AFFINE SPHERES AND CONVEX $\mathbb{RP}^n$-MANIFOLDS

JOHN C. LOFTIN

ABSTRACT. We outline consequences of a theorem of Cheng-Yau in affine differential geometry for manifolds locally modeled on $\mathbb{RP}^n$. In particular, for properly convex $\mathbb{RP}^n$-manifolds, we observe that there are two canonical projectively flat connections and a canonical Riemannian metric. One connection represents the given $\mathbb{RP}^n$-structure and the other the $\mathbb{RP}^n$-structure of the projective dual manifold. When n is 2, we use this approach together with a result of C.-P. Wang to show that a compact oriented convex $\mathbb{RP}^2$-surface of genus at least two is equivalent to a conformal structure on the surface together with a holomorphic tricanonical form. This recovers a theorem of Goldman on the deformation space of such surfaces, and yields a description of the moduli space.

1. INTRODUCTION

Affine differential geometry and the study of $\mathbb{RP}^n$-structures on manifolds are two fields in geometry each with a long and rich history. In short, affine geometry is the study of those properties of hypersurfaces in $\mathbb{R}^{n+1}$ which are invariant under the unimodular affine group generated by $\mathbf{SL}(n+1,\mathbb{R})$ and translations. The field was very active in the early part of this century with Blaschke's monograph on the subject [3], and subsequently, mathematicians such as Calabi, Cheng, and Yau have made important contributions.

An $\mathbb{RP}^n$-structure on an n-manifold, on the other hand, is a system of coordinate charts in $\mathbb{RP}^n$ glued together by transition maps in $\mathbf{PGL}(n+1,\mathbb{R})$. Ehresmann studied these in the 1930s. The study of $\mathbb{RP}^2$-structures has been particularly strong. Kuiper, Benzécri, Kobayashi, and Thurston have all done important work in the field. Recently, the field has been quite active, led by Goldman, Choi and Benoist.

The connection between the two fields is this: For a large and important class of manifolds M with $\mathbb{RP}^n$-structure, the convex ones, $M = \Omega/\Gamma$, with Ω a convex domain in some $\mathbb{R}^n \subset \mathbb{RP}^n$ and $\Gamma \subset \mathbf{PGL}(n+1,\mathbb{R})$ an appropriate subgroup. Cheng and Yau prove that any bounded convex Ω uniquely determines a special hypersurface called a hyperbolic affine sphere which is asymptotic to the cone over Ω

in $\mathbb{R}^{n+1}$. (For example, if Ω is a ball, then the affine sphere is just a hyperboloid and we recover the familiar Klein model of hyperbolic space on the ball.) The $\mathbf{SL}(n+1,\mathbb{R})$ invariance the affine geometry provides upstairs means that, under the natural map $\mathbb{R}^{n+1}\setminus 0 \to \mathbb{RP}^n$, we get a lot of canonical structure on Ω that is invariant under $\mathbf{PGL}(n+1,\mathbb{R})$. Therefore, all of it descends to our convex $\mathbb{RP}^n$-manifold M.

Theorem 1. *Any properly convex $\mathbb{RP}^n$-manifold M carries a canonical complete Riemannian metric and two canonical projectively flat connnections. One projectively flat connection determines the given $\mathbb{RP}^n$-structure on M and the other determines the $\mathbb{RP}^n$-structure of the dual manifold. The $\mathbb{RP}^n$-structure comes from a complete hyperbolic metric if and only if these two connections both coincide with the Levi-Civita connection.*

The main application is to convex $\mathbb{RP}^2$-surfaces.

Theorem 2. *Let S be a compact oriented surface of genus $g>1$. There is a natural bijective correspondence between convex $\mathbb{RP}^2$-structures on S and pairs (M,U), where M is a Riemann surface homeomorphic to S and U is a holomorphic cubic differential on M.*

Let $\mathcal{G}(S)$ denote the deformation space of convex $\mathbb{RP}^2$-structures on a compact oriented surface S of genus $g>1$. Therefore, $\mathcal{G}(S)$ has the structure of a holomorphic $5g-5$ dimensional vector bundle over Teichmüller space (which of course has complex dimension $3g-3$). We thereby recover

Corollary 1.0.1 (Goldman [15])**.** *The deformation space $\mathcal{G}(S)$ is topologically a real $16g-16$ dimensional ball.*

C.-P. Wang [34] shows that an invariant called the Pick form on a two-dimensional hyperbolic affine sphere is equivalent to a holomorphic section of K^3, with the conformal structure given by a natural invariant metric. Using this fact, together with a classification theorem of Wang on hyperbolic affine spheres which cover a surface of genus $g>1$ in an appropriate way, we get Theorem 2.

In particular, when the Pick form vanishes, the $\mathbf{RP}^2$ structure is the one descending from the hyperbolic disk Δ. (The Klein model of the hyperbolic metric on Δ shows that the hyperbolic isometries of Δ are given exactly by the action of those elements of $\mathbf{PGL}(3,\mathbb{R})$ which preserve the set $\Delta\subset\mathbb{RP}^2$.) Therefore, the uniformization theorem shows that the structure preserved is exactly the conformal structure of the Riemann surface Σ. From this, we see that Teichmüller space naturally sits in $\mathcal{G}(S)$ as the zero section of the vector bundle with fiber $H^0(\Sigma,K^3)$.

Also, using result on the conormal map of affine spheres, we find that by replacing a section U of K^3 by $-U$, we recover the projective dual surface. This recovers the fact that Teichmüller space is exactly the fixed locus of the action of projective duality on $\mathcal{G}(S)$.

Darvishzadeh and Goldman [11] use different canonical structures on cones to study convex $\mathbb{RP}^2$-surfaces. They show find an almost-complex structure on the $\mathcal{G}(S)$ which, when matched with a symplectic form found by Goldman [16], defines an almost-Kähler metric. (We hope to address the problem of finding a Kähler structure in future work; see the author's dissertation [28].) It is unclear how this almost-complex structure is related to the complex structure induced from Theorem 2, and it seems unlikely that Darvishzadeh and Goldman's almost-complex structure is integrable.

Another approach is given by Hitchin in [18]. By means of stable Higgs bundles on a Riemann surface Σ, he studies the connected components of the space of representations $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ for a simple Lie group G split over $\mathbb{R}$. An $\mathbb{RP}^2$-structure on an oriented surface induces a holonomy representation into $G = \mathbf{PSL}(3, \mathbb{R})$. Using this fact, Goldman and Choi [9] prove that $\mathcal{G}(S)$ coincides with one component of this representation space, which Hitchin had shown is parametrized by the space of sections $H^0(\Sigma, K^2) \oplus H^0(\Sigma, K^3)$. This gives another complex structure on $\mathcal{G}(S)$, but again it is not clear how it is related to the one we find in Theorem 2. In particular, Hitchin's construction depends on an a priori choice of conformal structure on Σ, while in the construction used in Theorem 2, the $\mathbb{RP}^2$-structure determines a metric and therefore a conformal structure on Σ.

For more results on $\mathbb{RP}^2$-surfaces, the reader should consult the survey article of Choi and Goldman [10].

Note Added in Proof. After this work was complete, the author learned of the work of François Labourie [23, 24], which contain versions of Theorems 2 and 4. Prof. Labourie has also kindly informed me that he and J. Dymara have found a straightforward relationship between Wang's and Hitchin's cubic forms [22].

