Section 6.3: Arc Length - Worksheet Solutions

#23. Calculate the arc length of the given curves.

(a) $y = 11 - 2(x-5)^{3/2}$, $5 \leqslant x \leqslant 6$.

Solution: We have

$$\frac{dy}{dx} = -2 \cdot \frac{3}{2}(x-5)^{1/2} = -3\sqrt{x-5}.$$

So

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(-3\sqrt{x-5}\right)^2 = 1 + 9(x-5) = 9x - 44.$$

Therefore the arc length is given by

$$\begin{aligned} L &= \int_5^6 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_5^6 \sqrt{9x-44}\, dx. \end{aligned}$$

We use the substitution $u = 9x - 44$, so that $du = 9dx$. The bounds become

$$\begin{aligned} x = 5 &\Rightarrow u = 1, \\ x = 6 &\Rightarrow u = 10. \end{aligned}$$

So the integral becomes

$$\begin{aligned} L &= \int_1^{10} \frac{1}{9}\sqrt{u}\, du \\ &= \frac{1}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10} \\ &= \boxed{\frac{2}{27}\left(10^{3/2} - 1\right) \text{ units}}. \end{aligned}$$

(b) $y = \dfrac{1}{3}\ln(\cos(3x)) + 4$, $\frac{\pi}{4} \leqslant x \leqslant \frac{\pi}{3}$.

Solution: We have

$$\frac{dy}{dx} = \frac{1}{3} \cdot \frac{-\sin(3x)}{\cos(3x)}(3) = \tan(3x).$$

So

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2(3x) = \sec^2(3x).$$

Hence

$$\begin{aligned}
L &= \int_{\pi/4}^{\pi/3} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= \int_{\pi/4}^{\pi/3} \sqrt{\sec^2(3x)}dx \\
&= \int_{\pi/4}^{\pi/3} |\sec(3x)|dx \\
&= \int_{\pi/4}^{\pi/3} -\sec(3x)dx \quad \left(\sec(3x) < 0 \text{ for } \frac{\pi}{4} \leqslant x \leqslant \frac{\pi}{3}\right) \\
&= \left[-\frac{1}{3}\ln|\sec(3x)+\tan(3x)|\right]_{\pi/4}^{\pi/3} \\
&= -\frac{1}{3}\left(\ln|-1| - \ln|-\sqrt{2}-1|\right) \\
&= \boxed{\frac{\ln(\sqrt{2}+1)}{3} \text{ units}}.
\end{aligned}$$

(c) $x = \frac{1}{4}\sqrt[3]{y} - \frac{9}{5}\sqrt[3]{y^5}$, $1 \leqslant y \leqslant 2$.

Solution: We have

$$\frac{dx}{dy} = \frac{1}{4}\cdot\frac{1}{3}y^{-2/3} - \frac{9}{5}\cdot\frac{5}{3}y^{2/3} = \frac{1}{12}y^{-2/3} - 3y^{2/3}.$$

So

$$\begin{aligned}
1+\left(\frac{dx}{dy}\right)^2 &= 1+\left(\frac{1}{12}y^{-2/3} - 3y^{2/3}\right)^2 \\
&= 1+\frac{1}{144}y^{-4/3} + 9y^{4/3} - 2\cdot\frac{1}{12}y^{-2/3}\cdot 3y^{2/3} \\
&= 1+\frac{1}{144}y^{-4/3} + 9y^{4/3} - \frac{1}{2} \\
&= \frac{1}{144}y^{-4/3} + 9y^{4/3} + \frac{1}{2} \\
&= \left(\frac{1}{12}y^{-2/3}\right)^2 + \left(3y^{2/3}\right)^2 + 2\cdot\frac{1}{12}y^{-2/3}\cdot 3y^{2/3} \\
&= \left(\frac{1}{12}y^{-2/3} + 3y^{2/3}\right)^2.
\end{aligned}$$

