Section 8.3: Trigonometric Integrals - Worksheet Solutions

#36. Calculate the following integrals. **Note:** some of these problems use integration techniques from earlier sections.

(a) $\int \sin^2(5x)dx$

Solution: We use the double angle formula

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2},$$

which gives

$$\begin{aligned}\int \sin^2(5x)dx &= \int \frac{1 - \cos(10x)}{2}dx \\ &= \boxed{\frac{x}{2} - \frac{\sin(10x)}{20} + C}.\end{aligned}$$

(b) $\int \cos^3(7\theta)d\theta$

Solution: The exponent of cosine is odd, so we can split off a factor $\cos(7\theta)$, rewrite the rest of the integrand in terms of $\sin(7\theta)$ using the Pythagorean identity $\cos^2(7\theta) = 1 - \sin^2(7\theta)$ and use the substitution $u = \sin(7\theta)$, $du = 7\cos(7\theta)d\theta$. This gives

$$\begin{aligned}\int \cos^3(7\theta)d\theta &= \int \cos^2(7\theta)\cos(7\theta)d\theta \\ &= \int (1 - \sin^2(7\theta))\cos(7\theta)d\theta \\ &= \int \frac{1 - u^2}{7}du \\ &= \frac{u}{7} - \frac{u^3}{21} + C \\ &= \boxed{\frac{\sin(7\theta)}{7} - \frac{\cos^3(7\theta)}{21} + C}.\end{aligned}$$

(c) $\int \sec^4(2x)\tan^6(2x)dx$

Solution: The exponent of secant is even, so we can split off a factor $\sec^2(2x)$, rewrite the rest of the integrand in terms of $\tan(2x)$ using the Pythagorean identity $\sec^2(2x) = \tan^2(2x) + 1$, and use the substitution $u = \tan(2x)$, $du = 2\sec^2(2x)dx$. This gives

$$\begin{aligned}
\int \sec^4(2x)\tan^6(2x)dx &= \int \sec^2(2x)\tan^6(2x)\sec^2(2x)dx \\
&= \int (\tan^2(2x)+1)\tan^6(2x)\sec^2(2x)dx \\
&= \int (u^2+1)u^6\frac{du}{2} \\
&= \frac{1}{2}\int (u^8+u^6)du \\
&= \frac{1}{2}\left(\frac{u^9}{9}+\frac{u^7}{7}\right)+C \\
&= \boxed{\frac{1}{2}\left(\frac{\tan^9(2x)}{9}+\frac{\tan^7(2x)}{7}\right)+C}.
\end{aligned}$$

(d) $\int (x^3+8x)\sin(3x)dx.$

Solution: We use integration by parts as many times as needed to reduce the degree of the polynomial factor to zero, so three times here. For the first IBP, we choose the parts

$$\begin{aligned}
u &= x^3+8x \;\Rightarrow\; du = (3x^2+8)dx, \\
dv &= \sin(3x)dx \;\Rightarrow\; v = -\frac{1}{3}\cos(3x).
\end{aligned}$$

We obtain

$$\begin{aligned}
\int (x^3+8x)\sin(3x)dx &= -\frac{(x^3+8x)\cos(3x)}{3} - \int -\frac{\cos(3x)}{3}(3x^2+8)dx \\
&= -\frac{(x^3+8x)\cos(3x)}{3} + \frac{1}{3}\int \cos(3x)(3x^2+8)dx.
\end{aligned}$$

The next IBP will use

$$\begin{aligned}
u &= 3x^2+8 \;\Rightarrow\; du = 6xdx, \\
dv &= \cos(3x)dx \;\Rightarrow\; v = \frac{1}{3}\sin(3x).
\end{aligned}$$

This gives

$$\int (x^3+8x)\sin(3x)dx = -\frac{(x^3+8x)\cos(3x)}{3} + \frac{1}{3}\left(\frac{(3x^2+8)\sin(3x)}{3} - \int \frac{\sin(3x)}{3}6xdx\right)$$

$$= -\frac{(x^3+8x)\cos(3x)}{3} + \frac{1}{3}\left(\frac{(3x^2+8)\sin(3x)}{3} - 2\int x\sin(3x)dx\right).$$

For the last IBP, we take

$$u = x \;\Rightarrow\; du = dx,$$
$$dv = \sin(3x)dx \;\Rightarrow\; v = -\frac{1}{3}\cos(3x).$$

We get

$$\int (x^3+8x)\sin(3x)dx = -\frac{(x^3+8x)\cos(3x)}{3} + \frac{1}{3}\left(\frac{(3x^2+8)\sin(3x)}{3} - 2\left(-\frac{x\cos(3x)}{3} + \frac{1}{3}\int \cos(3x)dx\right)\right)$$
$$= -\frac{(x^3+8x)\cos(3x)}{3} + \frac{1}{3}\left(\frac{(3x^2+8)\sin(3x)}{3} - 2\left(-\frac{x\cos(3x)}{3} + \frac{\sin(3x)}{9}\right)\right) +$$
$$= \boxed{\frac{(9x^2+22)\sin(3x)}{27} - \frac{(3x^3+22x)\cos(3x)}{9} + C}.$$

