, x_j) of pointed convex domains *converges in the Hausdorff sense* to a pointed convex domain (Ω, x) if $\Omega_j \to \Omega$ with respect to the Hausdorff topology and $x_j \to x$. More generally $(\Omega_j, x_j) \to (\Omega, x)$ *in the Benzécri sense* if there is a sequence $\rho_j \in \mathbf{SL}(n+1, \mathbb{R})$ so that $\rho_j(\Omega_j, x_j) \to (\Omega, x)$ in the Hausdorff sense. Benzécri proved

Theorem 5.2.1. [**8**] *The space of pointed convex domains in $\mathbb{RP}^n$, modulo the action of $\mathbf{SL}(n+1, \mathbb{R})$, is compact and Hausdorff.*

The theorem we prove for the remainder of this subsection is

Theorem 5.2.2. *The map* $\Phi^{-1}\colon \mathcal{R}_S^{\rm aug} \to \mathcal{V}_g$ *is continuous.*

Outline of proof. In order to show convergence of (Σ_j, U_j) given the convergence of the corresponding regular convex $\mathbb{RP}^2$ structures, we first show that the sequence $\{(\Sigma_j, U_j)\}$ is precompact in $\mathcal{V}_g$ by using the compactness of $\overline{\mathcal{M}}_g$ and proving in Proposition 5.2.3 that the U_j are uniformly bounded in the space of regular cubic differentials over Σ_j. Thus we have a convergent subsequence $(\Sigma_{j_k}, U_{j_k}) \to (\Sigma_\infty, U_\infty)$ in $\mathcal{V}_g$. Then we prove that there is a unique limit of the corresponding regular convex $\mathbb{RP}^2$ structure and thus Theorem 4.3.1 above shows $\Phi(\Sigma_\infty, U_\infty)$ must be the original limit of the sequence of regular convex $\mathbb{RP}^2$ structures.

Many of the technical details below concern proving the uniqueness of the limit in Proposition 5.2.6. In particular, for simplicity consider on a surface S the case of convex $\mathbb{RP}^2$ structures $(\Omega_i, \Gamma_i) \to (\mathcal{O}_1, G_1) \oplus (\mathcal{O}_2, G_2)$ in $\mathcal{G}_S^{\rm aug}$ along a path in which a single loop ℓ is separated and $S - \ell = S_1 \sqcup S_2$. There are $\rho_i, \sigma_i \in \mathbf{SL}(3,\mathbb{R})$ so that $\rho_i\Omega_i \to \mathcal{O}_1$, $\sigma_i\Omega_i \to \mathcal{O}_2$, $\rho_i(\Gamma_i|_{S_1})\rho_i^{-1} \to G_1$, and $\sigma_i(\Gamma_i|_{S_2})\sigma_i^{-1} \to G_2$. We first treat the convergence of the domains by considering sequences of pointed domains $(\Omega_i, x_i^1) \to (\mathcal{O}_1, x_\infty^1)$ and $(\Omega_i, x_i^2) \to (\mathcal{O}_2, x_\infty^2)$ converging in the Benzécri sense. For $k = 1, 2$, consider the sequence $[x_j^k]$ in the Riemann surface induced by the Blaschke metric on $\Gamma_i\backslash\Omega_i$.

Then we show that there is an $\epsilon > 0$ so that each $[x_j^k]$ for large finite j lies in the same connected component C_j^k of $\mathrm{Thick}_\epsilon \subset \Sigma_j$ as corresponds to $\mathcal{O}_k$. Assume otherwise, and prove this statement by contradiction. First note that Proposition 4.2.3 shows that the Blaschke distance from C_j^k to $[x_j^k]$ must diverge to infinity. Then we use Proposition 3.3.2 and an analysis of the possible actions of Γ_j on Ω_j following Goldman [**27**] to show that the limit G_k of $\Gamma_j|_{S_k}$ cannot act on the limiting domain $\mathcal{O}_k$. This contradicts the definition of $(\mathcal{O}_1, G_1) \oplus (\mathcal{O}_2, G_2)$ representing a point in $\mathcal{G}_S^{\rm aug}$.

Inside the component C_j^k of Thick_ϵ, we have uniform bounds on the cubic differential U_j and uniform C^1 bounds on the conformal factors u_j. Thus the ODE techniques of Theorem 4.4.1 apply to show that $\mathcal{O}_k$ must be projectively equivalent to a corresponding component of the projective domains in $\Phi(\Sigma_\infty, U_\infty)$.

Finally, to show that the groups G_k also come from $\Phi(\Sigma_\infty, U_\infty)$, we use the fact that an element of $\mathbf{SL}(3,\mathbb{R})$ is determined by its action on 4 points in general position. We may take these 4 points to be in a neighborhood of an image $\tilde{x}_\infty^k$ of x_∞^k by an element of G_k and use the uniform bounds and the Implicit Function Theorem to derive the convergence of $(\Omega_j, \Gamma_j|_{S_k}) \to (\mathcal{O}_k, G_k)$ up to the action of $\mathbf{SL}(3,\mathbb{R})$. q.e.d.

Now we introduce the details of the proof of Theorem 5.2.2. $\mathcal{G}_S^{\rm aug}$ is a stratified space with strata $\mathcal{G}_S^c$ for $c \in C(S)$. The closure of each stratum

$$\overline{\mathcal{G}_S^c} = \bigsqcup_{d \supset c} \mathcal{G}_S^d.$$

The lowest strata $\mathcal{G}_S^c$, in which c splits $S - c$ into a disjoint collection of pairs of pants, are closed. Thus, by considering subsequences, we may assume the limit either remains within one stratum or the limit point is on a smaller stratum (by separating one or more necks). This means that in considering limits of sequences in $\mathcal{G}_S^{\rm aug}$, we may consider, by taking subsequences if necessary, only the case of elements of $\mathcal{G}_S^c$ approaching a limit in $\mathcal{G}_S^{d\cup c}$ for d and c fixed disjoint sets of free homotopy classes of simple loops satisfying $d \cup c \in C(S)$. In particular, we may focus precisely on separating the necks in d.