Therefore the arc length is given by

$$L = \int_1^2 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$
\begin{aligned}
&= \int_1^2 \sqrt{\left(\frac{1}{12}y^{-2/3} + 3y^{2/3}\right)^2}\, dy \\
&= \int_1^2 \left|\frac{1}{12}y^{-2/3} + 3y^{2/3}\right| dy \\
&= \int_1^2 \left(\frac{1}{12}y^{-2/3} + 3y^{2/3}\right) dy \\
&= \left[\frac{1}{4}y^{1/3} + \frac{9}{5}y^{5/3}\right]_1^2 \\
&= \left(\frac{1}{4}2^{1/3} + \frac{9}{5}2^{5/3}\right) - \left(\frac{1}{4} + \frac{9}{5}\right) \\
&= \boxed{\frac{5\sqrt[3]{2} + 72\sqrt[3]{4} - 41}{20} \text{ units}}.
\end{aligned}
$$

(d) $y = \sqrt{36 - x^2}$, $0 \leqslant x \leqslant 3$.

Solution: We have

$$
\frac{dy}{dx} = \frac{1}{2\sqrt{36 - x^2}}(-2x) = -\frac{x}{\sqrt{36 - x^2}}.
$$

Therefore

$$
\begin{aligned}
1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \left(-\frac{x}{\sqrt{36 - x^2}}\right)^2 \\
&= 1 + \frac{x^2}{36 - x^2} \\
&= \frac{36 - x^2 + x^2}{36 - x^2} \\
&= \frac{36}{36 - x^2}.
\end{aligned}
$$

So

$$
\begin{aligned}
L &= \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_0^3 \sqrt{\frac{36}{36 - x^2}}\, dx \\
&= 6\int_0^3 \frac{dx}{\sqrt{6^2 - x^2}} \\
&= 6\left[\sin^{-1}\left(\frac{x}{6}\right)\right]_0^3
\end{aligned}
$$

$$= 6\left(\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right)$$
$$= \boxed{\pi \text{ units}}.$$

(e) $y = \dfrac{1}{3}(2x+1)^{3/2}$, $0 \leqslant x \leqslant 1$.

Solution: We have

$$\begin{aligned}
L &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_0^1 \sqrt{1 + \left(\frac{1}{3} \cdot \frac{3}{2} \cdot 2(2x+1)^{1/2}\right)^2}\, dx \\
&= \int_0^1 \sqrt{1 + 2x + 1}\, dx \\
&= \int_0^1 \sqrt{2x+2}\, dx \\
&= \left[\frac{1}{3}(2x+2)^{3/2}\right]_0^1 \\
&= \frac{4^{3/2} - 2^{3/2}}{3} \\
&= \boxed{\frac{8 - 2\sqrt{2}}{3} \text{ units}}.
\end{aligned}$$

(f) $x = \sqrt{16y - y^2}$, $4 \leqslant y \leqslant 12$.

Solution: We have

$$\frac{dx}{dy} = \frac{16 - 2y}{2\sqrt{16y - y^2}} = \frac{8 - y}{\sqrt{16y - y^2}}.$$

Therefore

$$\begin{aligned}
1 + \left(\frac{dx}{dy}\right)^2 &= 1 + \left(\frac{8 - y}{\sqrt{16y - y^2}}\right)^2 \\
&= 1 + \frac{(8 - y)^2}{16y - y^2} \\
&= \frac{16y - y^2 + (64 - 16y + y^2)}{16y - y^2} \\
&= \frac{64}{16y - y^2}
\end{aligned}$$

$$= \frac{64}{64-(y-8)^2}$$

where we have completed the square in the denominator for the last step. So the arc length integral becomes

$$\begin{aligned}
L &= \int_4^{12} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \\
&= \int_4^{12} \sqrt{\frac{64}{64-(y-8)^2}}\, dy \\
&= 8\int_4^{12} \frac{dy}{\sqrt{64-(y-8)^2}} \\
&= 8\int_4^{12} \frac{dy}{8\sqrt{1-\left(\frac{y-8}{8}\right)^2}} \\
&= 8\int_{-1/2}^{1/2} \frac{du}{\sqrt{1-u^2}} \qquad \left(u = \frac{y-8}{8}\right) \\
&= 8\left[\sin^{-1}(u)\right]_{-1/2}^{1/2} \\
&= 8\left(\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right) \\
&= 8\left(\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right) \\
&= \boxed{\frac{8\pi}{3} \text{ units}}.
\end{aligned}$$