Acknowledgements. This paper contains part of the author's Harvard University thesis. I would like to thank my advisor S.-T. Yau for his kind support and for introducing me to affine differential geometry. I am grateful to Curt McMullen, Eugenio Calabi, and S.-Y. Cheng for useful conversations. I would like to thank Joe Harris and Ravi Vakil for explaining some facts about algebraic curves to me. Finally, I would also like to thank Bill Goldman for inspiring discussions and for his interest in my work.

2. $\mathbb{RP}^n$-STRUCTURES

$\mathbb{RP}^n$ is defined as the space of all lines passing through 0 in $\mathbb{R}^{n+1}$. There is a natural map from $\mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$ given by $p \mapsto \ell$, where ℓ is the unique line through p. We also use the notation $[p]$ to denote this line. The linear automorphisms of $\mathbb{RP}^n$ are given by the group $\mathbf{PGL}(n+1,\mathbb{R})$, which is equivalence classes of matrices $A \in \mathbf{GL}(n+1,\mathbb{R})$ with $A \sim \lambda A$ for real constants λ.

We say that an n-dimensional manifold M has an $\mathbb{RP}^n$-*structure* if it admits coordinate charts represented by open sets in $\mathbb{RP}^n$ and the transition maps between these coordinate charts are given by maps in $\mathbf{PGL}(n+1,\mathbb{R})$. We also say M is an $\mathbb{RP}^n$-*manifold.* A path in M is called a *geodesic* with respect to the $\mathbb{RP}^n$-structure if it is a straight line in each of the coordinate charts.

The $\mathbb{RP}^n$-structure on M can clearly be lifted to an $\mathbb{RP}^n$-structure on its universal cover $\tilde{M}$. Then we can define the *developing map* as a local diffeomorphism from $\tilde{M} \to \mathbb{RP}^n$ in the following manner. Any coordinate map for a neighborhood $\mathcal{U}_0$ of $x \in \tilde{M}$ serves to define the developing map $\mathcal{U}_0 \to \mathbb{RP}^n$. For any adjacent coordinate chart $\mathcal{U}$, the transition map ensures that there is a unique way to define a map from $\mathcal{U}$ to $\mathbb{RP}^n$ which agrees on the overlap of $\mathcal{U}$ and $\mathcal{U}_0$. Repeating this process, we define the developing map from $\tilde{M}$ to $\mathbb{RP}^n$. Deck transformations of $\tilde{M}$ are taken to linear automorphisms of $\mathbb{RP}^n$ by the developing map, and so define a holonomy map $\pi_1(M) \to \mathbf{PGL}(n+1,\mathbb{R})$. The developing map is unique up to the action of $\mathbf{PGL}(n+1,\mathbb{R})$. For more details about $\mathbb{RP}^n$-structures, there are lecture notes of Goldman [14].

2.1. Convex $\mathbb{RP}^n$-structures.

An $\mathbb{RP}^n$-manifold is *convex* if its developing map is a diffeomorphism onto a domain Ω convex in some affine $\mathbb{R}^n \subset \mathbb{RP}^n$. In this case, we can realize $M = \Omega/\Gamma$, where Γ is a subgroup of $\mathbf{PGL}(n+1,\mathbb{R})$ which acts discretely and properly discontinously on Ω. M is *properly convex* if Ω is bounded in some such $\mathbb{R}^n$. Below we find a canonical projectively flat connection on a properly convex $\mathbb{RP}^n$-manifold.

2.2. The tautological bundle.

We define $\mathbb{RP}^n$ as the space of all lines ℓ passing through 0 in $\mathbb{R}^{n+1}$. Then the subset of $\mathbb{RP}^n \times \mathbb{R}^{n+1}$ consisting of all (p,ℓ) with $p \in \ell$ is the total space for the *tautological line bundle* τ of $\mathbb{RP}^n$. Given an $\mathbb{RP}^n$-manifold M, $\mathrm{dev}^{-1}\tau$ defines the tautological bundle on $\tilde{M}$. We say M *admits a tautological bundle* if this structure descends to M, i.e. if there is a line bundle on M which pulls back to $\text{dev}^{-1}\tau$ on $\tilde{M}$ under the universal covering map. For simplicity, we denote this line bundle as τ also.

In general, there are obstructions to the existence of τ if n is odd. We are primarily concerned with $\mathbb{RP}^n$-manifolds with convex structure. In this case, we have

Proposition 2.2.1. *A manifold M with convex $\mathbb{RP}^n$-structure admits an oriented tautological bundle.*

Proof. Since M is convex, we have $M = \Omega/\Gamma$, where $\Omega \subset \mathbb{R}^n \subset \mathbb{RP}^n$ and $\Gamma \subset \mathbf{PGL}(n+1, \mathbb{R})$ is a representation of $\pi_1 M$ which acts discretely and properly discontinously. Introduce inhomogeneous coordinates on $\mathbb{R}^n \supset \Omega$, and consider the cone over Ω in $\mathbb{R}^{n+1}$ defined by

$$C(\Omega) = \{(tx, t) : x \in \Omega \subset \mathbb{R}^n,\, t > 0\}.$$

It is enough to prove that we can lift the action of Γ to a linear action Γ' on $C(\Omega) \subset \mathbb{R}^{n+1}$:

We assume we can lift Γ to Γ'. Then let $C^{\pm}(\Omega) = C(\Omega) \sqcup \{0\} \sqcup -C(\Omega)$ be the union of all lines in Ω. Then Γ' acts on $C^{\pm}(\Omega)$. The total space of the tautological bundle is all pairs

$$\{(p, \ell) \in C^{\pm}(\Omega) \times \Omega : p \in \ell\}$$

modulo the action of $\Gamma' \times \Gamma$. Since Γ' preserves the set $C(\Omega)$, the line bundle is oriented. Sections which take values in $C(\Omega)/\Gamma'$ can be thought of as positive sections.

Now we prove we can lift the action of Γ: Consider the natural map $\pi : \mathbf{SL}(n+1, \mathbb{R}) \to \mathbf{PGL}(n+1, \mathbb{R})$ induced by the projection on $\mathbf{GL}(n+1, \mathbb{R})$. Then there are two cases: If n is even, then π is an isomorphism; if n is odd, then $\mathbf{PGL}(n+1, \mathbb{R})$ has two components and π is a two-to-one map onto the identity component of $\mathbf{PGL}(n+1, \mathbb{R})$. However, we can define

$$\mathbf{SL}^{\pm}(n+1, \mathbb{R}) = \{A \in \mathbf{GL}(n+1, \mathbb{R}) : |\det A| = 1\}.$$

$$\tilde{\pi} \colon \mathbf{SL}^{\pm}(n+1, \mathbb{R}) \to \mathbf{PGL}(n+1, \mathbb{R})$$

is always a two-to-one map with kernel $\{\pm I\}$.

