

# SINGULAR SEMI-FLAT CALABI-YAU METRICS ON $S^2$

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## 1. INTRODUCTION

This paper is motivated by recent work of Gross and Wilson [14], in which they construct degenerate limits of families of K3 surfaces equipped with Calabi-Yau metrics (Ricci-flat Kähler metrics). Upon proper rescaling, the metric limit of such a family is a two-dimensional sphere equipped with a Riemannian metric with prescribed singularities at 24 points. Away from singularities, this limit metric is an affine Kähler metric. In other words, there are natural affine flat coordinates  $(\alpha^j)$  and a local potential function  $\phi$  so that the metric is given by

$$(1) \quad \frac{\partial^2 \phi}{\partial \alpha^j \partial \alpha^k} d\alpha^j d\alpha^k.$$

Moreover, the potential  $\phi$  satisfies the Monge-Ampère equation

$$(2) \quad \det \frac{\partial^2 \phi}{\partial \alpha^j \partial \alpha^k} = 1.$$

In this case, the metric is naturally a real slice of a Calabi-Yau metric. We refer to such a metric as a *semi-flat Calabi-Yau metric*.

Such singular semi-flat Calabi-Yau metrics on surfaces were first constructed by Greene-Shapere-Vafa-Yau [11]. We construct many examples of such metrics. Our main theorem is this:

**Theorem 1.** *Given any holomorphic cubic differential  $U$  on  $\mathbb{CP}^1$  which has poles of order 1 at a finite number of points  $p_j$  and is not identically zero, there exists an affine flat structure and a semi-flat Calabi-Yau metric on  $\mathbb{CP}^1 \setminus \{p_j\}$ . The singularities of the affine flat structure and metric at the  $p_j$  are asymptotically the same as those in [14].*

The details of the nature of the singularities are given in Theorem 4 below.

A semi-flat Calabi-Yau metric can naturally be seen as a real slice of a Calabi-Yau metric. For a Kähler metric given by a potential function  $\Phi$ , the equation for the metric to be Ricci flat is

$$\partial \bar{\partial} \log \det \Phi_{j\bar{k}} = 0.$$

A convex function  $\phi$  on a domain  $\Omega \subset \mathbb{R}^n$  can be extended to be constant along the imaginary fibers of the tube domain  $\Omega + i\mathbb{R}^n \subset \mathbb{C}^n$ . Then up to a constant factor, the Kähler metric  $\phi_{j\bar{k}} dz^j \overline{dz^k}$  extends the affine Kähler metric  $\phi_{jk} d\alpha^j d\alpha^k$  on  $\Omega$ . If in addition  $\phi$  satisfies  $\det \phi_{jk} = 1$ , then  $\phi_{j\bar{k}} dz^j \overline{dz^k}$  is Calabi-Yau. This Kähler metric is commonly known as semi-flat since it is flat along the imaginary fibers. We then extend the terminology a small bit to call the real metric  $\phi_{jk} d\alpha^j d\alpha^k$  semi-flat Calabi-Yau in this case. The conjecture of Strominger-Yau-Zaslow [31] (as explicated by Hitchin, Gross, Leung and others) implies that in this degenerate limit, mirror symmetry reduces to the Legendre transform of the affine Kähler potential function  $\phi$ .

The graph in  $\mathbb{R}^{n+1}$  of a convex function  $\phi$  satisfying (2) is a hypersurface classically studied in affine differential geometry, a parabolic affine sphere. The study of such surfaces dates back to T̃ițeica and Blaschke. A parabolic affine sphere whose metric (1) is complete must be the graph of a quadratic polynomial (and thus the metric is flat). This was proved by Jörgens [20] in the case  $n = 2$  and by Calabi [1] in higher dimensions. Cheng and Yau [4] studied affine Kähler manifolds and produced semi-flat Calabi-Yau metrics on many compact manifolds (on any compact affine Kähler manifold which admits a volume form covariantly constant with respect to the the canonical affine flat connection). We note affine Kähler metrics are also studied in the works of Shima e.g. [29], where they are called Hessian metrics. Cheng and Yau observed that Calabi's estimates imply all semi-flat Calabi-Yau metrics on a compact manifold are flat. Thus in the present work it is important that the metric we produce is not complete near the singularities: if it were, then Calabi's theorem would imply that it is flat. We should also mention that the Bernstein problem has for parabolic affine spheres has also been solved by Jörgens ( $n = 2$ ), Calabi ( $n = 3, 4, 5$ ), Pogorelov [28], and Cheng-Yau [5].

Given an affine Kähler metric with local potential function  $\phi$ , the affine Kähler metric transforms as a tensor under affine flat coordinate changes. A manifold  $M$  built of coordinate charts in  $\mathbb{R}^n$  with gluing maps locally constant maps in  $\mathbf{GL}(n, \mathbb{R}) \times \mathbb{R}^n$  is called an affine flat manifold. This system of canonical affine flat coordinates is equivalent to the existence of a torsion-free flat connection  $\nabla$  on the tangent bundle (the coordinate vector fields  $\partial/\partial\alpha^j$  are parallel with respect to this connection). In our case, equation (2) demands a little more structure on the manifold  $M$ . The 1 on the right side of (2) is parallel under  $\nabla$  and is the square of a volume form. Thus in general there is a parallel density on  $M$ , and if  $M$  is oriented (as are the examples in this

paper)  $M$  admits a parallel volume form. Then the gluing maps are elements of  $\mathbf{SL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$ . We can think of  $\nabla$  as an affine connection (a connection on a principal bundle modelled on the affine group  $\mathbf{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$ ). Thus we have a natural holonomy representation  $\gamma$  on an affine flat manifold  $M$  with parallel volume form:

$$\gamma: \pi_1 M \rightarrow \mathbf{SL}(n, \mathbb{R}) \ltimes \mathbb{R}^n.$$

In the present work, we calculate the conjugacy class of the holonomy around each singular point.

In dimension  $n = 2$ , there are two special techniques for analyzing parabolic affine spheres. Both use the conformal structure induced by the affine metric (1). The first involves a semilinear equation used by Simon and Wang [30] to perform a conformal change to find the affine metric. (We call this semilinear equation  $\mathbb{T}$ ıteica's equation.) Then Simon and Wang introduce an initial-value problem (a developing map) to deduce the local structure of the parabolic affine sphere. The second technique, which goes back to Blaschke, is a representation of the parabolic affine sphere in terms of two holomorphic functions. We recall both these theories, as well as some general facts about parabolic affine spheres and affine flat coordinates, in Section 2 below.

In Section 3, we introduce a model metric solution to  $\mathbb{T}$ ıteica's equation near each first-order pole of a cubic differential  $U$  on  $\mathbb{CP}^1$ . This model comes from Gross-Wilson [14]. Then we proceed to perturb the model metric by a conformal factor  $e^u$  and get asymptotic bounds on  $u$  near each singularity.

Then in Section 4, using the bounds on  $u$ , we analyze the affine flat structure induced on  $M = \mathbb{CP}^1 \setminus \{p_j\}$  induced by the parabolic affine sphere we've constructed. In particular, we use Simon-Wang's developing map and techniques of ODEs to calculate the holonomy and other natural invariants of affine flat structure. (This basic plan of first solving for a conformal factor and then applying a developing map and ODE techniques to characterize the relevant geometric structure near a singular point on a surface was first carried out in [24], where asymptotics for singular convex real projective structures were investigated using hyperbolic affine spheres.) Finally we use Blaschke's holomorphic characterization to find precise asymptotics of the metric and the affine flat structure.

Then in Section 5, we recall Leung's picture of mirror symmetry without correction terms [23, 22], and we write down the manner in which mirror symmetry should work in this degenerate limit via the Legendre transform.

We note that we are not able to reproduce one relevant feature required by mirror symmetry, the integrality of the affine holonomy. In the picture of Strominger-Yau-Zaslow, the total space of a Calabi-Yau manifold is formed by a fibration of special Lagrangian tori over a singular affine flat manifold  $B$ . (In the present work,  $B$  is  $S^2 = \mathbb{C}\mathbb{P}^1$  with singularities at the  $p_j$ .) These tori are naturally quotients of the imaginary fibers of the tube domain construction above. In order for such a quotient to make sense globally on an affine flat manifold, the linear part of holonomy should be integral (so that there is a lattice in the tangent bundle preserved by  $\nabla$ ). In other words, the holonomy representation should be conjugate to one in the group  $\mathbf{SL}(2, \mathbb{Z}) \ltimes \mathbb{R}^2$ . In this paper we determine the holonomy only near the singular points, and it seems the methods of this paper are insufficient to determine such a global integrality condition. On the other hand, there are combinatorial constructions of integral affine manifolds with singularities due to Haase-Zharkov [15] and Gross-Siebert [13, 12], which discuss mirror symmetry from combinatorial and algebro-geometric points of view. Haase-Zharkov [16] also construct affine Kähler metrics on their examples, but these do not satisfy the Monge-Ampère equation. Recently Zharkov [32] has conjectured a detailed picture of how Calabi-Yau metrics degenerate to semi-flat Calabi-Yau metrics.