(e) $\int_0^{\pi/21} \tan^3(7\theta)d\theta$

Solution: We can split off a factor $\tan^2(7\theta)$, replace it by $\sec^2(7\theta)-1$ and distribute. This gives

$$\int_0^{\pi/21} \tan^3(7\theta)d\theta = \int_0^{\pi/21} \tan(7\theta)\tan^2(7\theta)d\theta$$
$$= \int_0^{\pi/21} \tan(7\theta)\left(\sec^2(7\theta)-1\right)d\theta$$
$$= \int_0^{\pi/21} \tan(7\theta)\sec^2(7\theta)d\theta - \int_0^{\pi/21} \tan(7\theta)d\theta.$$

The first integral can be evaluated using the substitution $u = \tan(7\theta)$, which gives $du = 7\sec^2(7\theta)d\theta$. The bounds become

$$\theta = 0 \;\Rightarrow\; u = \tan(0) = 0,$$
$$\theta = \frac{\pi}{21} \;\Rightarrow\; u = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}.$$

So we get

$$\int_0^{\pi/21} \tan^3(7\theta)d\theta = \int_0^{\sqrt{3}} \frac{u}{7}du - \int_0^{\pi/21} \tan(7\theta)d\theta$$

$$= \left[\frac{u^2}{21}\right]_0^{\sqrt{3}} - \left[\frac{\ln|\sec(7\theta)|}{7}\right]_0^{\pi/21}$$
$$= \frac{\sqrt{3}^2}{21} - \frac{1}{7}\left(\ln\left(\sec\left(\frac{\pi}{3}\right)\right) - \ln(\sec(0))\right)$$
$$= \boxed{\frac{1-\ln(2)}{7}}.$$

(f) $\displaystyle\int_0^{\pi/6} x\cos(3x)\sin(3x)dx.$

Solution: We can calculate this integral with an integration by parts, choosing

$$u = x \;\Rightarrow\; du = dx,$$
$$dv = \cos(3x)\sin(3x)dx \;\Rightarrow\; v = \frac{\sin^2(3x)}{6}.$$

This gives

$$\int_0^{\pi/6} x\cos(3x)\sin(3x)dx = \left[\frac{x\sin^2(3x)}{6}\right]_0^{\pi/6} - \int_0^{\pi/6}\frac{\sin^2(3x)}{6}dx$$
$$= \frac{\pi}{36} - \frac{1}{6}\int_0^{\pi/6}\frac{1-\cos(6x)}{2}dx$$
$$= \frac{\pi}{36} - \frac{1}{6}\left[\frac{x}{2} - \frac{\sin(6x)}{12}\right]_0^{\pi/6}$$
$$= \frac{\pi}{36} - \frac{\pi}{72}$$
$$= \boxed{\frac{\pi}{72}}.$$

(g) $\displaystyle\int_{\pi/3}^{2\pi/3} \sqrt{1-\cos(6x)}dx$

Solution: We can rewrite the inside of the square root as a perfect square using the double angle identity $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$, which gives

$$1 - \cos(6x) = 2\sin^2(3x).$$

Therefore, we have

$$\int_{\pi/3}^{2\pi/3}\sqrt{1-\cos(6x)}dx = \int_{\pi/3}^{2\pi/3}\sqrt{2\sin^2(3x)}dx$$

$$
\begin{aligned}
&= \int_{\pi/3}^{2\pi/3} \sqrt{2}\,|\sin(3x)|\,dx \\
&= \int_{\pi/3}^{2\pi/3} -\sqrt{2}\sin(3x)dx \quad \left(\sin(3x) \leqslant 0 \text{ on } \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]\right) \\
&= -\sqrt{2}\left[-\frac{\cos(3x)}{3}\right]_{\pi/3}^{2\pi/3} \\
&= \frac{\sqrt{2}}{3}\left(\cos(2\pi) - \cos(\pi)\right) \\
&= \boxed{\frac{2\sqrt{2}}{3}}.
\end{aligned}
$$

(h) $\int \tan^3(\theta)\sec^5(\theta)d\theta$

Solution: Since the exponent of tan is odd, we can split off a factor $\tan(\theta)\sec(\theta)$, replace the remaining factors of $\tan(\theta)$ using the trigonometric identity $\tan^2(\theta) = \sec^2(\theta) - 1$ and then use the substitution $u = \sec(\theta)$, $du = \tan(\theta)\sec(\theta)d\theta$. This gives

$$
\begin{aligned}
\int \tan^3(\theta)\sec^5(\theta)d\theta &= \int \tan^2(\theta)\sec^4(\theta)^4\tan(\theta)\sec(\theta)d\theta \\
&= \int \left(\sec^2(\theta) - 1\right)\sec^4(\theta)\tan(\theta)\sec(\theta)d\theta \\
&= \int \left(u^2 - 1\right)u^4 du \\
&= \int \left(u^6 - u^4\right)du \\
&= \frac{u^7}{7} - \frac{u^5}{5} + C \\
&= \boxed{\frac{\sec^7(\theta)}{7} - \frac{\sec^5(\theta)}{5} + C}.
\end{aligned}
$$