For the case of families, consider a sequence of regular convex $\mathbb{RP}^2$ structures converging in $\mathcal{R}_S^{\rm aug}$. The associated unmarked conformal structures must converge (subsequentially) in $\overline{\mathcal{M}}_g$, as it is compact. Our first task is to show that this subsequential convergence can be extended to the regular cubic differentials as well. For a convergent sequence in $\mathcal{R}_S^{\rm aug}$, we consider by Lemma 2.4.2 convergent sequences of the form

$$\oplus_k (\Omega_j, \Gamma_j|_{S_k}).$$

The induced conformal structures given by the Blaschke metric associates to each pair $(\Omega_j, \Gamma_j|_{S_k})$ a conformal structure of finite type. We then pairwise attach the ends of the Riemann surface by adding a node for each pair. Thus we have a sequence Σ_j of noded Riemann surfaces. Since $\overline{\mathcal{M}}_S$ is compact, there is a convergent subsequence Σ_{j_ℓ}. Note that this gluing to form noded Riemann surfaces is purely complex-analytic, and so generalized twist parameters are not necessary.

For the cubic differentials, we have by Theorem 3.2.2 that the norm-squared of the cubic tensor with respect to the Blaschke metric converges in $C^\infty_{\rm loc}$. We can also prove the following universal bound

Proposition 5.2.3. *Let S be an oriented surface of genus $g \geq 2$. Consider a convergent sequence in $\mathcal{R}_S^{\rm aug}$. Assume, by possibly taking a subsequence, that the associated conformal structures Σ_j converge to a limit Σ_∞ in $\overline{\mathcal{M}}_g$. Assume Σ_∞ is an element of the chart V^α as above in Subsection 2.9. In terms of the metrics m_j^α, there is a constant C so that the cubic differentials U_j satisfy $\|U_j\|_{m_j^\alpha} \leq C$ pointwise for all j large.*

Proof. First of all, the convergence in $\mathcal{R}_S^{\rm aug}$ implies by Lemma 2.4.2 that we may lift to a convergent sequence in $\mathcal{G}_S^{\rm aug}$. Let n be the number of connected components of the regular limit $\mathbb{RP}^2$ surface. Assume, by taking subsequences, that all the surfaces in the sequence lie in the same stratum of $\mathcal{G}_S^{\rm aug}$. In particular, assume that the surface S is the disjoint union of a set of loops $c \in C(S)$ and open subsurfaces $S_1, \dots, S_n$. Along each loop in c, there is a regular separated neck, and for $k = 1, \dots, n$, there are pairs (Ω_j^k, Γ_j^k) of properly convex domains and discrete subgroups of $\mathbf{SL}(3,\mathbb{R})$ acting on the domains so that the quotient $\Gamma_j^k \backslash \Omega_j^k$ is diffeomorphic to S_k. Moreover, the induced projective structure at each end of an S_k is regular and is paired appropriately with another end of an $S_{\tilde{k}}$ to form a regular separated neck.

In the limit as $j \to \infty$, we may have more necks being separated. Consider a set of homotopy classes of loops d so that d and c are disjoint and $d \cup c \in C(S)$. We will separate the necks across d. For simplicity, we only consider the case of a single loop in d which separates S_1 into two pieces $\tilde{S}_0$ and $\tilde{S}_1$. In this case, we have $\rho_j, \sigma_j \in \mathbf{SL}(3,\mathbb{R})$ so that $\rho_j(\Omega_j^1, \Gamma_j^1|_{\tilde{S}_0}) \to (\mathcal{O}, G)$ and $\sigma_j(\Omega_j^1, \Gamma_j^1|_{\tilde{S}_1}) \to (\mathcal{U}, H)$, so that $\tilde{S}_0$ and $\tilde{S}_1$ are diffeomorphic to $G\backslash\mathcal{O}$ and $H\backslash\mathcal{U}$ respectively.

In particular, $\rho_j\Omega_j^1 \to \mathcal{O}$ in the Hausdorff topology. Theorem 5.1.6 implies that $(\mathcal{O}, G)$ is topologically conjugate to a non-elementary finitely-generated Fuchsian group of the first kind. In particular, for $\mathcal{D}$ the Poincaré disk, there is a diffeomorphism $\phi : \mathcal{O} \to \mathcal{D}$, conformal with respect to the Blaschke metric on $\mathcal{O}$, so that $\phi \circ G \circ \phi^{-1}$ is a Fuchsian group, and the Riemann surface $(\phi \circ G \circ \phi^{-1})\backslash\mathcal{D}$ has finite hyperbolic area. Consider the Dirichlet domain, which a convex ideal polygonal fundamental region $\mathcal{P}$ for $\phi \circ G \circ \phi^{-1}$ with finitely many sides and for which each ideal vertex corresponds to an end of the quotient surface $\tilde{S}_0$; see e.g. [**4**]. Let $K \subset \mathcal{O}$ be a compact set large enough so that all of $\mathcal{P}$ outside neighborhoods of the ideal vertices is in the interior of $\phi(K)$. Theorem 3.2.2 implies the Blaschke metrics and cubic tensors of $\rho_j\Omega_j^1$ converge on K in C^∞ to those on $\mathcal{O}$.

Upon passing to the quotient surface $\tilde{S}_0$, the convergence on K descends to the quotient surface to show $C^\infty_{\rm loc}$ convergence of the Blaschke metrics and cubic tensors on $\tilde{S}_0$ outside the ends (which are topological annuli). The same sort of convergence is true on $\tilde{S}_1$ and all the other connected components of $S - (d \cup c)$. On all of S, then, there exist disjoint annular neighborhoods $\mathcal{A}_k$, one for each homotopy class of loops in $d \cup c$, so that the Blaschke metrics and cubic tensors converge in C^∞ on $S - \cup_k \mathcal{A}_k$.

By our assumption, the necks in d are precisely those which are conformally pinched as $\Sigma_j \to \Sigma_\infty$. So we may assume each $\mathcal{A}_k$ contains the thin part of each collar neighborhood in Σ_j. In other words, there is an $\epsilon > 0$ so that $\Sigma_j - \cup_k \mathcal{A}_k \subset {\rm Thick}_\epsilon$. The Blaschke metric, the hyperbolic metric, and the modified metrics m^α are thus all uniformly equivalent (depending on ϵ) on $\Sigma_j - \cup_k \mathcal{A}_k$. Therefore, the uniform convergence of the cubic tensors and Blaschke metrics on $\Sigma_j - \cup_k \mathcal{A}_k$ implies the uniform convergence of $\|U_j\|_{m^{\alpha,j}}$ on $\Sigma_\infty - \cup_k \mathcal{A}_k$. So for large enough j, there is a uniform bound on $\|U_j\|_{m^{\alpha,j}}$ when restricted to $\Sigma_j - \cup_k \mathcal{A}_k$.

The next lemma shows this uniform bound can be extended to a uniform bound of $\|U_j\|_{m^{\alpha,j}}$ on all of Σ_j^{reg}. q.e.d.