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## 2. PARABOLIC AFFINE SPHERES

**2.1. Affine flat coordinates.** Given any locally strictly convex immersed hypersurface  $H \subset \mathbb{R}^{n+1}$ , there is a natural transversal vector field  $\xi = \xi_H$  which is invariant under affine volume-preserving automorphisms of  $\mathbb{R}^{n+1}$  (elements of the group  $\mathbf{SL}(n+1, \mathbb{R}) \ltimes \mathbb{R}^{n+1}$ ).  $\xi$  is called the *affine normal* to  $H$ . In other words, if  $\Psi \in \mathbf{SL}(n+1, \mathbb{R}) \ltimes \mathbb{R}^{n+1}$  and  $p \in H$ , then

$$\Psi_* \xi_H(p) = \xi_{\Psi(H)}(\Psi(p)).$$

$H$  is called a *parabolic affine sphere* if  $\xi$  is a constant vector. It is standard to choose coordinates on  $\mathbb{R}^{n+1}$  so that  $\xi = (0, \dots, 0, 1)$  in this case.

In these coordinates  $H$  can be locally represented as the graph of a strictly convex function  $\phi$ —so that  $H = \{(\alpha, \phi(\alpha))\}$  for  $\alpha$  in a domain in  $\mathbb{R}^n$ . The condition that  $H$  be a parabolic affine sphere is then the real Monge-Ampère equation

$$\det \frac{\partial^2 \phi}{\partial \alpha^j \partial \alpha^k} = 1.$$

The Legendre transform is quite natural in this context. Recall that if  $\beta_j = \partial \phi / \partial \alpha^j$ , then the Legendre transform  $\chi$  of  $\phi$  is given by

$$\chi + \phi = \beta_j \alpha^j.$$

(We use the usual summation convention.) We primarily think of  $\chi$  as a function of the  $\beta_j$ , which are coordinates on the dual vector space  $\mathbb{R}_n$  of  $\mathbb{R}^n$ . The graph  $(\beta, \chi) \subset \mathbb{R}_{n+1}$  is again a parabolic affine sphere.

The natural group action in this setting consists of those elements  $\Psi \in \mathbf{SL}(n+1, \mathbb{R}) \times \mathbb{R}^{n+1}$  which preserve  $\xi$  in the sense that  $\Psi_* \xi = \xi$ . In other words, we are interested in the group  $G$  of transformations of the form

$$(\tilde{\alpha} \ \tilde{\gamma}) = \Psi(\alpha \ \gamma) = (\alpha \ \gamma) \begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix} + (b \ d),$$

where  $\alpha$  and  $b$  are row vectors in  $\mathbb{R}^n$ ,  $c$  is a column vector,  $\gamma$  and  $d \in \mathbb{R}$ , and  $A \in \mathbf{SL}(n, \mathbb{R})$ .

The Legendre transform is natural with respect to these coordinates. In terms of the  $(\tilde{\alpha} \ \tilde{\gamma})$  coordinates,  $H$  is the graph of a function  $\tilde{\phi}$  so that  $\tilde{\gamma} = \tilde{\phi}(\tilde{\alpha})$ . Form the Legendre transform:  $\tilde{\beta}_j = \partial \tilde{\phi} / \partial \tilde{\alpha}^j$  and  $\tilde{\chi} + \tilde{\phi} = \tilde{\beta}_j \tilde{\alpha}^j$ . Then it is straightforward to verify that

$$\begin{pmatrix} \tilde{\beta} \\ \tilde{\chi} \end{pmatrix} = \begin{pmatrix} A^{-1} & 0 \\ bA^{-1} & 1 \end{pmatrix} \begin{pmatrix} \beta \\ \chi \end{pmatrix} + \begin{pmatrix} A^{-1}c \\ bA^{-1}c - d \end{pmatrix}.$$

This verifies that the Legendre transform naturally transforms under the action of  $G$ . Moreover, the Legendre transform coordinates  $\tilde{\alpha}$  are independent of the potential function  $\phi$ .

**2.2. Simon-Wang's developing map.** U. Simon and C.P. Wang [30] formulate the condition for a two-dimensional surface to be an affine sphere in terms of the conformal geometry given by the affine metric. Since we rely heavily on this work, we give a version of the arguments here for the reader's convenience. For basic background on affine differential geometry, see Calabi [2], Cheng-Yau [5] and Nomizu-Sasaki [25].

Consider a 2-dimensional parabolic affine sphere in  $\mathbb{R}^3$ . Then the affine metric gives a conformal structure, and we choose a local conformal coordinate  $z = x + iy$  on the hypersurface. Then the affine metric

is given by  $h = e^\psi |dz|^2$  for some function  $\psi$ . Parametrize the surface by  $f : \mathcal{D} \rightarrow \mathbb{R}^3$ , with  $\mathcal{D}$  a domain in  $\mathbb{C}$ . Since  $\{e^{-\frac{1}{2}\psi} f_x, e^{-\frac{1}{2}\psi} f_y\}$  is an orthonormal basis for the tangent space, the affine normal  $\xi$  must satisfy this volume condition (see e.g. [25])

$$(3) \quad \det(e^{-\frac{1}{2}\psi} f_x, e^{-\frac{1}{2}\psi} f_y, \xi) = 1,$$

which implies

$$(4) \quad \det(f_z, f_{\bar{z}}, \xi) = \frac{1}{2} i e^\psi.$$

Now only consider parabolic affine spheres. In this case, the affine normal  $\xi$  is a constant vector, and we have

$$(5) \quad \begin{cases} D_X Y = \nabla_X Y + h(X, Y) \xi \\ D_X \xi = 0 \end{cases}$$

Here  $D$  is the canonical flat connection on  $\mathbb{R}^3$ ,  $\nabla$  is a flat connection, and  $h$  is the affine metric.

It is convenient to work with complexified tangent vectors, and we extend  $\nabla$ ,  $h$  and  $D$  by complex linearity. Consider the frame for the tangent bundle to the surface  $\{e_1 = f_z = f_*(\frac{\partial}{\partial z}), e_{\bar{1}} = f_{\bar{z}} = f_*(\frac{\partial}{\partial \bar{z}})\}$ . Then we have

$$(6) \quad h(f_z, f_z) = h(f_{\bar{z}}, f_{\bar{z}}) = 0, \quad h(f_z, f_{\bar{z}}) = \frac{1}{2} e^\psi.$$

Consider  $\theta$  the matrix of connection one-forms

$$\nabla e_j = \theta_j^k e_k, \quad j, k \in \{1, \bar{1}\},$$

and  $\hat{\theta}$  the matrix of connection one-forms for the Levi-Civita connection. By (6)

$$(7) \quad \hat{\theta}_1^1 = \hat{\theta}_{\bar{1}}^{\bar{1}} = 0, \quad \hat{\theta}_1^{\bar{1}} = \partial\psi, \quad \hat{\theta}_{\bar{1}}^1 = \bar{\partial}\psi.$$

The difference  $\hat{\theta} - \theta$  is given by the Pick form. We have

$$\hat{\theta}_\ell^j - \theta_\ell^j = C_{\ell k}^j \rho^k,$$

where  $\{\rho^1 = dz, \rho^{\bar{1}} = d\bar{z}\}$  is the dual frame of one-forms. Now we differentiate (4) and use the structure equations (5) to conclude

$$\theta_1^1 + \theta_{\bar{1}}^{\bar{1}} = d\psi.$$

This implies, together with (7), the apolarity condition

$$C_{1k}^1 + C_{\bar{1}k}^{\bar{1}} = 0, \quad k \in \{1, \bar{1}\}.$$

Then, when we lower the indices, the expression for the metric (6) implies that

$$C_{\bar{1}1k} + C_{1\bar{1}k} = 0.$$

Now  $C_{jkl}$  is totally symmetric on three indices [5, 25]. Therefore, the previous equation implies that all the components of  $C$  must vanish except  $C_{111}$  and  $C_{\bar{1}\bar{1}\bar{1}} = \overline{C_{111}}$ .