Now consider $\gamma \in \Gamma$ which acts on Ω. Then there are two lifts of γ in $\mathbf{SL}^{\pm}(n+1, \mathbb{R})$. One of them will preserve $C(\Omega)$ and the other will interchange $C(\Omega)$ and $-C(\Omega)$. The former gives a canonical choice of lift which acts on $C(\Omega)$. ☐

Remark. Define $\mathbb{S}^n$ as $\mathbb{R}^{n+1} \setminus 0$ mod $\mathbb{R}^+$ with automorphisms given by $\mathbf{GL}(n+1, \mathbb{R})$ mod $\mathbb{R}^+$. Then an $\mathbb{RP}^n$-structure on M with oriented tautological bundle is equivalent to an $\mathbb{S}^n$-structure on M by considering the positive part of the total space mod $\mathbb{R}^+$. These $\mathbb{S}^n$-structures are the projective structures studied by Benzécri in [2].

2.3. **Projectively flat connections.** There is an equivalent way of defining $\mathbb{RP}^n$-manifolds in terms of affine and projective connections. An $\mathbb{RP}^n$-structure on a manifold M is equivalent to a projective equivalence class of projectively flat connections. The geodesics for the $\mathbb{RP}^n$-structure are the geodesics for ∇. A key element in this correspondence follows from Cartan's theory of normal projective connections. We outline a simplified version which works for connections which are projectively flat. For the general case, see Kobayashi [19] or Hermann [17].

Two torsion-free connections ∇^1 and ∇^2 on TM are *projectively equivalent* if there is a one-form ρ such that

$$\nabla^1_X Y = \nabla^2_X Y + \rho(X)Y + \rho(Y)X,$$

where X and Y are tangent vector fields. This action of ρ is called a *projective transformation.* An equivalent condition is that the geodesics of ∇^1 and ∇^2 are the same as sets. A connection is *projectively flat* if it is locally projectively equivalent to a flat connection.

Proposition 2.3.1. *Let L be a trivial line bundle over M and ξ be a nonvanishing section of L. A torsion-free connection ∇ on TM is projectively flat if and only if there exists a connection $\tilde\nabla$ on the vector bundle $E = TM \oplus L$ such that for some $(0,2)$-tensor β*

$$\begin{cases} \tilde\nabla_X Y = \nabla_X Y + \beta(X,Y)\xi \\ \tilde\nabla_X \xi = X \end{cases} \tag{2.3.1}$$

and the curvature $\tilde R$ of $\tilde\nabla$ must satisfy

$$\tilde R = \lambda \otimes I, \tag{2.3.2}$$

where λ is a two-form on M and I is the identity in End$(TM \oplus L)$. The connection $\tilde\nabla$ satisfying these conditions is unique.

We call this $\tilde\nabla$ the *normal connection* associated to ∇.

Sketch of proof. This result is standard. The *curvature tensor* R is defined in the usual way:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

R is an $\mathrm{End}(TM)$-valued two-form. The *Ricci tensor* is defined by

$$\mathrm{Ric}(Y,Z) = \mathrm{tr}\{X \mapsto R(X,Y)Z\}.$$

Note that unlike in the case of Riemannian curvature, Ric is not always symmetric in Y and Z. Define $P(X,Y)$ by

$$P(X,Y) = \tfrac{1}{n^2-1}[n\mathrm{Ric}(X,Y) + \mathrm{Ric}(Y,X)].$$

Then the *Weyl tensor*, which is invariant under projective transformations, is defined by

$$W(X,Y)Z = R(X,Y)Z - P(X,Y)Z + P(Y,X)Z - P(Y,Z)X + P(X,Z)Y.$$

The condition on projective flatness, i.e. the existence of ρ giving a projective transformation to a flat connection, is equivalent to $W=0$ and

$$(\nabla_X P)(Y,Z) = (\nabla_Y P)(X,Z). \tag{2.3.3}$$

(see e.g. [33]). We note that if $n>2$, then $W=0$ implies (2.3.3).

On the other hand, compute the $\tilde{R}$ from (2.3.1)

$$\begin{aligned} \tilde{R}(X,Y)Z &= R(X,Y)Z + \beta(Y,Z)X - \beta(X,Z)Y + \gamma(X,Y,Z)\xi, \\ \tilde{R}(X,Y)\xi &= [\beta(X,Y) - \beta(Y,X)]\xi, \end{aligned}$$

where γ is a certain skew-symmetrization of $\nabla\beta$. Using these equations for $\tilde{R}$, we see the curvature constraint (2.3.2) is equivalent to (1) $W=0$, (2) $\beta = -P$, and (3) $\gamma(X,Y,Z) = -(\nabla_X P)(Y,Z) + (\nabla_Y P)(X,Z) = 0$. Therefore, by (2.3.3) the projective flatness of ∇ is equivalent to the existence of the normal connection, which satisfies the curvature condition (2.3.2). □

This proposition allows us to construct an $\mathbb{RP}^n$-structure on a manifold equipped with a projectively flat connection ∇. Consider a base point x in M, and the universal cover $\tilde{M}$. Any two paths from x to y in $\tilde{M}$ induce by $\tilde{\nabla}$-parallel transport linear maps between E_x and E_y. Then (2.3.2) and the Ambrose-Singer holonomy theorem [20] show that these maps are equivalent up to homothety; i.e. they define a projective isomorphism from $P(E_x)$ to $P(E_y)$. Then ξ, as a section of E, defines the developing map from $\tilde{M}$ to $\mathbb{RP}^n$ by $[\xi] \in P(E_y)$ for all $y \in \tilde{M}$. See e.g. Goldman [16] for details.

3. Affine differential geometry

The material in Subsections 3.1, 3.2, and 3.3 is standard. Most of it can be readily found in the book of Nomizu and Sasaki [29]. Other good sources are the papers of Calabi [4, 5] and Cheng and Yau [8], and Blaschke's monograph [3].

Affine differential geometry is concerned with those properties of hypersurfaces in $\mathbb{R}^{n+1}$ which remain invariant under the *unimodular affine group* consisting of affine transformations $x \mapsto Ax + b$ with $\det A = 1$.

While much of the formal theory works for any hypersurface which is nondegenerate (i.e. one which can be locally written as the graph of a function with nondegenerate Hessian), we only consider hypersurfaces which are strictly convex. Only in this case is the affine metric a Riemannian metric.

Given a hypersurface immersion $f : H \to \mathbb{R}^{n+1}$, consider a transversal vector field ξ on H. We have the equations:

$$\left\{ \begin{array}{l} D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi, \\ D_X \xi = -f_*(SX) + \tau(X)\xi. \end{array} \right. \tag{3.0.1}$$

X and Y are tangent vectors, D is the canonical flat connection induced from $\mathbb{R}^{n+1}$, ∇ is a torsion-free connection, h is a symmetric form on $T_x(H)$, S is an endomorphism of $T_x(M)$, and τ is a one-form. (When it is not confusing, we will drop the f_* and just consider X and Y as vectors in $\mathbb{R}^{n+1}$.)

A good choice of transversal field ξ allows us to study the geometry of $f(M)$. For example, the standard metric on $\mathbb{R}^{n+1}$ allows us to define a normal vector field (at least if H is oriented). If ξ is this normal field, the first equation of (3.0.1) becomes the familiar Gauss equation in Riemannian geometry. ∇ in this case is the Levi-Civita connection on H with respect to the induced metric and h is the second fundamental form. This choice of ξ, since it respects the metric on $\mathbb{R}^{n+1}$, induces a lot of structure on H which is invariant under isometries of $\mathbb{R}^{n+1}$. In our case, we want to study properties invariant under the much larger unimodular affine group. Clearly this Riemannian normal field is not invariant under this group, but it turns out that there is a transversal vector field which is invariant, the affine normal.