(g) $y = \frac{1}{6}\ln\left(\sin(3x)\cos(3x)\right), \ \frac{\pi}{18} \leqslant x \leqslant \frac{\pi}{9}$.

Solution: We have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{6}\cdot\frac{3\cos(3x)^2 - 3\sin(3x)^2}{\sin(3x)\cos(3x)} \\
&= \frac{3\cos(3x)^2}{6\sin(3x)\cos(3x)} - \frac{3\sin(3x)^2}{6\sin(3x)\cos(3x)} \\
&= \frac{1}{2}\cot(3x) - \frac{1}{2}\tan(3x).
\end{aligned}$$

So

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{1}{2}\cot(3x) - \frac{1}{2}\tan(3x)\right)^2$$

$$
\begin{aligned}
&= 1 + \frac{1}{4}\cot(3x)^2 + \frac{1}{4}\tan(3x)^2 - 2\cdot\frac{1}{2}\cot(3x)\cdot\frac{1}{2}\tan(3x) \\
&= 1 + \frac{1}{4}\cot(3x)^2 + \frac{1}{4}\tan(3x)^2 - \frac{1}{2} \\
&= \frac{1}{4}\cot(3x)^2 + \frac{1}{4}\tan(3x)^2 + \frac{1}{2} \\
&= \frac{1}{4}\cot(3x)^2 + \frac{1}{4}\tan(3x)^2 + 2\cdot\frac{1}{2}\cot(3x)\cdot\frac{1}{2}\tan(3x) \\
&= \left(\frac{1}{2}\cot(3x) + \frac{1}{2}\tan(3x)\right)^2.
\end{aligned}
$$

Therefore the arc length is given by

$$
\begin{aligned}
L &= \int_{\pi/18}^{\pi/9} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\
&= \int_{\pi/18}^{\pi/9} \sqrt{\left(\frac{1}{2}\cot(3x) + \frac{1}{2}\tan(3x)\right)^2}\,dx \\
&= \int_{\pi/18}^{\pi/9} \left|\frac{1}{2}\cot(3x) + \frac{1}{2}\tan(3x)\right|\,dx \\
&= \frac{1}{2}\int_{\pi/18}^{\pi/9} \left(\cot(3x) + \tan(3x)\right)dx \\
&= \frac{1}{6}\left[\ln|\sin(3x)| + \ln|\sec(3x)|\right]_{\pi/18}^{\pi/9} \\
&= \frac{1}{6}\left(\ln\left|\sin\left(\frac{\pi}{3}\right)\right| + \ln\left|\sec\left(\frac{\pi}{3}\right)\right| - \ln\left|\sin\left(\frac{\pi}{6}\right)\right| - \ln\left|\sec\left(\frac{\pi}{6}\right)\right|\right) \\
&= \frac{1}{6}\left(\ln\left(\frac{\sqrt{3}}{2}\right) + \ln(2) - \ln\left(\frac{1}{2}\right) - \ln\left(\frac{2}{\sqrt{3}}\right)\right) \\
&= \boxed{\frac{1}{6}\ln(3)\text{ units}}.
\end{aligned}
$$

(h) $y = \dfrac{e^{5x} + e^{-5x}}{10}$, $0 \leqslant x \leqslant \dfrac{1}{5}$.