This discussion completely determines  $\theta$ :

$$(8) \quad \begin{pmatrix} \theta_1^1 & \theta_{\bar{1}}^1 \\ \theta_{\bar{1}}^1 & \theta_1^1 \end{pmatrix} = \begin{pmatrix} \partial\psi & C_{\bar{1}\bar{1}}^1 d\bar{z} \\ C_{11}^{\bar{1}} dz & \bar{\partial}\psi \end{pmatrix} = \begin{pmatrix} \partial\psi & \bar{U}e^{-\psi}d\bar{z} \\ Ue^{-\psi}dz & \bar{\partial}\psi \end{pmatrix},$$

where we define  $U = C_{11}^{\bar{1}}e^\psi$ .

Recall that  $D$  is the canonical flat connection induced from  $\mathbb{R}^3$ . (Thus, for example,  $D_{f_z}f_z = D_{\frac{\partial}{\partial z}}f_z = f_{zz}$ .) Using this statement, together with (6) and (8), the structure equations (5) become

$$(9) \quad \begin{cases} f_{zz} = \psi_z f_z + Ue^{-\psi}f_{\bar{z}} \\ f_{\bar{z}\bar{z}} = \bar{U}e^{-\psi}f_z + \psi_{\bar{z}}f_{\bar{z}} \\ f_{z\bar{z}} = \frac{1}{2}e^\psi\xi \end{cases}$$

Then, together with the equations  $\xi_z = \xi_{\bar{z}} = 0$ , these form a linear first-order system of PDEs in  $\xi$ ,  $f_z$  and  $f_{\bar{z}}$ :

$$(10) \quad \frac{\partial}{\partial z} \begin{pmatrix} \xi \\ f_z \\ f_{\bar{z}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \psi_z & Ue^{-\psi} \\ \frac{1}{2}e^\psi & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ f_z \\ f_{\bar{z}} \end{pmatrix},$$

$$(11) \quad \frac{\partial}{\partial \bar{z}} \begin{pmatrix} \xi \\ f_z \\ f_{\bar{z}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2}e^\psi & 0 & 0 \\ 0 & \bar{U}e^{-\psi} & \psi_{\bar{z}} \end{pmatrix} \begin{pmatrix} \xi \\ f_z \\ f_{\bar{z}} \end{pmatrix}.$$

In order to have a solution of the system (9), the only condition is that the mixed partials must commute (by the Frobenius theorem). Thus we require

$$(12) \quad \begin{aligned} \psi_{z\bar{z}} + |U|^2 e^{-2\psi} &= 0, \\ U_{\bar{z}} &= 0. \end{aligned}$$

The system (9) is an initial-value problem, in that given (A) a base point  $z_0$ , (B) initial values  $f(z_0) \in \mathbb{R}^3$ ,  $f_z(z_0)$  and  $f_{\bar{z}}(z_0) = \overline{f_z(z_0)}$ , and (C)  $U$  holomorphic and  $\psi$  which satisfy (12), we have a unique solution  $f$  of (9) as long as the domain of definition  $\mathcal{D}$  is simply connected. We then have that the immersion  $f$  satisfies the structure equations (5). In order for  $\xi$  to be the affine normal of  $f(\mathcal{D})$ , we must also have the volume condition (4), i.e.  $\det(f_z, f_{\bar{z}}, \xi) = \frac{1}{2}ie^\psi$ . We require this at the base point  $z_0$  of course:

$$(13) \quad \det(f_z(z_0), f_{\bar{z}}(z_0), \xi) = \frac{1}{2}ie^{\psi(z_0)}.$$

Then use (9) to show that the derivatives with respect to  $z$  and  $\bar{z}$  of  $\det(f_z, f_{\bar{z}}, \xi)e^{-\psi}$  must vanish. Therefore the volume condition is

satisfied everywhere, and  $f(\mathcal{D})$  is a parabolic affine sphere with affine normal  $\xi$ .

Using (9), we compute

$$(14) \quad \det(f_z, f_{zz}, \xi) = \frac{1}{2}iU,$$

which implies that  $U$  transforms as a section of  $K^3$ , and  $U_{\bar{z}} = 0$  means it is holomorphic. [Note that equation (12) is in local coordinates. In other words, if we choose a local conformal coordinate  $z$ , then the Pick form  $U = U dz^3$ , and the metric is  $h = e^\psi |dz|^2$ . Then plug  $U$ ,  $\psi$  into (12).]

**2.3. Blaschke's holomorphic representation.** It was known to Blaschke that two-dimensional parabolic affine spheres may be represented by two holomorphic functions (much as minimal surfaces can). Yau and Zaslow have recently related this representation to the stringy cosmic string model [11] and Hitchin's work on the moduli space of special Lagrangian submanifolds [19]. We mention two fairly recent generalizations of this: Calabi found that affine maximal surfaces in  $\mathbb{R}^3$  may be represented by holomorphic data [3] (an affine maximal surface is one whose area with respect to the affine metric is at a critical point; parabolic affine spheres are prominent examples). Also Cortés has described special Kähler metrics in higher dimensions by holomorphic functions [6] (each special Kähler metric locally lives on a parabolic affine sphere of real dimension  $2n$ ).

We recall the version of Blaschke's result in Ferrer-Martínez-Milán [7]. A parabolic affine sphere with affine normal  $\xi = (0, 0, 1)$  is given locally by the graph  $\{(\alpha^1, \alpha^2, \phi(\alpha^1, \alpha^2))\}$ , where  $\phi$  is a convex function satisfying the Monge-Ampère equation  $\det \frac{\partial^2 \phi}{\partial \alpha^j \partial \alpha^k} = 1$ . Define

$$(15) \quad G = \left( \alpha^1 + \frac{\partial \phi}{\partial \alpha^1} \right) + i \left( \alpha^2 + \frac{\partial \phi}{\partial \alpha^2} \right),$$

$$(16) \quad F = \left( \alpha^1 - \frac{\partial \phi}{\partial \alpha^1} \right) + i \left( -\alpha^2 + \frac{\partial \phi}{\partial \alpha^2} \right).$$

$F$  and  $G$  are holomorphic with respect to the local conformal coordinate  $z$  introduced above. Moreover the affine metric

$$(17) \quad e^\psi |dz|^2 = \frac{1}{4}(|dG|^2 - |dF|^2)$$

(so note that  $dG \neq 0$  and  $|dG| > |dF|$  everywhere).

We will also need a formula for the cubic form  $U$  in terms of  $F$  and  $G$ . Note that  $f = (\alpha^1, \alpha^2, \phi)$ ,  $\xi = (0, 0, 1)$ . Use (15-16) to write  $\alpha^1$  and  $\alpha^2$  in terms of  $F$  and  $G$ . Then compute using (4)

$$(18) \quad U = -2i \det(f_z, f_{zz}, \xi) = \frac{1}{4}(G_z F_{zz} - F_z G_{zz}).$$

## 3. SOLVING TITEICA'S EQUATION

Let  $U$  be a meromorphic section of  $K^3$  over  $\mathbb{CP}^1$  with poles of order 1 at points  $p_j \in \mathbb{CP}^1$ , and no other poles. Assume  $U$  is not identically zero. It is easy to see that the number of poles is at least 6. Then we want a metric  $e^\psi |dz|^2$  so that

$$\psi_{z\bar{z}} + |U|^2 e^{-2\psi} = 0,$$

in local coordinates on  $M = \mathbb{CP}^1 \setminus \{p_j\}$ . If we choose a background metric  $h$  on  $M$  and then solve for a conformal factor  $e^u$  so that  $e^\psi |dz|^2 = e^u h$ , the equation becomes

$$(19) \quad L_h(u) = \Delta_h u + 4e^{-2u} \|U\|_h^2 - 2\kappa_h = 0.$$

Here  $\Delta_h$  is the Laplacian,  $\|\cdot\|_h$  denotes the metric on  $K^3$ , and  $\kappa_h$  is the Gauss curvature. Note that for another conformal metric  $k = e^v h$ ,

$$(20) \quad L_k(u) = e^{-v} L_h(u + v).$$

Near each pole  $p_j$ , we can always choose a holomorphic coordinate  $z = z_j$  so that  $p_j = \{z = 0\}$  and  $U = \frac{1}{z} dz^3$  near  $z = 0$ . For each  $p_j$ , we call this coordinate the *canonical holomorphic coordinate*. Then  $\psi = \log |\log |z|^2|$  solves equation (12). This will be our local model near  $z = 0$ .

**3.1. Barriers.** We will solve equation (19) on  $M$  by construction upper and lower barriers. We start with the lower barrier  $s$ , which is easier to construct. Let the background metric be a smooth conformal metric  $h$  so that

$$(21) \quad h = |\log |z_j|^2| |dz_j^2|$$

for the canonical holomorphic coordinate  $z_j$  in a neighborhood of each singularity  $p_j$ .