Lemma 5.2.4. *Let Σ be a noded Riemann surface represented by a point in $V^\alpha \subset \overline{\mathcal{M}}_g$ with metric m^α. Let U be a regular cubic differential on Σ. Let $\mathcal{A}_k$ be a collection of disjoint sets of the following forms: either 1) an annular subset of Σ or 2) a neighborhood of a node which is homeomorphic to $\{zw = 0 : |z|, |w| < 1\}$ with respect to the plumbing coordinates. Assume $\mathcal{A}_k$ contains a component of the locus where the m^α metric is flat. Then there is a constant C depending only on the genus so that for all $x \in \Sigma^{\mathrm{reg}}$,*

$$\|U(x)\|_{m^\alpha} \le C \sup\{\|U(z)\|_{m^\alpha} : z \in \Sigma^{\mathrm{reg}} - \cup_k \mathcal{A}_k\}.$$

Proof. See e.g. [**77**, **78**]. For simplicity, we consider the case of a single domain $\mathcal{A}$.

We consider two cases. First of all, let $\mathcal{A}$ be an annulus. If $\mathcal{A}$ is equal to $\mathcal{F} \equiv \{\ell : m^\alpha = (2\log c)^{-2}|d\ell|^2\}$, then the m^α metric is flat on $\mathcal{A}$. For the quasi-coordinate $\ell = \log z$, we have $m^\alpha = 2(\log c)^{-2}|d\ell|^2$. Thus $\|U\|_{m^\alpha}$ is, up to a constant, the same as $|\tilde U|$, for U represented locally as $\tilde U \, d\ell^3$. Thus the maximum modulus principle implies that $\sup\{\|U(x)\|_{m^\alpha} : x \in \mathcal{A}\} \le \sup\{\|U(x)\|_{m^\alpha} : x \in \partial\mathcal{A}\}$.

On the other hand, if the annulus $\mathcal{A}$ is not contained in $\mathcal{F}$, then outside this set, the metric m^α is uniformly equivalent to the hyperbolic metric. Thus if we attempt to extend the flat metric $(2\log c)^{-2}|d\ell|^2$ to all of $\mathcal{A}$, the hyperbolic metric differs from the flat metric by a conformal discrepancy whose size is bounded by a function only of the hyperbolic distance to the flat part, as the metric is given by (3) above. The universal bound on the hyperbolic diameter on the thick part (see e.g. p. 9 in Wolpert [**77**]) then provides the constant C as needed.

The remaining case is in which $\mathcal{A}$ is a neighborhood of a node, the regular part of which is two punctured disks. If $\mathcal{A}$ is exactly the locus $\mathcal{F}$ in which $m^\alpha = 2(\log c)^{-2}|d\ell|^2$, the maximum of $\|U\|_{m^\alpha}$ must occur at the boundary. Moreover, the asymptotic value $\|U\|_{m^\alpha}$ at the node when $z = w = t = 0$ is equal to $|R| \cdot |\log c|^3$, where R is the residue of U and c is a uniform constant. But R is determined by a Cauchy integral formula for $\tilde U$ integrated along the boundary of the disk. Thus in this case, we have the same sort of bounds as above. The analysis involving the hyperbolic distance is also valid by (2) above, and we may produce the uniform constant C needed. q.e.d.

Now as the cubic differentials remain uniformly bounded in the $m^{\alpha,j}$ metrics, they subsequentially converge to a regular limit $(\Sigma_j, U_j) \to (\Sigma_\infty, U_\infty)$ (ignoring the subsequence in the notation). Then Theorems

4.3.2 and 4.4.1 above imply that $(\Omega_j, \Gamma_j|_{S_k}) \to (\mathcal{O}_k, G_k)$ for $k = 1, \dots, n$, where n is the number of components of $\Sigma_\infty^{\rm reg}$. Let $\{\Sigma_\infty^k\}$ denote the corresponding components of $\Sigma_\infty^{\rm reg}$.

We investigate these regular limits of convex $\mathbb{RP}^2$ structures. Assume again that the regular convex $\mathbb{RP}^2$ structures lie in a single stratum, in which the surface S is already separated into pieces as $S - c$ for $c \in C(S)$. Then additional necks may be separated by choosing d disjoint from c and $d \cup c \subset C(S)$. We consider a single connected component S_1 of $S - c$, and let $\tilde S_k$ be a connected component of $S_1 - d$. In this case, we have (Ω_j, Γ_j) so that $\Gamma_j \backslash \Omega_j$ is diffeomorphic to S_1 and $(\Omega_j, \Gamma_j|_{\tilde S_k}) \to (\mathcal{O}, G)$ with G acting properly discontinuously and discretely on $\mathcal{O}$ so that the quotient is diffeomorphic to $\tilde S_k$. We will prove any other limit is equivalent up to the action of $\mathbf{SL}(3,\mathbb{R})$ and the mapping class group.

First of all, recall that, given a basepoint in $\tilde S_k$, the fundamental group of the open subsurface $\tilde S_k$ of S_1 can be represented as a conjugacy class of subgroups of $\pi_1 S_1$. We have shown above that we may pick one element of this conjugacy class to represent $\Gamma_j|_{\tilde S_k}$ as a sub-representation of Γ_j. We further characterize the boundary of $\tilde S_k \subset S_1$ to be given by a collection of principal geodesics (and so for this discussion S_1 is *cut* along geodesics into pieces $\tilde S_k$, rather than being pulled). Choose the image of the developing map Ω_j^k as a subset of Ω_j on which $\Gamma_j|_{\tilde S_k}$ acts so that the quotient $\Gamma_j|_{\tilde S_k} \backslash \Omega_j^k$ is diffeomorphic to $\tilde S_k$ with principal geodesic boundary. Goldman's Theorem 2.3.1 then shows that the surface S_1 is reconstructed from gluing the subsurfaces $\tilde S_k$ together in a standard combinatorial way, which we detail in the next four paragraphs. In the four paragraphs which follow, we suppress the dependence on the index j in our sequence of domains.