Solution: We have

$$
\frac{dy}{dx} = \frac{5e^{5x} - 5e^{5x}}{10} = \frac{e^{5x} - e^{5x}}{2}.
$$

So

$$
\begin{aligned}
1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \left(\frac{e^{5x} - e^{-5x}}{2}\right)^2 \\
&= 1 + \frac{1}{4}e^{10x} + \frac{1}{4}e^{-10x} - 2\cdot\frac{1}{2}e^{5x}\cdot\frac{1}{2}e^{-5x}
\end{aligned}
$$

$$
\begin{aligned}
&= 1 + \frac{1}{4}e^{10x} + \frac{1}{4}e^{-10x} - \frac{1}{2} \\
&= \frac{1}{4}e^{10x} + \frac{1}{4}e^{-10x} + \frac{1}{2} \\
&= \left(\frac{1}{2}e^{5x}\right)^2 + \left(\frac{1}{2}e^{-5x}\right)^2 + 2 \cdot \frac{1}{2}e^{5x} \cdot \frac{1}{2}e^{-5x} \\
&= \left(\frac{e^{5x} + e^{-5x}}{2}\right)^2.
\end{aligned}
$$

Therefore the arc length is given by

$$
\begin{aligned}
L &= \int_0^{1/5} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
&= \int_0^{1/5} \sqrt{\left(\frac{e^{5x} + e^{-5x}}{2}\right)^2} dx \\
&= \int_0^{1/5} \left|\frac{e^{5x} + e^{-5x}}{2}\right| dx \\
&= \int_0^{1/5} \left(\frac{e^{5x} + e^{-5x}}{2}\right) dx \\
&= \frac{1}{10}\left[e^{5x} - e^{-5x}\right]_0^{1/5} \\
&= \frac{1}{10}\left(e - e^{-1} - 1 + 1\right) \\
&= \boxed{\frac{e - e^{-1}}{10} \text{ units}}.
\end{aligned}
$$

(i) $x = \frac{4}{5}y^{5/4}$, $0 \leqslant y \leqslant 9$.

Solution: We have $\frac{dx}{dy} = y^{1/4}$, so

$$
\begin{aligned}
L &= \int_0^{9} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \\
&= \int_0^{9} \sqrt{1 + y^{1/2}} dy.
\end{aligned}
$$

In this last integral, we substitute $u = 1 + y^{1/2}$, so $du = \frac{dy}{2y^{1/2}}$. Observing that $y^{1/2} = u - 1$, we have

$$dy = 2y^{1/2} du = 2(u - 1) du.$$

The bounds become

$$y = 0 \;\Rightarrow\; u = 1 + 0^{1/2} = 1,$$
$$y = 9 \;\Rightarrow\; u = 1 + 9^{1/2} = 4.$$

We obtain

$$\begin{aligned} L &= \int_1^4 \sqrt{u}2(u-1)du \\ &= 2\int_1^4 \left(u^{3/2} - u^{1/2}\right) du \\ &= 2\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^4 \\ &= 4\left(\frac{4^{5/2}}{5} - \frac{4^{3/2}}{3} - \frac{1}{5} + \frac{1}{3}\right) \\ &= \boxed{\frac{232}{15}}. \end{aligned}$$

(j) $x = \frac{6}{7}y^{7/6}$, $0 \leqslant y \leqslant 1$.

Solution: We have
$$\frac{dx}{dy} = y^{1/6},$$
so
$$L = \int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}dy = \int_0^1 \sqrt{1 + y^{1/3}}dy.$$
We can calculate this integral with the substitution $u = \sqrt{1 + y^{1/3}}$. This gives $du = \frac{dy}{6y^{2/3}\sqrt{1+y^{1/3}}}$, or $dy = 6y^{2/3}\sqrt{1 + y^{1/3}}du$. To express this in terms of u, note that $\sqrt{1 + y^{1/3}} = u$, and $y^{2/3} = (u^2 - 1)^2$. So we obtain $dy = 6(u^2 - 1)^2 udu$. The integral becomes