First, conformally modify  $h$  so that  $\tilde{h} = e^v h$  has negative curvature in a neighborhood of each zero of  $U$  and  $v$  is compactly supported in  $M$  (i.e. supported away from the  $p_j$ ).

Near each  $p_j$ , consider  $z = z_j$ ,  $r = |z|$ ,  $\alpha < 0$ , and

$$u = \beta |\log r|^\alpha.$$

Compute near  $z = 0$

$$(22) \quad L_h(u) = \frac{1}{2r^2 |\log r|^3} [\alpha(\alpha - 1)u + e^{-2u} - 1].$$

For  $\beta < 0$ ,  $\alpha \in (-1, 0)$ , then  $u < 0$  and it is easy to check  $L_h(u) \geq 0$ .

Choose  $\alpha \in (-1, 0)$ . Let  $f$  be a smooth positive function on  $M$  that is equal to  $|\log |z_j|^\alpha|$  on a neighborhood of each  $p_j$ , and is constant

outside a larger neighborhood of each  $p_j$ . In particular, we may choose  $f$  to be constant in a neighborhood of each zero of  $U$ . Then if  $s = v + \beta f$  for  $\beta \ll 0$ , then  $L_h(s) > 0$  on all of  $M$ .

The upper barrier is harder to obtain. Equation (22) above shows that for the same model  $L_h(u) \not\leq 0$  for  $\beta \gg 0$ . In particular, the upper barrier we obtain is only bounded at the  $p_j$  (it does not go to 0). As we'll see below in Corollary 8, the solution  $u$  to  $L_h(u) = 0$  does go to zero at each  $p_j$ .

Consider a smooth conformal background metric  $k$  on  $\mathbb{CP}^1$ , which is equal to  $|dz_j|^2$  near each singularity  $p_j$ . Let  $\kappa_k$  be the Gaussian curvature of  $k$ . Let  $\tilde{\kappa}$  be a smooth positive function on  $M$  which is equal to

$$-\frac{1}{2}\Delta_k \log |\log |z_j|^2| = \frac{1}{4|z_j|^2(\log |z_j|)^2}$$

on a neighborhood of each singularity  $p_j$ . Note  $\tilde{\kappa}$  is integrable. We also require that

$$\int_M (\kappa_k - \tilde{\kappa}) dV_k = 0,$$

where  $dV_k$  is the volume element of the metric  $k$ . (This is possible by Gauss-Bonnet.) Now use the Green's function of the Laplacian  $\Delta_k$  to find  $f$  so that

$$\Delta_k f = 2\kappa_k - 2\tilde{\kappa}$$

A fairly straightforward computation with the Green's function (which we put in Appendix A below) shows that near each  $p_j$ ,

$$f = \log |\log |z_j|^2| + O(1).$$

Then compute for any constant  $c$

$$\begin{aligned} L_k(f + c) &= \Delta_k f + 4e^{-2f-2c} \|U\|_k^2 - 2\kappa_k \\ &= 4e^{-2f-2c} \|U\|_k^2 - 2\tilde{\kappa}. \end{aligned}$$

By our choice of  $\tilde{\kappa}$ , we have that if  $c \gg 0$ ,  $L_k(f + c) < 0$  always. If  $h$  is the model metric from above, and  $k = e^v h$ , then by (20),  $L_h(f + c + v) < 0$ . Let the upper barrier  $S = f + c + v$ . By the asymptotics of  $f$ , we know that  $S$  is bounded. Moreover, by choosing  $c$  large enough, we can ensure that  $S > s$  on  $M$ .

Now we solve the equation by exhausting  $M$  by  $M = \bigcup_n M_n$ , where  $M_n$  is a smooth manifold with boundary. For example, for  $n$  large, take

$$M_n = M \setminus \left( \bigcup_j \left\{ |z_j| < \frac{1}{n} \right\} \right).$$

Let  $u_n$  be the solution to the Dirichlet problem on  $M_n$

$$L_h(u) = 0, \quad u = S \quad \text{on} \quad \partial M_n.$$

This can be done by e.g. Theorem 12.5 in [8]. Then the strong maximum principle shows that  $S \geq u_n \geq s$ , and that the sequence  $\{u_n\}$  is decreasing pointwise to a function  $u$  on  $M$ . Therefore we have  $C^0$  bounds on  $u_n$ . Then equation (19) shows that we have local  $W^{2,p}$  bounds on  $u_n$ , which imply  $C^{0,\alpha}$  bounds. Then (19) again gives local  $C^{2,\alpha}$  bounds on  $u_n$  independent of  $n$ . Therefore, by Ascoli-Arzelá, the convergence is  $C^2$ , and  $u$  is a bounded solution of  $L_h(u) = 0$  on  $M$ .  $u$  is smooth by further bootstrapping. We record this as a proposition.

**Proposition 1.** *Let  $h$  be the background metric above on  $M$ . Then there exists a bounded smooth solution  $u$  to*

$$\Delta_h u + 4e^{-2u} \|U\|_h^2 - 2\kappa_h = 0.$$

*Remark.* Here is an interpretation of the semi-flat Calabi-Yau metric  $m = e^u h$  due to Robert Bryant. Consider the tensor

$$g = \frac{|U|^2}{m^2}.$$

Then  $m$  is a semi-flat Calabi-Yau metric with cubic differential  $U$  if and only if  $g$  has constant Gaussian curvature  $-4$ . So away from the zeroes of  $U$ ,  $g$  is a constant curvature Riemannian metric. In the particular case where  $U$  has exactly 6 poles of order 1 on  $\mathbb{CP}^1$ , then  $U$  has no zeroes, and  $g$  is the unique complete metric of constant curvature  $-4$  on  $\mathbb{CP}^1 \setminus \{p_j\}$ .

**3.2. Blow-up analysis.** Let  $z = z_j$  be the canonical holomorphic coordinate for a singular point  $p_j$ . Then for  $z$  near 0,

$$h = |\log |z|^2| |dz|^2, \quad U = \frac{1}{z} dz^3.$$

Then  $u$  satisfies

$$u_{z\bar{z}} + \frac{e^{-2u} - 1}{|z|^2 (\log |z|^2)^2} = 0.$$

For  $\lambda \geq 1$ , let

$$u_\lambda(z) = u\left(\frac{z}{\lambda}\right).$$

Then compute

$$(23) \quad u_{\lambda,z\bar{z}} + \frac{e^{-2u_\lambda} - 1}{|z|^2 (\log |z|^2 - 2 \log \lambda)^2} = 0.$$

Since  $u$  is bounded, this implies that  $\lim_{\lambda \rightarrow \infty} u_{\lambda,z\bar{z}} = 0$  uniformly on compact subsets of  $\mathbb{C} \setminus \{0\}$ . By the same bootstrapping estimates

as above, we have uniform  $C^{2,\alpha}$  bounds on  $u_\lambda$  on compact subsets of  $\mathbb{C} \setminus \{0\}$ . Then by Ascoli-Arzelá and a diagonalization argument, there is a sequence  $\lambda_j \rightarrow \infty$  so that  $u_{\lambda_j}$  converges locally in  $C^2$  to a limit  $u_\infty$  on  $\mathbb{C} \setminus \{0\}$ . By letting  $\lambda \rightarrow \infty$  in equation (23), we see

$$u_{\infty, z\bar{z}} = 0 \quad \text{on } \mathbb{C} \setminus \{0\}.$$

Therefore,  $u_\infty$ , as a bounded harmonic function on  $\mathbb{C} \setminus \{0\}$ , must be a constant function. Moreover,  $u_{\infty, z} = 0$  so that  $\lim_j u_{\lambda_j, z} = 0$ . For any sequence  $\lambda_j \rightarrow \infty$ , this argument shows that there is a subsequence  $\lambda_{j_k}$  so that  $\lim_k u_{\lambda_{j_k}, z} = 0$ . Thus

$$\lim_{\lambda \rightarrow \infty} u_{\lambda, z} = 0$$

uniformly on compact subsets of  $\mathbb{C} \setminus \{0\}$ . (We cannot yet conclude that  $u_\lambda$  converges, since the constant limit functions for each such subsequence may be different; see Corollary 8 below.) In terms of the unrescaled solution  $u$ , this is equivalent to

**Proposition 2.**  $\lim_{z \rightarrow 0} z u_z = 0$ .

#### 4. AFFINE FLAT STRUCTURE

Choose coordinates on  $\mathbb{R}^3$  so that the affine normal  $\xi = (0, 0, 1)$ . Let  $\alpha^1, \alpha^2$  be coordinates on  $\mathbb{R}^3$  transverse to  $\xi$ . Then locally an immersed parabolic affine sphere is the graph of a convex function  $\phi(\alpha^1, \alpha^2)$  satisfying

$$\det \frac{\partial^2 \phi}{\partial \alpha^j \partial \alpha^k} = 1.$$

The affine sphere is given in coordinates by  $(\alpha^1, \alpha^2, \phi)$ .