We can represent S_1 as the disjoint union of open subsurfaces $\tilde S_k$, $k = 1, \dots, m$ and free homotopy class of loops $\ell_i \in c$. Combinatorially, we may represent S_1 as a connected graph with nodes $\tilde S_k$ and connected by edges ℓ_i. Now consider an image of the developing map Ω^k for each $\tilde S_k$. Then we follow Goldman [**27**] to reconstruct the image Ω of the developing map of S_1 from many copies of the Ω^k. Begin by analyzing a single loop ℓ_1 which connects $\tilde S_1$ to $\tilde S_2$. Fix Ω^1 and pick a lift $b \subset \partial\Omega^1$ of ℓ_1. Then choose $\gamma \in \mathbf{SL}(3,\mathbb{R})$ which acts on Ω^1 by a hyperbolic action on the principal segment b. Similarly, there is a $\rho \in \mathbf{SL}(3,\mathbb{R})$ so that the closures $\overline{\Omega^1} \cap \overline{\rho\Omega^2} = \bar b$ and $\overline{\Omega^1} \cup \overline{\rho\Omega^2}$ is convex. We may repeat this attaching process along all copies of ℓ_1 in order to glue $\tilde S_1$ and $\tilde S_2$ along ℓ_1. Then this process can be repeated for all the other copies of the same principal segment, which can be enumerated by δb for δ in the coset space $\Gamma(\pi_1 S_1)/\langle\gamma\rangle$. (We have assumed in our notation that $\tilde S_1 \neq \tilde S_2$. The case in which $\tilde S_1 = \tilde S_2$, and thus $\tilde S_1$ is attached to itself across ℓ_1, is essentially the same.)

We emphasize that as a part of this gluing process, the endpoints of the geodesic segment b remain in the boundary of the glued domain. This can be seen purely from the point of view of the representations. As an interior loop in the convex $\mathbb{RP}^2$ surface S_1, ℓ_1 must have hyperbolic holonomy $\Gamma(\ell_1)$ [**52**]. So there are a unique attracting and a unique repelling fixed point, the endpoints of b. The dynamics of any point in Ω^1 under $\Gamma(\ell_1)$ then show these endpoints must be in the boundary $\partial\Omega^1$. As we glue $\tilde{S}_2$ to $\tilde{S}_1$ across ℓ_1, the induced representation from $\pi_1(\tilde{S}_1 \cup_{\ell_1} \tilde{S}_2)$ still includes the element $\Gamma(\ell_1)$ [**27**]. Thus the endpoints of b remain in the boundary of the larger domain, and indeed in the boundary of Ω. This shows that the geodesic segment b partitions Ω into two open convex pieces.

We repeat this process with other loops in d, and then describe Ω as a disjoint union of copies of $\Omega^1, \ldots, \Omega^m$ and lifts of loops in d. The combinatorial structure of this union can be described by an infinite-valence tree, with each vertex corresponding to a copy of an Ω^k and each edge corresponding to a lift of a principal geodesic segment across which the two domains represented by the vertices are attached. The fact that this graph is a *tree* is a consequence of the injectivity of the developing map [**27**]. For Ω^1 as in the previous paragraph, there is one adjacent edge for each $\delta \in \Gamma(\pi_1 S_1)/\langle\gamma\rangle$, which corresponds to the principal geodesic segment δb. The other vertex of this edge corresponds to the domain $\delta\rho\Omega^2$. (If there are other loops in c which border $\tilde{S}_1$, then there will be corresponding edges from Ω^1 as well.) Denote the domains represented by vertices in the graph by $\mathcal{O}_i$. Each $\mathcal{O}_i = \sigma\Omega^k$ for $\sigma \in \mathbf{SL}(3,\mathbb{R})$ and $1 \le k \le m$. Note that a simple induction argument shows that all the geodesic segments along which we have glued are disjoint.

Now we consider the action of Γ^1 on Ω. Γ^1 acts on the sub-domain Ω^1. For I the identity matrix, we have

Lemma 5.2.5.
- $\gamma \in \Gamma^1(\pi_1 S_1) - \{I\}$ *acts on the boundary segment b of Ω^1 if and only if γ is a hyperbolic action on the principal geodesic segment b.*
- $\gamma \in \Gamma^1(\pi_1 S_1) - \{I\}$ *acts on $\mathcal{O}_i$ if and only if $\mathcal{O}_i$ is adjacent to Ω^1 and γ is a hyperbolic action on the principal geodesic segment separating Ω^1 and $\mathcal{O}_i$.*

With this combinatorial picture set up, we assume $(\Omega_j, \Gamma_j|_{S_k}) \to (\mathcal{O}, G)$ so that $G\backslash\mathcal{O}$ is diffeomorphic to S_k.

Recall Benzécri's compactness Theorem 5.2.1 above: For every sequence (Ω_j, x_j) for Ω_j properly convex and $x_j \in \Omega_j$, upon passing to a subsequence, there are $\rho_j \in \mathbf{SL}(3,\mathbb{R})$ so that $\rho_j(\Omega_j, x_j) \to (\mathcal{O}, x)$ in the Hausdorff topology. We analyze our limits of $(\Omega_j, \Gamma_j|_{S_k})$ in terms of these Benzécri limits of pointed convex domains. Recall that the surface S_1 has genus $\tilde{g}$ and n punctures, where $\tilde{g}+2n \le g$ the genus of S. Consider the conformal structure induced by the Blaschke metric on $\Gamma_j\backslash\Omega_j$ as an element of the compact space $\overline{\mathcal{M}}_{\tilde{g},n}$, and then consider sequences of points in the corresponding universal curve.

Consider a convergent sequence in Benzécri's sense $(\Omega_j, x_j) \to (\mathcal{O}, x)$. By taking a subsequence if necessary, assume x_j converges in the universal curve $\overline{\mathcal{C}}_{\tilde{g},n}$. Denote by R_j the noded Riemann surface containing x_j.

Proposition 5.2.6. *Up to the actions of* $\mathbf{SL}(3,\mathbb{R})$ *and the mapping class group, there is exactly one limit of the sequence of pairs* $(\Omega_j, \Gamma_j|_{S_k})$ *for each* k.

Proof. Consider a convergent Benzécri sequence $(\Omega_j, x_j) \to (\mathcal{U}, x)$. By choosing a subsequence if necessary, we consider two cases, as in Lemma 2.9.1 above.

First of all, consider the case in which $[x_j]$ converges to a node or a puncture in $\overline{\mathcal{C}}_{\tilde{g},n}$, where $[x_j]$ denotes the image of x_j in the Riemann surface conformal to the quotient $\Gamma_j\backslash\Omega_j$ equipped with the Blaschke metric. In this case, for all $\epsilon > 0$ the hyperbolic distance from $[x_i]$ to Thick_ϵ diverges to infinity as $j \to \infty$. Since the Blaschke metric is bounded from below by the hyperbolic metric (Proposition 4.2.3), the Blaschke distance from $[x_j]$ to any point in the thick part Thick_ϵ of R_j has an infinite limit as $j \to \infty$ while $\epsilon > 0$ is fixed. Then Lemma 5.2.5 implies that, for every $\delta_j \in \Gamma_j|_{S_k}(\pi_1 S_k) - \langle\gamma_j\rangle$, the Blaschke distance from x_j to $\delta_j x_j$ diverges to infinity, where γ_j is the projective holonomy around the neck determined by the conformal cusp/collar neighborhood of $[x_j]$. Proposition 3.3.2 implies that $\Gamma_j|_{S_k}$ cannot converge to act on the limiting domain $\mathcal{U}$. This rules out this case.