$$\begin{aligned} L &= \int_1^{\sqrt{2}} u \cdot 6(u^2-1)^2 udu \\ &= 6\int_1^{\sqrt{2}} u^2(u^4 - 2u^2 + 1)du \\ &= 6\int_1^{\sqrt{2}} (u^6 - 2u^4 + u^2)du \\ &= 6\left[\frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3}\right]_1^{\sqrt{2}} \end{aligned}$$

$$= \boxed{\frac{44\sqrt{2}-16}{35} \text{ units}}.$$

(k) $y = \cos\left(\frac{x}{2}\right) - \ln\left(\csc\left(\frac{x}{2}\right) + \cot\left(\frac{x}{2}\right)\right),\ \frac{\pi}{2} \leqslant x \leqslant \pi.$

Solution: We have

$$\frac{dy}{dx} = -\frac{1}{2}\sin\left(\frac{x}{2}\right) + \frac{1}{2}\csc\left(\frac{x}{2}\right).$$

So

$$\begin{aligned}
1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \left(-\frac{1}{2}\sin\left(\frac{x}{2}\right) + \frac{1}{2}\csc\left(\frac{x}{2}\right)\right)^2 \\
&= 1 + \frac{1}{4}\sin\left(\frac{x}{2}\right)^2 + \frac{1}{4}\csc\left(\frac{x}{2}\right)^2 - 2\cdot\frac{1}{2}\sin\left(\frac{x}{2}\right)\cdot\frac{1}{2}\csc\left(\frac{x}{2}\right) \\
&= 1 + \frac{1}{4}\sin\left(\frac{x}{2}\right)^2 + \frac{1}{4}\csc\left(\frac{x}{2}\right)^2 - \frac{1}{2} \\
&= \frac{1}{4}\sin\left(\frac{x}{2}\right)^2 + \frac{1}{4}\csc\left(\frac{x}{2}\right)^2 + \frac{1}{2} \\
&= \left(\frac{1}{2}\sin\left(\frac{x}{2}\right)\right)^2 + \left(\frac{1}{2}\csc\left(\frac{x}{2}\right)\right)^2 + 2\cdot\frac{1}{2}\sin\left(\frac{x}{2}\right)\cdot\frac{1}{2}\csc\left(\frac{x}{2}\right) \\
&= \left(\frac{1}{2}\sin\left(\frac{x}{2}\right) + \frac{1}{2}\csc\left(\frac{x}{2}\right)\right)^2.
\end{aligned}$$

Therefore the arc length is given by

$$\begin{aligned}
L &= \int_{\pi/2}^{\pi} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\
&= \int_{\pi/2}^{\pi} \sqrt{\left(\frac{1}{2}\sin\left(\frac{x}{2}\right) + \frac{1}{2}\csc\left(\frac{x}{2}\right)\right)^2}\,dy \\
&= \int_{\pi/2}^{\pi} \left|\frac{1}{2}\sin\left(\frac{x}{2}\right) + \frac{1}{2}\csc\left(\frac{x}{2}\right)\right|\,dy \\
&= \int_{\pi/2}^{\pi} \left(\frac{1}{2}\sin\left(\frac{x}{2}\right) + \frac{1}{2}\csc\left(\frac{x}{2}\right)\right)\,dy \\
&= \left[-\cos\left(\frac{x}{2}\right) - \ln\left|\csc\left(\frac{x}{2}\right) + \cot\left(\frac{x}{2}\right)\right|\right]_{\pi/2}^{\pi} \\
&= \left(-\cos\left(\frac{\pi}{2}\right) - \ln\left|\csc\left(\frac{\pi}{2}\right) + \cot\left(\frac{\pi}{2}\right)\right|\right) - \left(-\cos\left(\frac{\pi}{4}\right) - \ln\left|\csc\left(\frac{\pi}{4}\right) + \cot\left(\frac{\pi}{4}\right)\right|\right) \\
&= -0 - \ln(1+0) + \frac{\sqrt{2}}{2} + \ln\left(\sqrt{2}+1\right) \\
&= \boxed{\frac{\sqrt{2}}{2} + \ln\left(\sqrt{2}+1\right) \text{ units}}.
\end{aligned}$$