The solution of Titeica's equation in the last section gives induces an immersion  $f: \tilde{M} \rightarrow \mathbb{R}^3$ , where  $\tilde{M}$  is the universal cover of  $M$ . More specifically, let  $p \in \tilde{M}$  with a local coordinate  $z$  near  $p$ , and consider an initial vector  $\mathcal{V} \in \mathbb{R}^3 \otimes \mathbb{C}$  which satisfies

$$\det(\mathcal{V}, \bar{\mathcal{V}}, \xi) = \frac{1}{2} i e^{\psi(p)}$$

as in equation (4) above. Then there is a unique map  $f: \tilde{M} \rightarrow \mathbb{R}^3$  so that  $f_z(p) = \mathcal{V}$  and  $f$  satisfies the evolution equations (10-11).

In the coordinates above,  $f = (\alpha^1, \alpha^2, \phi)$ . If we leave  $\xi = (0, 0, 1)$  fixed and choose other transverse coordinates  $\tilde{\alpha}^1, \tilde{\alpha}^2$ , then we have a natural affine change of coordinates. The map  $\text{dev} = (\alpha^1, \alpha^2): \tilde{M} \rightarrow \mathbb{R}^2$  is the developing map of the affine flat structure induced on  $M$ .

In order to check this, we must show that the holonomy homomorphism behaves appropriately. Let  $\gamma \in \pi_1(M)$  be a deck transformation. Consider a loop representing  $\gamma$  which begins at a point  $q$ , and choose

a lift  $\tilde{q} \in \tilde{M}$ . Lift the loop so that the other endpoint is  $\tilde{q}' \in \tilde{M}$ . Then following a neighborhood of the lifted path induces a coordinate map from a neighborhood of  $\tilde{q}$  to a neighborhood of  $\tilde{q}'$ . This map is a constant element of  $\mathbf{SL}(2, \mathbb{R}) \times \mathbb{R}^2$ , by the affine invariance of the initial value problem: Let  $\Phi$  be the element of  $\mathbf{SL}(3, \mathbb{R}) \times \mathbb{R}^3$  which takes the initial data consisting of the position  $f$  and the frame  $\{f_z, f_{\bar{z}}, \xi\}$  at  $\tilde{q}$  to the corresponding data at  $\tilde{q}'$ . Then since  $f$  solves the initial value problem with initial data at  $\tilde{q}$ ,  $\Phi(f)$ ,  $\Phi_*\{f_z, f_{\bar{z}}, \xi\}$  must solve the initial value problem with initial data at  $\tilde{q}'$ . Thus  $\Phi \circ f = f \circ \gamma$  for the deck transformation  $\gamma$ . Equation (4) shows that  $\Phi$  preserve volume. Since the affine normal  $\xi$  is constant,  $\Phi$  induces an affine map  $\text{hol}_\gamma \in \mathbf{SL}(2, \mathbb{R}) \times \mathbb{R}^2$ . Thus we have a holonomy map  $\gamma \mapsto \text{hol}_\gamma$  from  $\pi_1(M)$  to  $\mathbf{SL}(2, \mathbb{R}) \times \mathbb{R}^2$ . The pair  $\{\text{dev}, \text{hol}\}$  is equivalent to the affine flat structure (see e.g. Goldman [9]). (We should also show that if we choose a different basepoint in  $\tilde{M}$  and different coordinates in  $\mathbb{R}^3$  transverse to  $\xi$ , that the pair  $\{\text{dev}, \text{hol}\}$  transforms appropriately. These points are again easy to check by the affine invariance of the problem in  $\mathbb{R}^3$  and the uniqueness of solutions to the initial value problem.)

We have proved

**Proposition 3.** *The immersion  $f : \tilde{M} \rightarrow \mathbb{R}^3$  induces an affine flat structure on  $M$  with a covariant constant volume form.*

**4.1. The holonomy type.** We focus on the affine structure around each pole of the cubic differential  $U$ . Choose a loop around such a singular point, represented by an element  $\gamma$  of  $\pi_1(M)$ .

Near a singularity  $z = z_j = 0$ , let  $z = e^{iw}$  so that for  $|z| \in (0, \epsilon)$ ,  $w = x + iy$  satisfies  $y > -\log \epsilon$ . Then in terms of the  $w$  coordinates,

$$\begin{aligned} e^\psi |dz|^2 &= e^u |\log |z|^2| |dz|^2 = 2ye^{-2y} e^u |dw|^2, \\ U &= \frac{1}{z} dz^3 = -ie^{2iw} dw^3. \end{aligned}$$

Now the equations (10), (11) imply that in terms of the real frame  $\{\xi, f_x, f_y\}$ ,

$$(24) \quad \begin{pmatrix} \xi \\ f_x \\ f_y \end{pmatrix}_x = A \begin{pmatrix} \xi \\ f_x \\ f_y \end{pmatrix},$$

where  $A = A(x, y) =$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2ye^{-2y} e^u & \frac{1}{2}u_x + e^{-u} \frac{1}{2y} \sin 2x & -\frac{1}{2}u_y - \frac{1}{2y} + 1 + e^{-u} \frac{1}{2y} \cos 2x \\ 0 & \frac{1}{2}u_y + \frac{1}{2y} - 1 + e^{-u} \frac{1}{2y} \cos 2x & \frac{1}{2}u_x - e^{-u} \frac{1}{2y} \sin 2x \end{pmatrix}.$$

In the  $w$  coordinate, Proposition 2 implies that

$$u_w = \frac{1}{2}u_x - \frac{i}{2}u_y \rightarrow 0 \quad \text{as } y \rightarrow \infty.$$

Therefore, since  $u$  is bounded, we find

$$\lim_{y \rightarrow \infty} A = A_\infty = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

uniformly in  $x$ .

To compute the affine flat structure, we only need the components of  $f$  transverse to  $\xi$ , which we call  $\text{dev} = (\alpha^1, \alpha^2)$  above. Then, we only need to consider  $\tilde{A}$ , the bottom right  $2 \times 2$  submatrix of  $A$  and we have

$$\begin{pmatrix} \text{dev}_x \\ \text{dev}_y \end{pmatrix}_x = \tilde{A} \begin{pmatrix} \text{dev}_x \\ \text{dev}_y \end{pmatrix}, \quad \lim_{y \rightarrow \infty} \tilde{A} = \tilde{A}_\infty = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The theory of ODEs [17] then implies that the fundamental solution to the ODE (24) must approach the fundamental solution to

$$(25) \quad X_x = \tilde{A}_\infty X,$$

where  $X = (\text{dev}_x, \text{dev}_y)^\perp$ . In other words, the solution  $\Psi(x, y)$  to the initial value problem

$$X(0, y) = X_0, \quad X_x = \tilde{A}X,$$

must approach the solution

$$X = X_0 e^{x\tilde{A}_\infty}$$

to (25) uniformly in  $x$  as  $y \rightarrow \infty$ .

For any  $y \gg 0$ , the linear path from  $(0, y)$  to  $(2\pi, y)$  corresponds to a loop  $|z| = e^{-y}$  around the singularity  $p_j$ . Also,  $\{\text{dev}_x, \text{dev}_y\}$  is a frame of the tangent space to hypersurface  $H$ . Therefore, integrating the initial value problem (10-11) along such path computes the linear part of the holonomy. So the solution to the ODE  $X_x = \tilde{A}X$  computes this part of the holonomy. Note that since the deck transformation is  $w \mapsto w + 2\pi$ , the frame  $\{\text{dev}_x, \text{dev}_y\}$  is appropriate for computing the holonomy (the frames at  $x = 0$  and  $x = 2\pi$  may be naturally identified).

Since the connection  $D$  on  $\mathbb{R}^3$  is flat, the conjugacy class of the holonomy is independent of the choice of loop in a free homotopy class. In particular, the linear holonomy matrix determined by our frame and the loop  $|z| = e^{-y}$  is given by  $\Psi(2\pi, y)$ , which satisfies

$$\lim_{y \rightarrow \infty} \Psi(2\pi, y) = e^{2\pi\tilde{A}_\infty} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This *does not* mean the linear part of the holonomy is trivial, as shown by the family of matrices

$$\begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

However, we do know that the set of eigenvalues of the linear part of the holonomy must be  $\{1\}$ . We record this as

**Proposition 4.** *The set of eigenvalues for the linear part of the affine holonomy around each puncture is  $\{1\}$ .*

There are four different conjugacy classes in  $\mathbf{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$  whose elements have unipotent linear part (we simply list a representative in each conjugacy class):

- (1)  $Y \mapsto Y$ .
- (2)  $Y \mapsto Y + b$ ,  $b \neq 0$ .
- (3)  $Y \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} Y$ .
- (4)  $Y \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} Y + b$ , which has no fixed point.