3.1. **The affine normal.** Perhaps the easiest way to describe the affine normal is the following geometric characterization due to Blaschke [3]. At a point $x \in H$, consider hyperplanes $P(t) \in \mathbb{R}^{n+1}$ displaced a distance t from and parallel to $T_x(M)$. Since we assume M is locally strictly convex, for $t > 0$, $P(t) \cap H$ is the boundary of a convex domain $D(t) \subset P(t)$. Let $y(t) \in D(t) \subset \mathbb{R}^{n+1}$ be the center of gravity of $D(t)$. We would then like to define the affine normal to be $\xi = \frac{dy}{dt}\big|_{t=0}$, but since distance is not affine invariant, we cannot. To get around this, define

$$s(t) = \left(\mathrm{Vol} \bigcup_{0<\tau<t} D(\tau) \right)^{\frac{2}{n+2}}, \qquad c_n = 2 \left(\frac{V_n}{n+2} \right)^{\frac{2}{n+2}},$$

where V_n is the volume of the unit ball in $\mathbb{R}^n$. The exponent in the definition of s makes s approximately linear as a funtion of t, and c_n is

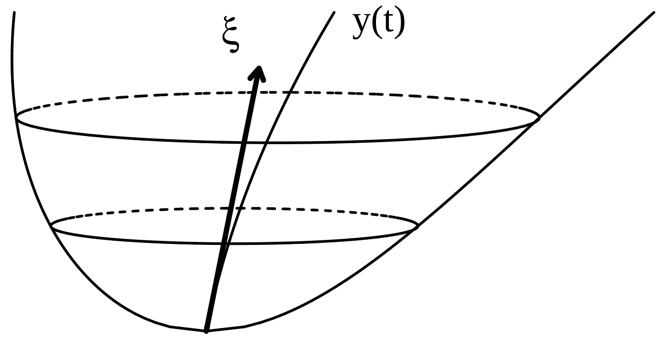

FIGURE 1. The Affine Normal

a volume-normalizing factor. Then the affine normal is defined to be $\xi = c_n \frac{dy}{ds}\Big|_{s=0}$. Notice that the affine normal points to the convex side of H. See Figure 1.

It is clear that this definition is invariant under the affine unimodular group, because this group preserves volumes. In fact it is invariant under the larger group given by all transformations $x \mapsto Ax + b$ with $A \in \mathbf{SL}^{\pm}(n+1, \mathbb{R})$.

3.2. **Three connections.** For the affine normal ξ, the structure equations for H (3.0.1) become

$$\left\{ \begin{array}{l} D_X Y = \nabla_X Y + h(X,Y)\xi, \\ D_X \xi = -SX. \end{array} \right. \tag{3.2.1}$$

The connection ∇ is called the *Blaschke connection*, or simply the *affine connection*. The bilinear form h is the *affine metric* and the endomorphism S is called the *affine shape operator*. Since H is strictly convex, h is a Riemannian metric on M. We can then also consider $\hat{\nabla}$ the Levi-Civita connection with respect to h. It is also useful to consider the *conjugate connection* $\bar{\nabla}$, which is defined to be the connection $2\hat{\nabla} - \nabla$. As we discuss below in Subsection 3.5, $\bar{\nabla}$ is always projectively flat.

Another important invariant is the Pick form, which is the tensor $C = \hat{\nabla} - \nabla$. In index notation, we have the following conditions

$$\left\{ \begin{array}{lll} \sum_i C^i_{ij} = 0 & \text{for all} & j, \\ C_{ijk} & \text{symmetric in} & i,j,k, \end{array} \right. \tag{3.2.2}$$

where we use h to lower the index. In addition, if C vanishes identically on H, then H must be an open subset of a hyperquadric in $\mathbb{R}^{n+1}$. Hyperquadrics are the trivial objects in affine differential geometry. There are three strictly convex cases—an ellipsoid, an elliptic paraboloid and one sheet of an elliptic hyperboloid. The affine metrics on these examples have constant curvature, which is respectively positive, zero, and negative.

We have the following curvature formula for the Blaschke connection ∇:

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY. \tag{3.2.3}$$

Also, the *affine mean curvature* is defined to be the quantity $\frac{1}{n}\mathrm{tr}S$. Another useful equation is the Codazzi equation for h:

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z). \tag{3.2.4}$$

3.3. **Affine spheres.** An affine sphere is a hypersurface in $\mathbb{R}^{n+1}$ all of whose affine normals point toward a given point in $\mathbb{RP}^{n+1}$, the *center* of the affine sphere. If the center lies on the convex side of H, the affine sphere is *elliptic*. If it lies on the line at infinity, it is called *parabolic*. If the center lies on the concave side of H, then the affine sphere is *hyperbolic*. We will be concerned exclusively with this case.

For an affine sphere, the shape operator satisfies $S = LI$, where the affine mean curvature L is a constant function on H and I is the identity map. L is positive, zero, or negative if H is elliptic, parabolic, or hyperbolic respectively. The center is given by the formula $x+\frac{1}{L}\xi(x)$, where x is any point in H. If H is an affine sphere, upon scaling $\mathbb{R}^{n+1}$ away from any point by a constant factor λ, the image of H after this transform remains an affine sphere, but the affine mean curvature becomes $\lambda^{-\frac{2}{n+2}}L$.

Thus by scaling, we can normalize any hyperbolic affine sphere to have $L=-1$. Also, we can translate so that the center is 0. Then the affine normal $\xi = f$, where f is the embedding of H into $\mathbb{R}^{n+1}$. The structure equations (3.2.1) and the curvature equation (3.2.3) then become

$$\left\{ \begin{array}{l} D_X Y = \nabla_X Y + h(X,Y)f \\ D_X f = X \\ R(X,Y)Z = -h(Y,Z)X + h(X,Z)Y \end{array} \right. \tag{3.3.1}$$

This implies the Ricci curvature of ∇ is given by

$$\mathrm{Ric}(X,Y) = (-n+1)h(X,Y) \tag{3.3.2}$$

The basic examples of affine spheres are the quadratic hypersurfaces introduced above. In fact, for the elliptic and parabolic cases, these are the only examples of affine spheres which are closed subsets of $\mathbb{R}^{n+1}$ [8].

Hyperbolic affine spheres are more complicated, however. The basic result for hyperbolic affine spheres was conjectured by Calabi [4] and proved by Cheng and Yau [6, 8], and Calabi and Nirenberg (with clarifications by Gigena [13], Sasaki [30] and A.-M. Li [25, 26]):

Theorem 3. *Conisder a convex, bounded domain $\Omega \subset \mathbb{R}^n$, where $\mathbb{R}^n$ is embedded in $\mathbb{R}^{n+1}$ as the affine space $x_{n+1} = 1$. Then, for any constant $L < 0$, there is a unique properly embedded hyperbolic affine sphere $H \subset \mathbb{R}^{n+1}$ of affine mean curvature L and center 0 asymptotic to the boundary of the cone $\{t\Omega : t > 0\} \subset \mathbb{R}^{n+1}$. For any immersed hyperbolic affine sphere $H \to \mathbb{R}^{n+1}$, properness of the immersion is equivalent to completeness of the affine metric, and any such H is a properly embedded hypersurface asymptotic to boundary of the cone given by the convex hull of H and its center.*

Outline of proof. By scaling, we may consider just the case $L = -1$.