#24. Find a curve of the form $y = f(x)$ passing through $(4, 13)$, having negative derivative, and whose length integral on $1 \leqslant x \leqslant 7$ is given by

$$L = \int_1^7 \sqrt{1 + \frac{25}{x^3}} dx.$$

Solution: The arc length being given by

$$L = \int_1^7 \sqrt{1 + f'(x)^2} dx,$$

we can deduce that $f'(x)^2 = \dfrac{25}{x^3}$. Taking square roots on both sides gives

$$\sqrt{f'(x)^2} = \sqrt{\frac{25}{x^3}}$$
$$\Rightarrow \quad \left|f'(x)\right| = \frac{5}{x^{3/2}}.$$

Since f has negative derivative, $|f'(x)| = -f'(x)$ and we deduce

$$f'(x) = -\frac{5}{x^{3/2}} = -5x^{-3/2}.$$

Taking an antiderivative gives

$$f(x) = \int -5x^{-3/2} dx = -5(-2)x^{-1/2} + C = \frac{10}{\sqrt{x}} + C.$$

To find the constant C, we can use the fact that the curve passes through $(4, 13)$, which gives

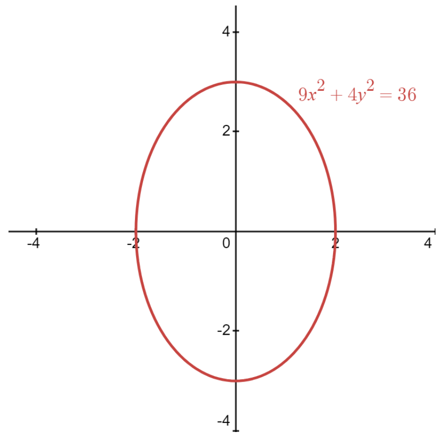
$$\frac{10}{\sqrt{4}} + C = 13 \;\Rightarrow\; C = 8.$$

Therefore, the solution to the problem is the curve

$$\boxed{y = \frac{10}{\sqrt{x}} + 8}.$$

#25. Consider the ellipse of equation $9x^2 + 4y^2 = 36$. Set-up an integral that calculates the perimeter of the ellipse.

Solution: The ellipse is sketched below.



Since the equation $9x^2 + 4y^2 = 36$ does not change when replacing x with $-x$ or y with $-y$, the ellipse is symmetric about the x-axis and the y-axis both. Therefore, it suffices to compute the perimeter of the part of the ellipse located in the first quadrant and multiply it by 4. To do this, we could use either integration with respect to x or y – we will choose integration with respect to x for this solution. We need to express the ellipse as a of function of x:

$$\begin{aligned} 9x^2 + 4y^2 = 36 \Rightarrow 4y^2 &= 36 - 9x^2 \\ \Rightarrow y^2 &= \frac{36 - 9x^2}{4} \\ \Rightarrow y &= \pm\frac{\sqrt{36 - 9x^2}}{2}. \end{aligned}$$

The positive solution corresponds to the top half of the ellipse, and the negative solution corresponds to the bottom half of the ellipse. Therefore, the part of the ellipse in the first quadrant is the graph of $y = \frac{\sqrt{36-9x^2}}{2}$, $0 \leqslant x \leqslant 2$. We have

$$\frac{dy}{dx} = \frac{-18x}{4\sqrt{36 - 9x^2}} = -\frac{9}{2\sqrt{36 - 9x^2}}.$$

So

$$\boxed{L = 4\int_0^2 \sqrt{1 + \frac{81}{36 - 9x^2}}\,dx}.$$