In order to address the affine part of the holonomy, we must consider  $\text{dev}$  itself instead of the derivatives  $\text{dev}_x, \text{dev}_y$ . To calculate  $\text{dev} = \int \text{dev}_y dy$ , compute from the structure equations (10-11)

$$(26) \quad \begin{pmatrix} \xi \\ f_x \\ f_y \end{pmatrix}_y = B \begin{pmatrix} \xi \\ f_x \\ f_y \end{pmatrix},$$

where  $B = B(x, y) =$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}u_y + \frac{1}{2y} - 1 + e^{-u} \frac{1}{2y} \cos 2x & \frac{1}{2}u_x - e^{-u} \frac{1}{2y} \sin 2x \\ 2ye^{-2y}e^u & -\frac{1}{2}u_x - e^u \frac{1}{2y} \sin 2x & \frac{1}{2}u_y + \frac{1}{2y} - 1 - e^{-u} \frac{1}{2y} \cos 2x \end{pmatrix}.$$

As above, let  $\tilde{B}$  be the bottom right  $2 \times 2$  submatrix of  $B$ , and we have

$$\begin{pmatrix} \text{dev}_x \\ \text{dev}_y \end{pmatrix}_y = \tilde{B} \begin{pmatrix} \text{dev}_x \\ \text{dev}_y \end{pmatrix}, \quad \lim_{y \rightarrow \infty} \tilde{B} = \tilde{B}_\infty = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A theorem of Perron [27] and Lettenmeyer [21] (see also Hartman-Wintner [18]) then gives the asymptotic behavior of any initial value problem  $X(y_0) = X_0, X_y = \tilde{B}y$ :

$$|X| = e^{-y+o(y)} \quad \text{as } y \rightarrow \infty.$$

So then if we choose  $X = (\text{dev}_x, \text{dev}_y)^\perp$  as above, with initial conditions  $X(x_0, y_0) = X_0$ , then as  $y \rightarrow \infty$ ,  $|X(x_0, y)| = e^{-y+o(y)}$ . This gives

the behavior of integrating (10-11) along a path  $x = x_0$ . If we have followed this path from  $(x_0, y_0)$  to  $(x_0, y)$ , we can then integrate in the  $x$  direction to find that

$$X(x, y) = \left[ \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} + o(1) \right] X(x_0, y)$$

as  $y \rightarrow \infty$  and bounded  $x$ . Therefore,

**Lemma 5.** *The solution  $X$  to the initial value problem  $X(x_0, y_0) = X_0$ ,  $X_x = \tilde{A}X$ ,  $X_y = \tilde{B}X$ , satisfies  $|X| = e^{-y+o(y)}$  as  $y \rightarrow \infty$  uniformly in any bounded interval in  $x$ .*

Therefore  $|\text{dev}_y| = e^{-y+o(y)}$  as  $y \rightarrow \infty$  uniformly in a bounded interval of  $x$  (and the same is true for  $\text{dev}_x$ ). Then if we augment the initial value problem to include an initial value for  $\text{dev}(x_0, y_0)$ , then

$$\text{dev}(\infty) = \text{dev}(x_0, y_0) + \int_{y_0}^{\infty} \text{dev}_y dy$$

exists. ( $x = x_0$  is constant in the path of this integral.) Moreover, we can determine how the holonomy acts on  $\text{dev}(\infty)$  by solving the initial value problem along the following path: First let  $y$  go from  $y_0$  to  $y'$  for some  $y' \gg 0$ . Then let  $x$  go from  $x_0$  to  $x_0 + 2\pi$ , and finally let  $y \rightarrow \infty$ . By the decay of  $\text{dev}_x$ , the contribution from integrating from  $x_0$  to  $x_0 + 2\pi$  is of order  $e^{-y'+o(y')}$ . Therefore, the limiting point value of  $\text{dev}$  for such a path must be  $\text{dev}(\infty)$  again as  $y' \rightarrow \infty$ . Moreover, the integral is the same upon integrating along any two homotopic paths; thus the limiting case as  $y' \rightarrow \infty$  is equal to the integral along any such path for  $y' \gg 0$ . These paths compute the action of the holonomy; so  $\text{dev}(\infty)$  is a fixed point of the affine holonomy. We record this as

**Proposition 6.** *The affine holonomy around each puncture has a fixed point.*

Note that this rules out the holonomy cases (2) and (4) above. Thus the affine holonomy either is the identity or is conjugate to  $Y \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} Y$ .

To rule out the identity holonomy, choose affine flat coordinates  $\alpha^1$ ,  $\alpha^2$ , and recall the representation in terms of local holomorphic functions  $F$  and  $G$ . The definitions (15-16) imply that

$$\alpha^1 = \frac{1}{2} \text{Re}(G + F), \quad \alpha^2 = \frac{1}{2} \text{Im}(G - F).$$

Since the holonomy is trivial,  $\alpha^1$  and  $\alpha^2$  are well-defined on some neighborhood of the puncture  $\{z : 0 < |z| < \epsilon\}$ . Therefore,  $\frac{\partial}{\partial z}(G+F) = 4\frac{\partial \alpha^1}{\partial z}$  is single valued. This is similarly true for  $\frac{\partial}{\partial z}(G-F)$ , and so  $dF/dz$

and  $dG/dz$  are single valued. We will derive a contradiction given the bounds on the metric. By formula (17) and Proposition 1 above, the metric  $e^\psi |dz|^2$  satisfies

$$(27) \quad C' |\log |z|^2| |dz|^2 \leq \frac{1}{4} (|dG|^2 - |dF|^2) \leq C |\log |z|^2| |dz|^2$$

for  $C, C' > 0$ . In particular,  $dG/dz$  cannot have an essential singularity since it goes to infinity as  $z \rightarrow 0$ ; so it must have a pole of some order  $n$  at  $z = 0$ .  $dF/dz$  then cannot have an essential singularity, since it must satisfy  $|dF/dz|^2 \geq |dG/dz|^2 - 4C |\log |z|^2|$ , which forces it to have a pole. Then look at the power series of  $dG/dz$  and  $dF/dz$  to derive a contradiction. Therefore, the affine holonomy cannot be trivial and we have proved

**Theorem 2.** *For any oriented loop around each pole  $p_j$  of  $U$ , the affine holonomy corresponding to the metric constructed in Proposition 1 is conjugate to*

$$Y \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} Y.$$

*Remark.* We call the holonomy type in the previous theorem *parabolic*.

**4.2. Fine structure.** To investigate the fine structure, we again write  $dF/dz$  and  $dG/dz$  in terms of holomorphic functions, this time involving log terms. We will use the following terminology: A function is *holomorphic on the disk* if it is holomorphic on  $\{z : |z| < \epsilon\}$  for  $\epsilon$  a small positive number. Similarly, a function is *holomorphic on the punctured disk* if it is holomorphic on  $\{z : 0 < |z| < \epsilon\}$ . The positive constant  $\epsilon$  will be unspecified, and it may be shrunk a little if necessary.

Introduce coordinates  $\{\alpha^1, \alpha^2\}$  so that the holonomy takes

$$(28) \quad \alpha^1 \mapsto \alpha^1 + \alpha^2, \quad \alpha^2 \mapsto \alpha^2,$$

and so that for any fixed  $x$ ,

$$(29) \quad \lim_{y \rightarrow \infty} \alpha^1 = \lim_{y \rightarrow \infty} \alpha^2 = 0.$$

Note these last equations are equivalent to  $\text{dev}(\infty) = 0$ . Then as above, since  $\alpha^2$  is single-valued and  $\text{Im}(G - F) = 2\alpha^2$ , we find

$$(30) \quad (G - F)'(z) = j(z),$$

for  $j(z)$  holomorphic on the punctured disk. Since  $\text{Re}(G + F) = 2\alpha^1$ , (28) and (30) imply that

$$(G + F)'(z) + \frac{\log z}{2\pi} j(z)$$

is holomorphic on the punctured disk. Therefore,

$$G'(z) = -\frac{1}{4\pi} j(z) \log z + \ell(z), \quad F'(z) = -\frac{1}{4\pi} j(z) \log z + \ell(z) - j(z).$$

for  $\ell$  holomorphic on the punctured disk.