The key to proving existence is to study the differential equation

$$\det(w_{ij}) = \left(-\frac{1}{w}\right)^{n+2}, \quad w|_{\partial\Omega} = 0 \tag{3.3.3}$$

Cheng and Yau in [6] prove that for each bounded, convex $\Omega \subset \mathbb{R}^n$, this Dirichlet problem has a unique C^∞ convex solution which is continuous to the boundary. (See also Loewner-Nirenberg [27] for earlier work.)

There are two affine spheres associated to the solution w. Calabi [4] observed that if α is the Legendre transform of w, then the (rectilinear) graph of α

$$\bar{H} = \{(x, \alpha(x))\} \subset \mathbb{R}_{n+1}$$

is an affine sphere in $\mathbb{R}_{n+1}$, the dual of $\mathbb{R}^{n+1}$, and this formulation (by choosing appropriate affine coordinates) locally determines all such affine spheres. The other affine sphere is given by

$$H = \left\{ \left(\frac{y}{w(y)}, -\frac{1}{w(y)} \right) : y \in \Omega \right\} \subset \mathbb{R}^{n+1}.$$

(Notice that $H = \pi(G)$, where π is the diagonal matrix with diagonal entries $(-1, \ldots, -1, 1)$ and G is the radial graph of $-\frac{1}{w}$.) H and $\bar{H}$ are mapped to each other by the conormal map (see Subsection 3.5 below), and H is an affine sphere by the duality result of Schirokov and Schirokov (Proposition 3.5.1 below) and Calabi's formulation. (The radial graph G is also an affine sphere since π is a linear map.) Note that it is clear that H is properly embedded in $\mathbb{R}^{n+1}$ and it is asymptotic to the cone over Ω. All this was known to experts in the 1970s, when Cheng and Yau completed their work, but this last observation did not appear in print until the paper of Gigena [13] (see also Sasaki [30] for a different argument).

The affine metric of these affine spheres is $-\frac{1}{w}w_{ij}$ in the coordinates induced from Ω.

Cheng and Yau in [8] prove (with a small gap) that an affine sphere is properly embedded if and only if its affine metric is complete. Moreover, any such hyperbolic affine sphere H must be asymptotic to the boundary of the cone given by the convex hull of H and its center. (Calabi and Nirenberg, in unpublished work, establish the same results.) In [25, 26], A.-M. Li clarified the proof of Cheng and Yau by using essentially the same estimates developed in [8]. □

Also, we have the following proposition that will be useful later.

Proposition 3.3.1. *The Blaschke connection of a hyperbolic affine sphere is projectively flat. Moreover, if the affine mean curvature is -1, the normal connection associated to ∇ can be realized as the canonical flat connection D induced from $\mathbb{R}^{n+1}$.*

Proof. The curvature equations (3.3.1) and (3.3.2), together with the Codazzi equation (3.2.4), when applied to the characterization of the normal connection in Subsection 2.3, provide the proof. (Here the trivial line bundle in Proposition 2.3.1 is spanned by the affine normal.)

We also provide a more geometric proof of the first part (see Nomizu-Sasaki [29, pp. 15-18]). Assume for simplicity that the affine mean curvature is -1 and the center is 0. Consider a positive smooth function λ on H and the new hypersurface $\check{H} = \lambda H \subset \mathbb{R}^{n+1}$. Then f is still a transversal vector field on $\check{H}$. Form the connection $\check{\nabla}$ by

$$D_X Y = \check{\nabla}_X Y + \check{h}(X,Y)f. \tag{3.3.4}$$

A simple computation shows

$$\check{\nabla}_X Y = \nabla_X Y + \rho(X)Y + \rho(Y)X, \quad \rho = d(\log \lambda).$$

Now we can take $\check{H}$ to be the intersection of a hyperplane with the cone to which H is asymptotic. Then (3.3.4) implies $\check{\nabla}$ is flat. Furthermore, the ∇-geodesics on H map to straight lines on $\check{H}$ under the map along rays from M to 0. See Figure 2. □

3.4. Quotients of hyperbolic affine spheres.

Let M be an $\mathbb{RP}^n$-manifold with oriented tautological bundle τ. Then the total space of the positive part of τ is locally a cone in $\mathbb{R}^{n+1}$, and, as in Proposition 2.2.1, the gluing maps from M lift to gluing maps in $\mathbf{SL}^{\pm}(n+1,\mathbb{R})$ to glue these cones together to form the positive part of the total space of τ. We say M *admits an affine sphere structure* if there is a positive section s of τ so that for each coordinate chart $\mathcal{U}$ of M, $s(\mathcal{U})$ is a

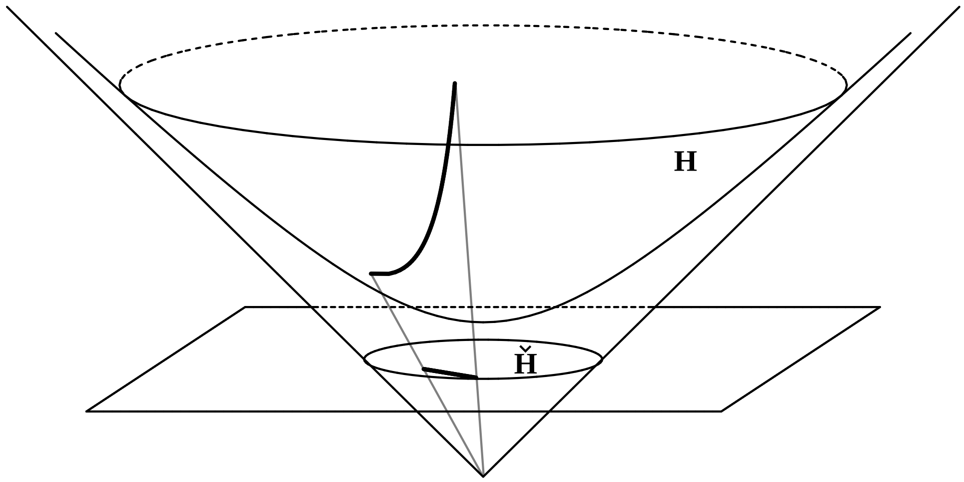

FIGURE 2

hyperbolic affine sphere with center 0 and affine mean curvature -1 in this cone.

Now we show that any properly convex $\mathbb{RP}^n$-manifold M admits an affine sphere structure. Write $M = \Omega/\Gamma$. Proposition 2.2.1 shows that M admits an oriented tautological bundle, and the action of Γ lifts to an action $\Gamma' \subset \mathbf{SL}^{\pm}(n+1,\mathbb{R})$ on the cone $C(\Omega)$. Theorem 3 shows there is a unique affine sphere asymptotic to the boundary of $C(\Omega)$ with center 0 and affine mean curvature -1. By uniqueness and the invariance of the affine normal under $\mathbf{SL}^{\pm}(n+1,\mathbb{R})$, this structure descends to the quotient, and M must admit an affine sphere structure. We record this result and related facts in the following theorem.