Second, consider the case in which $[x_j]$ converges to a limit in $\overline{\mathcal{C}}_{\tilde{g},n}$ which is not a node or puncture, then for all large j, the $[x_j]$ lie uniformly in the thick part of the Riemann surfaces R_j. Now if the $[x_j]$ lie in a different connected component of the thick part of R_j from S_k, then the same considerations as in the previous paragraph apply to rule this out.

Therefore, we may assume that $[x_j]$ converges to a limit in $\overline{\mathcal{C}}_{\tilde{g},n}$ so that $[x_j]$ is in a component of the thick part Thick_ϵ of the R_j which overlaps with S_k for some $\epsilon > 0$. Recall that we have already taken a subsequence to show $(\Sigma_j, U_j) \to (\Sigma_\infty, U_\infty)$ and that this convergence by Theorems 4.3.2 and 4.4.1 implies the convergence of $(\Omega_j, \Gamma_j|_{S_k}) \to (\mathcal{O}, G)$. The proofs of Theorems 4.3.2 and 4.4.1 show that we may fix diffeomorphisms $\phi_j : \Omega_j \to \mathcal{D}$ so that $\mathcal{D} \subset \mathbb{C}$ is the unit disk, ϕ_j is conformal with respect to the Blaschke metric, $\phi_j^{-1}(0)$ lies in each Ω_j, and $\phi_\infty^{-1}(0) \in \mathcal{O}$. We may rephrase our assumption to state that $[x_j]$ lies in the same component of the thick part as $[\phi_j^{-1}(0)]$.

But there are uniform bounds on the hyperbolic diameter of connected components of the thick part of Riemann surfaces. See e.g. [**77**], page 9. In particular, the hyperbolic distance from $[x_j]$ to $[\phi_j^{-1}(0)]$ is uniformly bounded by a constant C. Therefore, we may consider a lift $\tilde{x}_j$ of $[x_j]$ so that the hyperbolic distance from 0 to $\phi_j(\tilde{x}_j)$ is bounded by C. Passing from x_j to $\tilde{x}_j$ corresponds to the action of an element $\rho_j \in \Gamma_j|_{S_k}$. See Subsection 2.6 above. Lemma 5.2.4 above shows the cubic differentials U_j on R_j are uniformly bounded with respect to the $m^{\alpha,j}$ metric. Moreover, on the thick part Thick_ϵ, the hyperbolic metric and the $m^{\alpha,j}$ metrics are uniformly equivalent, and the conformal factors u_j of the Blaschke metrics are uniformly bounded in the C^1 norm, with all the uniform constants depending on ϵ.

Choosing an appropriate initial frame, we may integrate the structure equations for the affine sphere as in Theorem 4.4.1 to show that the limit $\tilde{x}_\infty \in \mathcal{O}$ and, as the previous paragraph shows the coefficients of the relevant ODE system are uniformly bounded, $\tilde{x}_j \to \tilde{x}_\infty$ (up to a subsequence). Now we already have assumed that (Ω_j, x_j) converges to $(\mathcal{U}, x)$ in the space of pointed convex domains in $\mathbb{RP}^2$ modulo the action of $\mathbf{SL}(3,\mathbb{R})$. We have just shown that a subsequence (Ω_{j_i}, x_{j_i}) converges to $(\mathcal{O}, \tilde{x}_\infty)$. Benzécri's Theorem 5.2.1 above shows the space of pointed properly convex domains in $\mathbb{RP}^2$ modulo $\mathbf{SL}(3,\mathbb{R})$ is Hausdorff; thus there is a $\rho \in \mathbf{SL}(3,\mathbb{R})$ so that $(\mathcal{U}, x) = \rho(\mathcal{O}, \tilde{x}_\infty)$. Moreover, every subsequence of (Ω_j, x_j) itself has a subsequence converging to $(\mathcal{U}, x)$ in the Benzécri sense, and so we find $(\Omega_j, x_j) \to (\mathcal{O}, \tilde{x}_\infty)$ up to the action of $\mathbf{SL}(3,\mathbb{R})$.

To address the convergence of the representations $\Gamma_j|_{S_k}$, recall that an element of $\mathbf{SL}(3,\mathbb{R})$ is determined by its action on 4 points in general position in $\mathbb{RP}^2$. Luckily, the estimates we have proved are strong enough to control the geometry of a uniformly large neighborhood of $\tilde{x}_\infty$, and points in this neighborhood will serve as our 4 points in general position. In particular, as $j \to \infty$, we have a neighborhood $\mathcal{N}$ of $\phi_\infty^{-1}(\tilde{x}_\infty)$ in the unit disk $\mathcal{D}$, and uniform estimates on $\mathcal{N}$ of the cubic differentials U_j, the conformal factors u_j, and their derivatives. (This is because there is a uniform ϵ so that $[x_j] \in \mathrm{Thick}_\epsilon$ for $j = 1, 2, \ldots, \infty$. This shows there is a uniformly large neighborhood $\mathcal{N}$ around $\tilde{x}_j$ for j large so that the projection of $\mathcal{N}$ to Σ_j^{reg} is contained in $\mathrm{Thick}_{\epsilon/2}$. See e.g. Lemma 1.1 in [**77**] for a justification. On the thick part, we have uniform bounds on U_j, u_j and du_j.)

Upon choosing a suitable initial frame, the diffeomorphism $\phi_j^{-1} \colon \mathcal{D} \to \Omega_j$ is constructed by solving the ODE system (17), choosing the component f of the frame F, and finally projecting from $\mathbb{R}^3 \to \mathbb{RP}^2$. The uniform estimates on $\mathcal{N}$ imply that there are open sets $\mathcal{A}$ and $\mathcal{B}$ so that $\phi_\infty^{-1}(\tilde{x}_\infty) \in \mathcal{A} \subset \mathcal{N}$, $\tilde{x}_\infty \in \mathcal{B} \subset \phi_\infty(\mathcal{N})$, and for all j large, $\tilde{x}_j \in \mathcal{B}$, $\phi_j(\tilde{x}_j) \in \mathcal{A}$, ϕ_j and its derivatives are bounded on $\mathcal{A}$, and ϕ_j^{-1} and its derivatives are bounded on $\mathcal{B}$. (This is just a quantitative version of the Inverse Function Theorem.)