Since the metric is  $\frac{1}{4}(|dG|^2 - |dF|^2)$ ,

$$(31) \quad |G'|^2 - |F'|^2 = -\frac{1}{4\pi}|j|^2 \log |z|^2 + \ell\bar{j} + \bar{\ell}j - |j|^2$$

is always positive. In particular,  $j$  can never be 0. Also notice that  $G'$  is never zero on the punctured disk, even though it's not single-valued. Then for

$$k(z) = -\frac{4\pi\ell(z)}{j(z)},$$

$\log z + k = -4\pi G'/j$  is never zero. Exponentiating, we find  $ze^k \neq 1$  on the punctured disk. Of course  $ze^k \neq 0$  there as well. Picard's Big Theorem then implies  $ze^k$  (and thus also  $e^k$ ) cannot have an essential singularity at  $z = 0$ . Now write  $k = k_1 + k_2$ , for

$$(32) \quad k_1 = \sum_{n=1}^{\infty} a_{-n}z^{-n}, \quad k_2 = \sum_{n=0}^{\infty} a_n z^n.$$

We want to show that  $k_1 = 0$ , (i.e.  $k$  extends to a holomorphic function on the disk). Since  $e^{k_2}$  is nonzero and holomorphic on the disk,  $e^{k_1} = e^k/e^{k_2}$  then cannot have an essential singularity at  $z = 0$ , and  $e^{k_1}$  must be a polynomial in  $1/z$ . Then

$$e^{k_1(z)} = z^{-n}p(z), \quad k_1(z) = -n \log z + \log p(z)$$

for  $p$  a polynomial in  $z$  which doesn't vanish on a neighborhood of  $z = 0$ . Unless  $n = 0$  (and thus  $k_1 = 0$ ),  $k_1$  cannot be written as a Laurent series convergent on the punctured disk. This contradicts (32); so  $k_1 = 0$ , and  $k$  extends over  $z = 0$  as a holomorphic function.

Equations (27) and (31) then give the following bounds on the metric  $e^\psi |dz|^2$ :

$$C' |\log |z|^2| \leq e^\psi = \frac{1}{4}|j|^2 \left(-\frac{1}{4\pi} \log |z|^2 - \frac{1}{2\pi} \operatorname{Re} k - 1\right) \leq C |\log |z|^2|$$

Since  $k$  is holomorphic on the disk, this implies  $j$  is bounded near  $z = 0$  and thus  $j$  extends over  $z = 0$  as a nonvanishing holomorphic function. So

$$(33) \quad G' = -\frac{j}{4\pi}(\log z + k), \quad F' = -\frac{j}{4\pi}(\log z + k + 4\pi).$$

We also know that the cubic form  $U = 1/z dz^3$ , and thus (18) gives

$$\frac{1}{z} = \frac{1}{4}(G'F'' - G''F') = -\frac{j^2}{16\pi} \left(\frac{1}{z} + k'\right).$$

Thus we have proved

**Proposition 7.** *Consider the canonical coordinate  $z = z_j$  around each pole  $p_j$  of  $U$ , and the semi-flat Calabi-Yau metric  $e^u h$  constructed in Proposition 1 above. For affine flat coordinates  $\alpha^1, \alpha^2$  satisfying (28-29), the holomorphic functions  $G$  and  $F$  defined in (15-16) depend on the following data: Let  $j(z), k(z)$  be two holomorphic functions on a neighborhood of  $p_j = \{z = 0\}$  so that*

$$(34) \quad -16\pi = (1 + zk')j^2.$$

*Then  $G$  and  $F$  satisfy (33).*

*Remark.* The normalization (29) also implies that along any radial path toward the origin  $z = 0$ ,  $\text{Im}(G - F)$  and  $\text{Re}(G + F)$  go to zero.

**Corollary 8.** *The conformal factor  $u$  constructed in Proposition 1 above approaches 0 at each of the poles of the cubic form  $U$ .*

*Proof.* By (34),  $j^2(0) = -16\pi$  and so the metric

$$e^\psi = \frac{1}{4}|j|^2 \left( -\frac{1}{4\pi} \log |z|^2 - \frac{1}{2\pi} \text{Re } k - 1 \right) = |\log |z|^2| + O(1).$$

But near  $z = 0$ , (21) implies  $e^\psi = e^u |\log |z|^2|$ . □

The description provided by Proposition 7 allows us to calculate another important invariant of the affine structure near each pole of  $U$ : a winding number.

Consider an affine flat structure on a punctured disk with parabolic holonomy and coordinates  $(\alpha^1, \alpha^2)$  satisfying (28-29). Then the line  $L = \{\alpha^2 = 0\}$  is preserved by the action of the holonomy. If possible, choose a point  $p$  in the preimage of  $L$  under the developing map. Then choose a path  $\mathcal{P}$  starting and ending at  $p$  which winds once around the puncture of the punctured disk. Since  $\text{dev}(p) \in L$ , the developing map takes any lift of  $\mathcal{P}$  to a path  $\text{dev}(\mathcal{P})$  in  $\mathbb{R}^2 \setminus \{0\}$  which begins and ends at same point. We define the *winding number* of the affine flat structure to be the winding number of  $\text{dev}(\mathcal{P})$  around the origin. If the developing map of the punctured disk does not intersect  $L$ , we define the winding number to be zero.

**Theorem 3.** *For the affine flat structure constructed above, the winding number around each pole of  $U$  is +1.*

*Proof.* Proposition 7 gives us that

$$\begin{aligned}
\alpha^1 &= \frac{1}{2} \operatorname{Re} (G + F) \\
&= \frac{1}{2} \operatorname{Re} \int -\frac{j}{2\pi} (\log z + k + 2\pi) dz \\
&= -\frac{1}{4\pi} \operatorname{Re} \int ic \log z + a + O(z \log z) dz \\
&= \frac{cr}{4\pi} [(\theta + b_1) \cos \theta + (\log r - b_2) \sin \theta] + O(z^2 \log z).
\end{aligned}$$

Here  $z = re^{i\theta}$ ,  $c = j(0) = \pm 4\sqrt{\pi}$ ,  $a = (k(0) + 2\pi)c$ , and  $b_1 + ib_2 = ic + a$ . There is no constant term in the integration by assumption (29). Similarly,

$$\alpha^2 = \frac{cr}{2} \cos \theta + O(z^2).$$

Fix  $r$  near 0. Then, as we show below, we may ignore higher order terms, and the winding number is easily computed to be +1 for

$$\check{\alpha} = (\check{\alpha}^1, \check{\alpha}^2) = \left( \frac{cr}{4\pi} (\log r - b_2) \sin \theta, \frac{cr}{2} \cos \theta \right).$$

More specifically, assume  $|\log r - b_2| \gg 2\pi$ . Then choose  $\Theta \in \frac{\pi}{2} + 2\pi\mathbb{Z}$  so that  $\Theta + b_1 \ll |\log r - b_2|$ . Then for  $\theta \in [\Theta, \Theta + 2\pi]$ , the path  $\check{\alpha}(\theta)$  winds around the origin once with orientation (note  $\log r - b_2 < 0$ ), and  $\check{\alpha}(\Theta) = \check{\alpha}(\Theta + 2\pi)$  is in the fixed line  $L$ .

Since as  $r \rightarrow 0$ ,  $\alpha = (\alpha^1, \alpha^2)$  is  $C^1$  close to  $\check{\alpha}$ , there is a  $\delta$  near zero so that  $\alpha(\Theta + \delta)$  is in the fixed line  $L$ , and the curve  $\alpha(\theta)$  meets  $L$  transversely at  $\theta = \Theta + \delta$ . Moreover, the winding number of  $\alpha$  for  $\theta \in [\Theta + \delta, \Theta + \delta + 2\pi]$  is +1, since the loops determined by  $\alpha$  and  $\check{\alpha}$  are homotopic via

$$\alpha_t(\theta) = t\check{\alpha}(\theta) + (1-t)\alpha(\theta - \delta), \quad \theta \in [\Theta, \Theta + 2\pi].$$

For ranges of  $\theta \in [\Theta + \delta, \Theta + \delta + 2\pi] + 2\pi m$ ,  $m \in \mathbb{Z}$ , the winding number is still +1 since the path  $\alpha(\theta)$  is just shifted  $m$  times by the action of the holonomy.  $\square$

All together, we have the following description of the metric and affine flat structure.