Theorem 4. *Let M be an $\mathbb{RP}^n$-manifold with oriented tautological bundle τ. Let τ^* denote the dual line bundle. The following are equivalent:*

1. *M is properly convex.*
2. *M admits a negative strictly convex section w of τ^* satisfying $\det(w_{ij}) = (-\frac{1}{w})^{n+2}$ so that the metric $-\frac{w_{ij}}{w}$ is complete.*
3. *M admits an affine sphere structure whose metric is complete.*

If any of these conditions are satisfied, then the $\mathbb{RP}^n$-structure on M is given by the Blaschke connection ∇. Also, the normal connection of ∇ is exactly the flat connection D on $TM \oplus \tau$ which is induced by the canonical flat connection on $\mathbb{R}^{n+1}$.

Proof. $2 \Leftrightarrow 3$ follows from the outline of the proof of Theorem 3, and $1 \Rightarrow 3$ is proved in the preceeding paragraphs.

We now prove $3 \Rightarrow 1$. An affine sphere structure on M induces one on the universal cover $\tilde{M}$. This is pushed down by the developing map to an affine sphere structure on a spread domain over $\mathbb{RP}^n$, which is equivalent, by looking at the total space of τ, to an immersed affine sphere in $\mathbb{R}^{n+1}$ with complete affine metric. Theorem 3 shows that this is asymptotic to a cone and we have $3 \Rightarrow 1$. □

Remark. Given two hyperbolic affine spheres $X \subset \mathbb{R}^{m+1}$ and $Y \subset \mathbb{R}^{n+1}$ asymptotic to cones C_1 and C_2, Calabi [4] has an explicit formula for the affine sphere in $\mathbb{R}^{m+n+2}$ asymptotic to $C_1 \times C_2$. If X and Y cover $\mathbb{RP}^n$-manifolds M and N, then this gives the affine sphere structures on $M \times N \times S^1$ corresponding to the convex $\mathbb{RP}^{m+n+1}$-structures constructed by Benzécri [2].

Remark. The canonical Kähler-Einstein metric on a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ studied by Cheng and Yau in [7] involves a Monge-Ampère equation similar to (3.3.3). If $g_{i\bar{j}}$ is the unique complete Kähler-Einstein metric on Ω normalized so that the Ricci $R_{i\bar{j}} = -g_{i\bar{j}}$, then we have a global Kähler potential function ϕ of $g_{i\bar{j}}$ satisfying

$$\det(\phi_{i\bar{j}}) = e^{-\phi}, \quad \phi_{i\bar{j}} > 0, \quad \phi|_{\partial\Omega} = \infty.$$

An affine sphere structure on a compact $\mathbb{RP}^n$-manifold is then analogous to the structure a Kähler-Einstein metric provides on a compact complex manifold with $c_1 < 0$. It is interesting to note by way of comparison that whenever we can define a complete affine sphere structure, Theorem 4 shows the universal cover is always a convex domain, but there are many compact simply-connected complex manifolds which admit negative Kähler-Einstein metrics.

3.5. **The conormal map.** Recall the definition of the conormal map. Let $H \subset \mathbb{R}^{n+1}$ be a hypersurface transverse to its position vector f. Define $\nu(x) \in \mathbb{R}_{n+1}$ by

$$\nu(x)(X) = 0 \quad \text{for} \quad X \in T_x(H), \quad \nu(x)(f) = 1. \tag{3.5.1}$$

(This construction can be done with any transverse vector field ξ in place of f.)

For any hypersurface $H \subset \mathbb{R}^{n+1}$ with f transverse to H, the image of the conormal map ν, is the dual hypersurface $\bar{H} \subset \mathbb{R}_{n+1}$. We have the following proposition found in Schirokov-Schirokov [31]. (There are also unpublished work of Calabi and the papers of Gigena [12, 13]).

Proposition 3.5.1. *The image of the conormal map of a hyperbolic affine sphere H with center 0 and affine mean curvature -1 is another such hyperbolic affine sphere $\bar{H}$ in the dual space $\mathbb{R}_{n+1}$.*

We call $\bar{H}$ the *dual sphere* of H.

Notice that in this case we have $f = \xi$ the affine normal. For conormal maps with respect to the affine normal, the conjugate connection $\bar{\nabla}$ and the metric h appear (see [29, p. 57])

$$D_X(\nu_* Y) = \nu_*(\bar{\nabla}_X Y) - h(SX, Y)\nu.$$

This shows, as in Proposition 3.3.1 above, that $\bar{\nabla}$ is always projectively flat. In our case, this equation becomes

$$D_X(\nu_* Y) = \nu_*(\bar{\nabla}_X Y) + h(X, Y)\nu.$$

Therefore, by comparison with (3.2.1),we have the following corollary:

Corollary 3.5.2. *The conormal map ν on H as above is an isometry with respect to the affine metrics. It takes the conjugate connection of H to the Blaschke connection of $\bar{H}$ and vice versa.*

Now consider the cone C formed by the convex hull of H and 0. Consider the *dual cone* $\bar{C} \subset \mathbb{R}_{n+1}$ consisting of all linear functionals y which are positive on C. Then $\bar{H}$ is asymptotic to the boundary of $\bar{C}$. Let C be the space of all rays in some $\Omega \subset \mathbb{RP}^n$. Similarly, projectivize $\bar{C}$ so it is the space of all rays in $\bar{\Omega} \subset \mathbb{RP}_n$. ($\mathbb{RP}_n$, the space of all lines in $\mathbb{R}_{n+1}$, is the *dual projective space*. $\bar{\Omega}$ is called the *projective dual region* to Ω.) Projecting along rays identifies H to Ω, and $\bar{H}$ to $\bar{\Omega}$. Therefore, the conormal map ν induces a map from Ω to $\bar{\Omega}$, which we also refer to as ν.

Now if $\Gamma \subset \mathbf{PGL}(n+1, \mathbb{R})$ acts on Ω, then we have a dual action on $\bar{\Omega}$: First lift the action of Γ to $\Gamma' \subset \mathbf{SL}^{\pm}(n+1, \mathbb{R})$ acting on the cone $C(\Omega)$ as in Proposition 2.2.1 above. Then for $y \in \mathbb{R}_{n+1}$ and $x \in \mathbb{R}^{n+1}$, define

$$(A \cdot y)(x) = y(A^{-1}x)$$

for $A \in \Gamma'$. This induces a projective action of Γ on $\bar{\Omega}$, which we denote by $\bar{\Gamma}$. The uniqueness of the affine sphere, the invariance of the affine normal under $\mathbf{SL}^{\pm}(n+1, \mathbb{R})$, and the definition (3.5.1) then show that ν is equivariant with respect to the action of Γ. Therefore, ν descends the the quotient and we have

Proposition 3.5.3. *Given a properly convex $\mathbb{RP}^n$-manifold $M = \Omega/\Gamma$, the conormal map ν with respect to the affine sphere structure induces a map to the dual manifold $\bar{M} = \bar{\Omega}/\bar{\Gamma}$. This map is an isometry of the affine metrics and interchanges the two projectively flat connections ∇ and $\bar{\nabla}$.*

Proof of Theorem 1. This is contained in Theorem 4 and the above proposition, together with basic facts about the Pick form in Subsection 3.2 above. ☐

3.6. **Hyperbolic affine spheres in $\mathbb{R}^3$.** C.-P. Wang formulates the condition for a two-dimensional surface to be the quotient of an affine sphere in $\mathbb{R}^3$ by an appropriate subgroup of $\mathbf{SL}(3,\mathbb{R})$ in terms of the conformal geometry given by the affine metric [34]. The details of this construction are also found in [28].