We assume $\rho_j(\Omega_j, \Gamma_j|_{S_k}) \to (\mathcal{O}, G)$, and we have shown that there is a sequence $x_j \in \Omega_j$ so that $\rho_j(x_j) \to x \in \mathcal{O}$ (this x is referred to as $\tilde{x}_\infty$ above), and x_j is in the same connected component of the thick part of $\Gamma_j \backslash \Omega_j$ as S_k is (this follows from the uniform estimates on $\mathcal{A}$ and $\mathcal{B}$ in the previous paragraph). Absorb the ρ_j into the notation for Ω_j and $\Gamma_j|_{S_k}$, so that $(\Omega_j, \Gamma_j|_{S_k}) \to (\mathcal{O}, G)$ and $x_j \to x$. Let $\gamma^1, \dots, \gamma^m$ be generators of G.

Let $x^a = x, x^b, x^c, x^d$ be in general position in $\mathcal{O}$, and assume that they are in a small neighborhood of x. In particular, for $p \in \{a, b, c, d\}$, let K^p be the convex hull of $\{x^a, x^b, x^c, x^d\} - \{x^p\}$. Assume for a choice of an affine coordinate patch in $\mathbb{RP}^2$ that there are six open disks $D_a, D_b, D_c, D_d, D_2, D_3$ so that

- The closure $\bar{D}_p$ is contained in the interior of K^p,
- $\bar{D}_2 \subset \mathcal{O}$ and each $x^p \in D_2$, and
- $\bar{\mathcal{O}} \subset D_3$.

Then for j large, all these points x^a, x^b, x^c, x^d will be in the same connected component as x_j of the thick part of $\Gamma_j \backslash \Omega_j$ (this is a consequence of the Inverse Function Theorem argument above). As $\Gamma_j|_{S_k} \to G$, let $\gamma^i_j \in \Gamma_j|_{S_k}$ converge to γ^i for $i = 1, \dots, m$ as $j \to \infty$. For large j, the $\gamma^1_j, \dots, \gamma^m_j$ still generate $\Gamma_j|_{S_k}$. Then the set $\{\gamma^i_j x^p : i = 1, \dots, m; p = a, b, c, d\}$ determines the generators of $\Gamma_j|_{S_k}$ and thus also the group $\Gamma_j|_{S_k}$ itself.

Now we prove the uniqueness of G (up to a possible $\mathbf{SL}(3, \mathbb{R})$ action). Recall we assume that $(\Omega_j, \Gamma_j|_{S_k}) \to (\mathcal{O}, G)$. We have established there is an x_j so that $\rho_j(\Omega_j, x_j) \to (\mathcal{O}, x)$. Now consider another sequence $\sigma_j(\Omega_j, \Gamma_j|_{S_k}) \to (\mathcal{O}, H)$. Consider the points x^a, x^b, x^c, x^d in general position in $\mathcal{O} \subset \mathbb{RP}^2$. Then, as above (recalling that x^a, x^b, x^c, x^d and their convex hull are uniformly contained in the thick part of $\Gamma_j \backslash \Omega_j$, for large j), $\{\sigma_j x^p\}$ remains in general position, and there are still uniformly large ellipses $D_a, D_b, D_c, D_d, D_2, D_3$ as above (this follows from a transversality argument based on the Inverse Function Theorem analysis above). The existence of these bounding ellipses shows that the $\sigma_j(x^p)$ remain uniformly in general position in $\mathbb{RP}^2$, and that the family σ_j lies in a compact subset of $\mathbf{SL}(3, \mathbb{R})$. Thus there is a limit $\sigma_j \to \sigma$ (upon taking a subsequence).

For $i = 1, \dots, m$, let $\eta^i = \sigma \gamma^i \sigma^{-1}$. These η^i generate H. Similarly, define $\eta^i_j = \sigma_j \gamma^i_j \sigma_j^{-1} \in \sigma_j(\Gamma_j|_{S_k})\sigma_j^{-1}$. Then for large j, η^i_j generate $\sigma_j(\Gamma_j|_{S_k})\sigma_j^{-1}$ and $\lim_{j\to\infty} \eta^i_j = \eta^i$. This implies $H = \sigma G \sigma^{-1}$. Moreover, since $\mathcal{O}$ has already been fixed, $\sigma \in \mathbf{SL}(3, \mathbb{R})$ is a projective automorphism of $\mathcal{O}$. Therefore, the two limits $(\mathcal{O}, G)$ and $(\mathcal{O}, H)$ are equivalent up to the action of $\mathbf{SL}(3, \mathbb{R})$. More precisely, for all subsequences of

$(\Omega_j, \Gamma_j|_{S_k})$, there is a further subsequence and an element $\sigma \in \mathbf{SL}(3,\mathbb{R})$ so that $(\mathcal{O}, G) = \sigma(\mathcal{O}, H)$. But then these two objects are the same modulo the action of $\mathbf{SL}(3,\mathbb{R})$. q.e.d.

Now we complete the proof of Theorem 5.2.2. Consider a convergent sequence of regular convex $\mathbb{RP}^2$ structures $\lim_{k\to\infty} \oplus_j(\Omega_{j,k}, \Gamma_{j,k}) = \oplus_m(\mathcal{O}_m, G_m)$ in $\mathcal{R}_S^{\mathrm{aug}}$, and their corresponding sequence $(\Sigma_k, U_k) = \Phi^{-1}[\oplus_j(\Omega_{j,k}, \Gamma_{j,k})]$ of noded Riemann surfaces and regular cubic differentials. Then there is a convergent subsequence $(\Sigma_{j_\ell}, U_{j_\ell}) \to (\Sigma_\infty, U_\infty)$. Moreover, $\lim_{k\to\infty} \oplus_k(\Omega_{j_\ell,k}, \Gamma_{j_\ell,k}) = \oplus_m(\mathcal{O}_m, G_m)$ in $\mathcal{R}_S^{\mathrm{aug}}$, and the regular convex $\mathbb{RP}^2$ structure corresponding to $(\Sigma_\infty, U_\infty)$ is $\Phi(\Sigma_\infty, U_\infty) = \oplus_m(\mathcal{O}_m, G_m)$. But then Proposition 5.2.6 shows that every subsequence of (Σ_j, U_j) has a subsequence which converges to the same limit. Recall $\mathcal{R}_S^{\mathrm{aug}}$ is first countable. This is enough to show that $(\Sigma_j, U_j) \to (\Sigma_\infty, U_\infty)$. Therefore, Φ^{-1} is continuous, and Theorem 5.2.2 is proved.