**Theorem 4.** *Given a holomorphic cubic differential  $U$  on  $\mathbb{CP}^1$  with poles of order 1 at  $\{p_j\}$ , there is a semi-flat Calabi-Yau metric on  $\mathbb{CP}^1 \setminus \{p_j\}$  asymptotically given by*

$$[|\log |z_j|^2| + o(1)] |dz_j|^2$$

for  $z_j$  the canonical holomorphic coordinate near the pole  $p_j$ . The affine flat structure has parabolic holonomy and winding number +1 around  $p_j$ .

*Remark.* In terms of the affine flat coordinates near the singularity  $p_j$ , the asymptotics of the metric and the affine Kähler potential function  $\phi$  can also be easily calculated from the holomorphic representation in Proposition 7.

## 5. MIRROR SYMMETRY

Consider the picture of Strominger-Yau-Zaslow in the simplest case (without instanton corrections or singular fibers). A Calabi-Yau manifold  $X$  admits a map

$$(35) \quad \pi X \rightarrow \bar{B},$$

where  $B$  is an affine flat manifold and  $\pi$  is a fibration with fiber  $T^n$  of special Lagrangian  $n$ -tori.  $B$  admits a semi-flat Calabi-Yau metric. Over an affine coordinate chart  $\Omega \subset B$ , form the tube domain  $\Omega + i\mathbb{R}^n$ . The special Lagrangian tori fibered over  $\Omega$  are then quotients of the imaginary fibers  $i\mathbb{R}^n$ . The mirror Calabi-Yau manifold should then be constructed by taking a Fourier transform in the fiber variables and a Legendre transform in affine coordinates and affine Kähler potential on  $B$ . The details of this construction may be found in Leung [23, 22].

The semi-flat Calabi-Yau metric we construct is singular at the poles  $p_j$  of  $U$ . As Gross-Wilson showed [14], a semi-flat metric with similar behavior at the singularities is obtained as the Gromov-Hausdorff limit of certain classes of elliptic K3 surfaces equipped with Calabi-Yau metrics. Near the 24 singular points, their model for nearby smooth Calabi-Yau metrics is not semi-flat (they glue in a model metric due to Ooguri-Vafa [26] there). In particular, the fibration (35) is not globally valid for any Calabi-Yau manifold near this singular limit. Therefore, the Fourier transform on the fibers is not relevant in our case, and mirror symmetry expresses itself only through the Legendre transform of the affine coordinates and the semi-flat Calabi-Yau potential function.

The Legendre transform appears naturally in the holomorphic representation in Subsection 2.3 above. In particular the dual affine coordinates  $\beta_j$  under the Legendre transform are by (15-16)

$$\beta_1 = \frac{\partial \phi}{\partial \alpha^1} = \frac{1}{2} \operatorname{Re}(G - F), \quad \beta_2 = \frac{\partial \phi}{\partial \alpha^2} = \frac{1}{2} \operatorname{Im}(G + F).$$

The Legendre transform potential is given by

$$\chi = \alpha^1 \beta_1 + \alpha^2 \beta_2 - \phi = \frac{1}{4}(|G|^2 - |F|^2) - \phi.$$

In terms  $\alpha^j$  and  $\beta_j$ ,  $G$  and  $F$  are given by

$$G = (\alpha^1 + \beta_1) + i(\alpha^2 + \beta_2), \quad F = (\alpha^1 - \beta_1) + i(\alpha^2 - \beta_2).$$

Passing a semi-flat Calabi-Yau metric to its mirror means that we switch the roles of the  $\alpha^j$  and the  $\beta_j$ . So the mirror transform becomes

$$G \mapsto G, \quad F \mapsto -F.$$

The metric remains the same under this mapping, and the cubic differential transforms as

$$U \mapsto -U$$

by (18).

We can also use Proposition 7 to find explicit asymptotics of the affine flat coordinates  $\beta_j$  and potential  $\chi$ . In particular, similarly to (29), we may assume that

$$(36) \quad \beta_1, \beta_2 \rightarrow 0$$

along any radial path as  $z \rightarrow 0$  for  $z$  the canonical holomorphic coordinate along each pole. Since  $\beta_j = \partial\phi/\partial\alpha^j$ , this may be accomplished by finding a new tilted set of coordinates in  $\mathbb{R}^3$  of the type allowed in Subsection 2.1 above. So together the assumptions (29) and (36) are equivalent to requiring the holomorphic functions  $F$  and  $G \rightarrow 0$  along any radial path as  $z \rightarrow 0$  (see the remark after Proposition 7). Moreover, we can read off the conjugacy class of the holonomy and the winding number to conclude

**Proposition 9.** *The affine flat structure for the mirror semi-flat Calabi-Yau has parabolic holonomy and winding number +1 around each pole of the cubic differential  $U$ . It is naturally isometric to its mirror.*

#### APPENDIX A. GREEN'S FUNCTION CALCULATION

Let

$$f(p) = \int_{S^2} G(p, q) [2\kappa_k(q) - 2\tilde{\kappa}(q)] dV_k(q)$$

for  $k$ ,  $\kappa_k$ ,  $\tilde{\kappa}$  defined as in subsection 3.1 above and  $G(p, q)$  the Green's function with respect to  $\Delta_k$ . Then in the coordinate  $z_j$  near each pole  $p_j$  of  $U$ ,

**Lemma 10.**  $f(z_j) = \log |\log |z_j|^2| + O(1)$ .

*Proof.* The Green's function on a compact Riemannian surface is of the form

$$G(P, Q) = \frac{1}{2\pi} \log d(P, Q) + O(1)$$

for  $d$  the Riemannian distance. Thus

$$f(P) = \int_{S^2} G(P, Q) [2\kappa_k(Q) - 2\tilde{\kappa}(Q)] dV_k(Q).$$

Represent  $P$  by the coordinate  $z_j$  near  $z_j = 0$  only. Then for a small positive  $\delta$ ,  $P = re^{i\theta}$ , and  $Q = \rho e^{i\varphi}$ ,

$$\begin{aligned} f(P) &= O(1) + g(r), \\ g(r) &= \int_{|Q| < \delta} \left( \frac{1}{2\pi} \log |P - Q| \right) \left( -\frac{1}{2|Q|^2 (\log |Q|)^2} \right) dq_1 dq_2 \\ &= -\frac{1}{4\pi} \int_{|Q| < \delta} \frac{\log |re^{i\theta} - \rho e^{i\varphi}|}{\rho^2 (\log \rho)^2} \rho d\rho d\varphi. \end{aligned}$$

We may change variables of integration to assume  $\theta = 1$  and compute

$$\begin{aligned} g'(r) &= -\frac{1}{4\pi} \int_{|Q| < \delta} \frac{\frac{\partial}{\partial r} \log \sqrt{(r - \rho \cos \varphi)^2 + \rho^2 \sin^2 \varphi}}{\rho (\log \rho)^2} d\rho d\varphi \\ &= -\frac{1}{4\pi} \int_0^\delta \frac{1}{\rho (\log \rho)^2} \int_0^{2\pi} \frac{2r - 2\rho \cos \varphi}{r^2 + \rho^2 - 2r\rho \cos \varphi} d\varphi d\rho. \end{aligned}$$

The inner integral can be integrated [10, §2.554], but we may also evaluate it as a contour integral for a new complex variable  $\zeta = \chi e^{i\varphi}$ : For  $|\zeta| = 1$ ,  $\cos \varphi = \frac{1}{2}(\zeta + \frac{1}{\zeta})$  and thus

$$\int_0^{2\pi} \frac{2r - 2\rho \cos \varphi}{r^2 + \rho^2 - 2r\rho \cos \varphi} d\varphi = \int_{|\zeta|=1} \frac{\rho\zeta^2 - 2r\zeta + \rho}{r\rho\zeta^2 - (r^2 + \rho^2)\zeta + r\rho} \left( -i \frac{d\zeta}{\zeta} \right)$$

There are poles of the integrand at  $\zeta = 0$ ,  $\rho/r$  and  $r/\rho$  with residues  $-i/r$ ,  $-i/r$  and  $i/r$  respectively. Thus if  $\rho < r$ , the sum of the residues inside the contour is  $-2i/r$ , and if  $r < \rho$ , the sum of the residues inside the contour is 0. Therefore,

$$\int_0^{2\pi} \frac{2r - 2\rho \cos \varphi}{r^2 + \rho^2 - 2r\rho \cos \varphi} d\varphi = \begin{cases} \frac{4\pi}{r} & \text{if } \rho < r \\ 0 & \text{if } r < \rho, \end{cases}$$

and for  $r < \delta$ ,

$$g'(r) = -\frac{1}{r} \int_0^r \frac{d\rho}{\rho (\log \rho)^2} = \frac{1}{r \log r}.$$

Therefore,

$$f(P) = O(1) + g(r) = O(1) + \log |\log r| = O(1) + \log |\log |P|^2|.$$

□

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