The main result we will need, expressed in the language of Subsection 3.4 above, is this

Proposition 3.6.1. *Let Σ be a Riemann surface, $g = e^{\phi}|dz|^2$ a metric, U a holomorphic section of K^3, and u a function on Σ which satisfies*

$$\Delta u + 4e^{-2u}\|U\|^2 - 2e^u - 2\kappa = 0, \tag{3.6.1}$$

where $\Delta = 4e^{-\phi}\partial_z\partial_{\bar z}$ is the Laplacian, $\|\cdot\|^2 = |\cdot|^2 e^{-3\phi}$ denotes the metric on K^3 induced by h, and $\kappa = -\frac{1}{2}\Delta\phi$ is the curvature. This induces a projectively flat connection on Σ and an associated affine sphere structure. The affine metric is given by $e^u g = e^{u+\phi}|dz|^2$, and the connection form of the (projectively flat) Blaschke connection is given by

$$\begin{pmatrix} \partial u + \partial\phi & \bar U e^{-u-\phi} d\bar z \\ U e^{-u-\phi} dz & \bar\partial u + \bar\partial\phi \end{pmatrix}$$

with respect to the basis $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar z}\}$ of the (complexified) tangent bundle.

Conversely, any affine sphere structure on an oriented surface induces by the affine metric a complex structure and a holomorphic section U of K^3 which satisfies (3.6.1).

Idea of proof. Let Δ be the universal cover of Σ. Consider maps $f : \Delta \to \mathbb{R}^3$ such that f is conformal with respect to the affine metric on $f(\Delta)$. Write the Blaschke connection in terms of the Levi-Civita connection and the Pick form. The symmetries of the Pick form (3.2.2) show it has only two linearly independent factors, which are realized as the real and complex parts of U. Also U transforms as a section of K^3. Then use the affine structure equations presented above to find necessary and sufficient conditions on f, u, and U so that f is an affine sphere immersion. We find that the structure equations of f are then an initial-value problem that can be solved if and only if certain integrability conditions are satisfied. These conditions are $U_{\bar z} = 0$ and equation (3.6.1). By uniqueness of the initial-value problem for f, we find that this structure must descend to the quotient Σ and that any deck transformation of Σ corresponds to an element in $\mathbf{SL}(3,\mathbb{R})$ acting on $f(\Delta) \subset \mathbb{R}^3$. □

4. The main theorem

Now we prove a result essentially due to Wang [34] that will allow us to determine the affine sphere structure on a given compact Riemann surface. We give a new proof since our techniques are simpler than Wang's original ones.

Proposition 4.0.2. *Let M be a compact manifold, g be a nonnegative C^∞ function on M, and Δ be the Laplacian with respect to a C^∞ Riemannian metric on M. Then the equation*

$$\Delta u + g(x)e^{-2u} - 2e^u + 2 = 0$$

has a unique C^∞ solution.

Proof. To show existence, by a standard result (see Schoen-Yau [32, Prop. V.1.1]), we only need to find a subsolution and a supersolution for the equation. It is straightforward to check that $s = 0$ is a subsolution. Also, if $G = \max g$, set m to be the positive root of the equation

$$2x^3 - 2x^2 - G = 0.$$

Then $s = \log m$ satisfies

$$\Delta s + g(x)e^{-2s} - 2e^s + 2 = g(x)m^{-2} - 2m + 2 \le 0,$$

and s is a supersolution. Smoothness follows by standard elliptic theory. Uniqueness follows from a standard maximum-principle argument, since $g(x)e^{-2u} - 2e^u + 2$ is strictly decreasing as a function of u. □

Proposition 4.0.3 (Wang [34])**.** *A conformal structure on a compact oriented surface S of genus $g \ge 2$ and a holomorphic section U of K^3 determine an affine sphere structure on S. All such affine sphere structures are determined in this way.*

Proof. Consider as a background metric the conformal metric on Σ with constant curvature -1. Then apply Proposition 4.0.2 and Proposition 3.6.1 above to construct the affine sphere structure. Conversely, the affine metric and the Pick form induce a conformal structure on Σ and a holomorphic section of K^3. □

Remark. This construction also works for surfaces of genus one [34, 28], and for certain noncompact surfaces [28].

We can now prove Theorem 2.

Proof of Theorem 2. Since S is compact, Theorem 4 and Proposition 4.0.3 then provide a convex $\mathbb{RP}^2$-structure on S. Any convex $\mathbb{RP}^2$-structure on such an S must be properly convex by Kuiper [21]. Then Theorem 4 and Proposition 4.0.3 provide the converse. □

Proposition 3.5.3 then gives us the following

Corollary 4.0.4. *If we replace the section U of K^3 by $-U$, we recover the projective dual surface (which is made by looking at the dual projective space of lines in $\mathbb{RP}^2$ and taking the dual gluing maps in the construction of the surface). The affine metric h is unchanged and the two projectively flat connections ∇ and $\bar{\nabla}$ are interchanged.*

When the Pick form $U = 0$, the affine sphere structure on the universal cover is the hyperboloid, and the Blaschke connection ∇ is the Levi-Civita connection coming from the metric of constant negative sectional curvature. Therefore, we have the $\mathbb{RP}^2$-structure is given by the hyperbolic structure on the disk, we recover the fact that Teichmüller space is exactly the fixed locus of the action of projective duality on $\mathcal{G}(S)$.

4.1. **Moduli problems.** In addition to the deformation space $\mathcal{G}(S)$ considered above, it is also useful to consider the moduli space of convex $\mathbb{RP}^2$-structures. While $\mathcal{G}(S)$ is given by projective equivalence classes of torsion-free, projectively flat connections modulo Diff_0, the identity component of the diffeomorphism group, for the *moduli space* of oriented $\mathbb{RP}^2$-structures, we replace Diff_0 by Diff^+, the group of all orientation-preserving diffeomorphisms. In other words, the moduli space is the quotient of $\mathcal{G}(S)$ by the mapping class group $\mathrm{Diff}^+/\mathrm{Diff}_0$.

Our main theorem immediately implies this corollary

Corollary 4.1.1. *The moduli space of convex $\mathbb{RP}^2$-structures on an oriented compact surface of genus $g \geq 2$ is equivalent to the moduli space of pairs (Σ, U), where Σ is a Riemann surface of genus g and $U \in H^0(\Sigma, K^3)$.*

Determining some basic facts about this space is an exercise in algebraic curve theory. We have the following proposition

Proposition 4.1.2. *Our moduli space, as a locally finite quotient of the deformation space, is a complex orbifold which is smooth on exactly those convex $\mathbb{RP}^2$-surfaces with no nontrivial automorphisms. The generic convex $\mathbb{RP}^2$-surface for each genus $g \geq 2$ has no nontrivial automorphisms. In fact, for a fixed complex structure on Σ of genus $g \geq 2$, the $\mathbb{RP}^2$-structure corresponding to a generic section in $H^0(\Sigma, K^3)$ has no automorphisms.*

Proof. The first statement follows from standard facts in algebraic curve theory, except for one point. The generic algebraic curve of genus 2 has a nontrivial automorphism, the hyperelliptic involution. We claim that a generic section of K^3 is not fixed by this involution.