The Main Theorem 1.0.1 follows from Theorems 4.1.2, 4.3.1, 5.1.1 and 5.2.2.

References

[1] W. Abikoff. *The real analytic theory of Teichmüller space*, volume 820 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[2] D. Alessandrini. Dequantization of real convex projective manifolds. In *Tropical and idempotent mathematics*, volume 495 of *Contemp. Math.*, pages 61–85. Amer. Math. Soc., Providence, RI, 2009.

[3] D. Baraglia. *G_2 Geometry and Integrable Systems*. PhD thesis, University of Oxford, 2009. arXiv:1002.1767.

[4] A. F. Beardon. *The geometry of discrete groups*, volume 91 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1983.

[5] R. Benedetti and C. Petronio. *Lectures on hyperbolic geometry*. Universitext. Springer-Verlag, Berlin, 1992.

[6] Y. Benoist and D. Hulin. Cubic differentials and finite volume convex projective surfaces. *Geom. Topol.*, 17(1):595–620, 2013.

[7] Y. Benoist and D. Hulin. Cubic differentials and hyperbolic convex sets. *J. Differential Geom.*, 98(1):1–19, 2014.

[8] J. P. Benzécri. Sur les variétés localement affines et projectives. *Bullétin de la Société Mathématique de France*, 88:229–332, 1960.

[9] L. Bers. Spaces of degenerating Riemann surfaces. In *Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973)*, pages 43–55. Ann. of Math. Studies, No. 79. Princeton Univ. Press, Princeton, N.J., 1974.

[10] W. Blaschke. *Vorlesungen über Differentialgeometrie II, Affine Differentialgeometrie*. Springer-Verlag, 1923.

[11] E. Calabi. Complete affine hyperspheres I. *Instituto Nazionale di Alta Matematica Symposia Mathematica*, 10:19–38, 1972.

[12] S.-Y. Cheng and S.-T. Yau. On the regularity of the Monge-Ampère equation $\det((\partial^2 u/\partial x^i \partial x^j)) = F(x, u)$. *Communications on Pure and Applied Mathematics*, 30:41–68, 1977.

[13] S.-Y. Cheng and S.-T. Yau. Complete affine hyperspheres. part I. The completeness of affine metrics. *Communications on Pure and Applied Mathematics*, 39(6):839–866, 1986.

[14] S. Choi. Convex decompositions of real projective surfaces. I: π-annuli and convexity. *Journal of Differential Geometry*, 40(1):165–208, 1994.

[15] S. Choi. Convex decompositions of real projective surfaces. II: Admissible decompositions. *Journal of Differential Geometry*, 40(2):239–283, 1994.

[16] S. Choi. The classification of ends of properly convex real projective orbifolds ii: Properly convex radial ends and totally geodesic ends. arXiv:1501.00352, 2015.

[17] S. Choi. The classification of radial or totally geodesic ends of real projective orbifolds I: a survey of results. arXiv:1501.00348, 2015.

[18] S. Choi and W. M. Goldman. Convex real projective structures on closed surfaces are closed. *Proceedings of the American Mathematical Society*, 118(2):657–661, 1993.

[19] S. Choi and W. M. Goldman. The classification of real projective structures on compact surfaces. *Bulletin (New Series) of the American Mathematical Society*, 34(2):161–171, 1997.

[20] V. Chuckrow. On Schottky groups with applications to kleinian groups. *Ann. of Math. (2)*, 88:47–61, 1968.

[21] D. Cooper and K. Delp. The marked length spectrum of a projective manifold or orbifold. *Proc. Amer. Math. Soc.*, 138(9):3361–3376, 2010.

[22] D. Dumas and M. Wolf. Polynomial cubic differentials and convex polygons in the plane. arXiv:1407.8149.

[23] S. Gigena. Integral invariants of convex cones. *Journal of Differential Geometry*, 13:191–222, 1978.

[24] S. Gigena. On a conjecture by E. Calabi. *Geometriae Dedicata*, 11:387–396, 1981.

[25] D. Gilbarg and N. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, 1983.

[26] W. M. Goldman. Geometric structures on manifolds and varieties of representations. In *Geometry of group representations (Boulder, CO, 1987)*, volume 74 of *Contemporary Mathematics*, pages 169–198. American Mathematical Society, 1988.

[27] W. M. Goldman. Convex real projective structures on compact surfaces. *Journal of Differential Geometry*, 31:791–845, 1990.

[28] W. M. Goldman. The symplectic geometry of affine connections on surfaces. *Journal für die reine und agnewandte Mathematik*, 407:126–159, 1990.

[29] H. Grauert and R. Remmert. *Coherent analytic sheaves*, volume 265 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984.

[30] O. Guichard and A. Wienhard. Convex foliated projective structures and the Hitchin component for $\mathrm{PSL}_4(\mathbf{R})$. *Duke Mathematical Journal*, 144(3):381–445, 2008.

[31] W. J. Harvey. Spaces of discrete groups. In *Discrete groups and automorphic functions (Proc. Conf., Cambridge, 1975)*, pages 295–348. Academic Press, London, 1977.

[32] N. J. Hitchin. Lie groups and Teichmüller space. *Topology*, 31(3):449–473, 1992.

[33] J. H. Hubbard and S. Koch. An analytic construction of the Deligne-Mumford compactification of the moduli space of curves. *J. Differential Geom.*, 98(2):261–313, 2014.

[34] I. Kim. Rigidity and deformation spaces of strictly convex real projective structures on compact manifolds. *J. Differential Geom.*, 58(2):189–218, 2001.

[35] I. Kim. Compactification of strictly convex real projective structures. *Geom. Dedicata*, 113:185–195, 2005.

[36] I. Kim. Erratum to Rigidity and deformation spaces of strictly convex real projective structures on compact manifolds [mr1913941]. *J. Differential Geom.*, 86(1):189, 2010.

[37] F. Labourie. in Proceedings of the GARC Conference in Differential Geometry, Seoul National University, Fall 1997, 1997.

[38] F. Labourie. Flat projective structures on surfaces and cubic holomorphic differentials. *Pure and Applied Mathematics Quarterly*, 3(4):1057–1099, 2007. Special issue in honor of Grisha Margulis, Part 1 of 2.

[39] F. Labourie. Cross ratios, Anosov representations and the energy functional on Teichmüller space. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(3):437–469, 2008.

[40] F. Labourie. Cyclic surfaces and Hitchin components in rank 2. arXiv:1406.4637, 2014.

[41] G.-S. Lee and T. Zhang. Collar lemma for hitchin representations. arXiv:1411.2082.

[42] A.-M. Li. Calabi conjecture on hyperbolic affine hyperspheres. *Mathematische Zeitschrift*, 203:483–491, 1990.

[43] A.-M. Li. Calabi conjecture on hyperbolic affine hyperspheres (2). *Mathematische Annalen*, 293:485–493, 1992.

[44] A.-M. Li, U. Simon, and G. Zhao. *Global Affine Differential Geometry of Hypersurfaces.* Walter de Gruyter, 1993.

[45] C. Loewner and L. Nirenberg. Partial differential equations invariant under conformal or projective transformations. In *Contributions to Analysis*, pages 245–272. Academic Press, 1974.

[46] J. Loftin. Affine spheres and convex $\mathbb{RP}^n$ manifolds. *American Journal of Mathematics*, 123(2):255–274, 2001.

[47] J. Loftin. Flat metrics, cubic differentials and limits of projective holonomies. *Geom. Dedicata*, 128:97–106, 2007.

[48] J. Loftin. Survey on affine spheres. In *Handbook of geometric analysis, No. 2*, volume 13 of *Adv. Lect. Math. (ALM)*, pages 161–191. Int. Press, Somerville, MA, 2010.

[49] J. Loftin and I. McIntosh. Cubic differentials in the differential geometry of surfaces. arXiv:1310.5120, to appear, Handbook of Teichmüller Theory, Volume V.

[50] J. C. Loftin. The compactification of the moduli space of $\mathbb{RP}^2$ surfaces, I. *Journal of Differential Geometry*, 68(2):223–276, 2004. math.DG/0311052.

[51] J. C. Loftin. Singular semi-flat Calabi-Yau metrics on S^2. *Communications in Analysis and Geometry*, 13(2):333–361, 2005. math.DG/0403218.

[52] L. Marquis. Espace des modules marqués des surfaces projectives convexes de volume fini. *Geom. Topol.*, 14(4):2103–2149, 2010.

[53] L. Marquis. Surface projective convexe de volume fini. *Ann. Inst. Fourier (Grenoble)*, 62(1):325–392, 2012.

[54] H. Masur. The extension of the Weil-Petersson metric to the boundary of Teichmüller space. *Duke Mathematical Journal*, 43(3):623–635, 1976.

[55] R. Mazzeo, J. Swoboda, H. Weiss, and F. Witt. Ends of the moduli space of Higgs bundles. *Duke Math. J.*, 165(12):2227–2271, 2016.

[56] T. Mochizuki. Asymptotic behaviour of certain families of harmonic bundles on Riemann surfaces. *J. Topol.*, 9(4):1021–1073, 2016.

[57] X. Nie. Entropy degeneration of convex projective surfaces. arXiv:1503.04420.

[58] X. Nie. Meromorphic cubic differentials and convex projective structures. arXiv:1503.02608.

[59] K. Nomizu and T. Sasaki. *Affine Differential Geometry: Geometry of Affine Immersions*. Cambridge University Press, 1994.

[60] R. Osserman. *A survey of minimal surfaces*. Dover Publications, Inc., New York, second edition, 1986.

[61] A. Parreau. *Dégénérescence de sous-groupes discrets des groupes de Lie semisimples et actions de groupes sur des immeubles affines*. PhD thesis, Université de Paris-sud, 2000.

[62] A. Parreau. Compactification d'espaces de représentations de groupes de type fini. *Math. Z.*, 272(1-2):51–86, 2012.

[63] T. Sasaki. Hyperbolic affine hyperspheres. *Nagoya Mathematical Journal*, 77:107–123, 1980.

[64] R. Schoen and S.-T. Yau. *Lectures on Differential Geometry*. International Press, 1994.

[65] C. T. Simpson. Harmonic bundles on noncompact curves. *J. Amer. Math. Soc.*, 3(3):713–770, 1990.

[66] J. Swoboda. The Hitchin fibration under degenerations to noded Riemann surfaces. arXiv:1609.04928.

[67] J. Swoboda. Moduli spaces of Higgs bundles on degenerating Riemann surfaces. *Adv. Math.*, 322:637–681, 2017.

[68] G. Tzitzéica. Sur une nouvelle classe de surfaces. *Rend. Circ. mat. Palermo*, 25:180–187, 1908.

[69] G. Tzitzéica. Sur une nouvelle classe de surfaces, $2^{\text{ème}}$ partie. *Rend. Circ. mat. Palermo*, 25:210–216, 1909.

[70] T. Y.-H. Wan. Constant mean curvature surface, harmonic maps, and universal Teichmüller space. *J. Differential Geom.*, 35(3):643–657, 1992.

[71] C.-P. Wang. Some examples of complete hyperbolic affine 2-spheres in $\mathbb{R}^3$. In *Global Differential Geometry and Global Analysis*, volume 1481 of *Lecture Notes in Mathematics*, pages 272–280. Springer-Verlag, 1991.

[72] A. Weil. On discrete subgroups of Lie groups. II. *Ann. of Math. (2)*, 75:578–602, 1962.

[73] M. Wolf and S. Wolpert. Real analytic structures on the moduli space of curves. *American Journal of Mathematics*, 114:1079–1102, 1992.

[74] S. Wolpert. The hyperbolic metric and the geometry of the universal curve. *Journal of Differential Geometry*, 31(2):417–472, 1990.

[75] S. Wolpert. Spectral limits for hyperbolic surfaces II. *Inventiones Mathematicae*, 108(1):91–129, 1992.

[76] S. A. Wolpert. Behavior of geodesic-length functions on Teichmüller space. *J. Differential Geom.*, 79(2):277–334, 2008.

[77] S. A. Wolpert. *Families of Riemann surfaces and Weil-Petersson geometry*, volume 113 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2010.

[78] S. A. Wolpert. On families of holomorphic differentials on degenerating annuli. In *Quasiconformal mappings, Riemann surfaces, and Teichmüller spaces*, volume 575 of *Contemp. Math.*, pages 363–370. Amer. Math. Soc., Providence, RI, 2012.

[79] T. Zhang. The degeneration of convex $\mathbb{RP}^2$ structures on surfaces. arXiv:1312.2452.