In order to prove this claim, we use the Riemann-Hurwitz and Riemann-Roch formulas to calculate the dimension of the subspace of $H^0(\Sigma, K^3)$ fixed by a given automorphism of Σ. Let σ be an automorphism of Σ of order d. Then we consider the quotient Ξ, which is another smooth Riemann surface. The quotient map $Q : \Sigma \to \Xi$ has degree d, and it is branched exactly at the fixed points of powers of σ.

Consider a point p where Q is branched to order n. Then a simple local calculation shows that local sections of K^3_Σ fixed by Q near p correspond exactly to local sections in K^3_Ξ with a certain pole order allowed at $q = Q(p)$. If $n = 2$, then we allow poles of order 1, and if $n > 2$, we allow poles of order at most 2.

If Q has degree d and is branched over points p_i with branching order n_i, then we can use Riemann-Hurwitz to determine the genus g' of Ξ:

$$2g - 2 = d(2g' - 2) + \sum_i (n_i - 1).$$

Also, if q_j are the images of the p_i under Q, m_j the allowed pole order at q_j, and D is line bundle determined by the divisor $\prod_j [q_j]^{m_j}$, Riemann-Roch gives us

$$\dim_{\mathbb{C}} H^0(\Xi, K^3 D) - \dim_{\mathbb{C}} H^0(\Xi, K^{-2} D^{-1}) = 5g' - 5 + \sum_j m_j.$$

These and other similar numerical statements (see e.g. [1, p. 45]) can be used to prove that if $g \geq 2$ then

$$\dim_{\mathbb{C}} H^0(\Xi, K^3 D) \leq \max\{\tfrac{5}{2}(g-1), \tfrac{7}{3}(g-1) + 2\}.$$

Notice that this bound is always less than $5g - 5$; therefore, a generic section U of K^3 is not fixed by any automorphism σ of Σ.

In the particular case where σ is the hyperelliptic involution of a curve of genus 2, the map Q is a double cover of $\mathbb{CP}^1$ branched over 6 points. Then $K^3 = \mathcal{O}(-6)$, $D = \mathcal{O}(6)$, and

$$\dim_{\mathbb{C}} H^0(\Xi, K^3 D) = \dim_{\mathbb{C}} H^0(\mathbb{CP}^1, \mathcal{O}) = 1.$$

Thus there is only a 1-dimensional subspace of the 5-dimensional space $H^0(\Sigma, K^3)$ which is fixed by σ. □

References

[1] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of Algebraic Curves: Volume I.* Springer-Verlag, 1985.

[2] J. P. Benzécri. Sur les variétés localement affines et projectives. *Bullétin de la Société Mathématique de France*, 88:229–332, 1960.

[3] W. Blaschke. *Vorlesungen über Differentialgeometrie II, Affine Differentialgeometrie.* Springer-Verlag, 1923.

[4] E. Calabi. Complete affine hyperspheres I. *Instituto Nazionale di Alta Matematica Symposia Mathematica*, 10:19–38, 1972.
[5] E. Calabi. Géometrie différentielle affine des hypersurfaces. *Séminaire Bourbaki*, 573:1–16, 1980/81.
[6] S.-Y. Cheng and S.-T. Yau. On the regularity of the Monge-Ampère equation $\det((\partial^2 u/\partial x^i \partial x^j)) = F(x,u)$. *Communications on Pure and Applied Mathematics*, 30:41–68, 1977.
[7] S.-Y. Cheng and S.-T. Yau. On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation. *Communications on Pure and Applied Mathematics*, 33(4):507–544, 1980.
[8] S.-Y. Cheng and S.-T. Yau. Complete affine hyperspheres. part I. The completeness of affine metrics. *Communications on Pure and Applied Mathematics*, 39(6):839–866, 1986.
[9] S. Choi and W. M. Goldman. Convex real projective structures on closed surfaces are closed. *Proceedings of the American Mathematical Society*, 118(2):657–661, 1993.
[10] S. Choi and W. M. Goldman. The classification of real projective structures on compact surfaces. *Bulletin (New Series) of the American Mathematical Society*, 34(2):161–171, 1997.
[11] M.-R. Darvishzadeh and W. M. Goldman. Deformation spaces of convex real projective and hyperbolic affine structures. *Journal of the Korean Mathematical Society*, 33:625–638, 1996.
[12] S. Gigena. Integral invariants of convex cones. *Journal of Differential Geometry*, 13:191–222, 1978.
[13] S. Gigena. On a conjecture by E. Calabi. *Geometriae Dedicata*, 11:387–396, 1981.
[14] W. M. Goldman. Projective geometry on manifolds. Lecture Notes for Mathematics 748B, Spring 1988, University of Maryland.
[15] W. M. Goldman. Convex real projective structures on compact surfaces. *Journal of Differential Geometry*, 31:791–845, 1990.
[16] W. M. Goldman. The symplectic geometry of affine connections on surfaces. *Journal für die reine und agnewandte Mathematik*, 407:126–159, 1990.
[17] R. Hermann. *Gauge Fields and Cartan-Ehresmann Connections, Part A*. Math-Sci Press, 1975.
[18] N. J. Hitchin. Lie groups and Teichmüller space. *Topology*, 31(3):449–473, 1992.
[19] S. Kobayashi. *Transformation Groups in Differential Geometry*. Springer-Verlag, 1972.
[20] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry: Volume I*. John Wiley and Sons, Inc., 1963.
[21] N. Kuiper. On convex locally-projective spaces. In *Convegno Int. Geometria Diff.*, pages 200–213, Italy, 1954.
[22] F. Labourie. Private communication.
[23] F. Labourie. $\mathbb{RP}^2$-structures et differentielles cubiques holomorphes. Preprint, 1995.
[24] F. Labourie. in Proceedings of the GARC Conference in Differential Geometry, Seoul National University, Fall 1997, 1997.
[25] A.-M. Li. Calabi conjecture on hyperbolic affine hyperspheres. *Mathematische Zeitschrift*, 203:483–491, 1990.

[26] A.-M. Li. Calabi conjecture on hyperbolic affine hyperspheres (2). *Mathematische Annalen*, 293:485–493, 1992.

[27] C. Loewner and L. Nirenberg. Partial differential equations invariant under conformal or projective transformations. In *Contributions to Analysis*, pages 245–272. Academic Press, 1974.

[28] J. C. Loftin. *Applications of Affine Differential Geometry to $\mathbb{RP}^2$ Surfaces.* PhD thesis, Harvard University, June 1999.

[29] K. Nomizu and T. Sasaki. *Affine Differential Geometry: Geometry of Affine Immersions.* Cambridge University Press, 1994.

[30] T. Sasaki. Hyperbolic affine hypersheres. *Nagoya Mathematical Journal*, 77:107–123, 1980.

[31] P. A. Schirokov and A. P. Schirokov. *Affine Differentialgeometrie.* Teubner, 1962.

[32] R. Schoen and S.-T. Yau. *Lectures on Differential Geometry.* International Press, 1994.

[33] J. A. Schouten. *Ricci-Calculus.* Springer-Verlag, 1954.

[34] C.-P. Wang. Some examples of complete hyperbolic affine 2-spheres in $\mathbb{R}^3$. In *Global Differential Geometry and Global Analysis*, volume 1481 of *Lecture Notes in Mathematics*, pages 272–280. Springer-Verlag, 1991.