(i) $\int \sin(5x)\sqrt{1+\cos(5x)}dx$

Solution: This integral can be computed using the substitution $u = 1 + \cos(5x)$, $du = -5\sin(5x)dx$, which gives

$$
\int \sin(5x)\sqrt{1+\cos(5x)}dx = \int -\frac{1}{5}\sqrt{u}du
$$

$$= -\frac{1}{5}\cdot\frac{2}{3}u^{3/2} + C$$
$$= \boxed{-\frac{2}{15}\left(1+\sin(5x)\right)^{3/2} + C}.$$

(j) $\displaystyle\int \sec^2(3x)\ln(\sec(3x))dx$

Solution: We can start with an IBP with parts

$$u = \ln(\sec(3x)) \;\Rightarrow\; du = \frac{3\sec(3x)\tan(3x)}{\sec(3x)}dx = 3\tan(3x)dx,$$
$$dv = \sec^2(3x)dx \;\Rightarrow\; v = \frac{1}{3}\tan(3x).$$

We get

$$\int \sec^2(3x)\ln(\sec(3x))dx = \frac{\tan(3x)\ln(\sec(3x))}{3} - \int 3\tan(3x)\frac{\tan(3x)}{3}dx$$
$$= \frac{\tan(3x)\ln(\sec(3x))}{3} - \int \tan^2(3x)dx.$$

This last integral can be computed using the Pythagorean identity $\tan^2(3x) = \sec^2(3x) - 1$, which gives

$$\int \sec^2(3x)\ln(\sec(3x))dx = \frac{\tan(3x)\ln(\sec(3x))}{3} - \int \left(\sec^2(3x) - 1\right)dx$$
$$= \boxed{\frac{\tan(3x)\ln(\sec(3x))}{3} - \frac{\tan(3x)}{3} - x + C}.$$

(k) $\displaystyle\int_{\pi}^{3\pi/2} \cos^5(z)\sin^8(z)dz$

Solution: Since the power of cos is odd, we can compute this integral by splitting off a factor $\cos(z)$, rewriting the remaining factors in terms of $\sin(z)$ using the Pythagorean identity $\cos^2(z) = 1 - \sin^2(z)$ and substituting $u = \sin(z)$, $du = \cos(z)dz$. The bounds will change to

$$z = \pi \;\Rightarrow\; u = \sin(\pi) = 0,$$
$$z = \frac{3\pi}{2} \;\Rightarrow\; u = \sin\left(\frac{3\pi}{2}\right) = -1.$$

This yields

$$\begin{aligned}
\int_{\pi}^{3\pi/2} \cos^5(z)\sin^8(z)dz &= \int_{\pi}^{3\pi/2} \cos^4(z)\sin^8(z)\cos(z)dz \\
&= \int_{\pi}^{3\pi/2} \left(1-\sin^2(z)\right)^2 \sin^8(z)\cos(z)dz \\
&= \int_{0}^{-1} \left(1-u^2\right)^2 u^8 du \\
&= \int_{0}^{-1} \left(u^8-u^{10}\right) du \\
&= \left[\frac{u^9}{9}-\frac{u^{11}}{11}\right]_0^{-1} \\
&= \frac{(-1)^9}{9}-\frac{(-1)^{11}}{11} \\
&= \boxed{-\frac{2}{99}}.
\end{aligned}$$

(l) $\displaystyle\int_{\pi/3}^{\pi/2} \sqrt{\frac{1+\cos(t)}{1-\cos(t)}}dt$

Solution: We can rewrite the inside of the square root as a perfect square using the double angle formulas. From

$$\cos^2(\theta)=\frac{1+\cos(2\theta)}{2}, \quad \sin^2(\theta)=\frac{1-\cos(2\theta)}{2},$$

we obtain

$$1+\cos(t)=2\cos^2\left(\frac{t}{2}\right), \quad 1-\cos(t)=2\sin^2\left(\frac{t}{2}\right).$$

Using this for the integral gives

$$\begin{aligned}
\int_{\pi/3}^{\pi/2} \sqrt{\frac{1+\cos(t)}{1-\cos(t)}}dt &= \int_{\pi/3}^{\pi/2} \sqrt{\frac{2\cos^2\left(\frac{t}{2}\right)}{2\sin^2\left(\frac{t}{2}\right)}}dt \\
&= \int_{\pi/3}^{\pi/2} \sqrt{\cot^2\left(\frac{t}{2}\right)}dt \\
&= \int_{\pi/3}^{\pi/2} \left|\cot\left(\frac{t}{2}\right)\right| dt \\
&= \int_{\pi/3}^{\pi/2} \cot\left(\frac{t}{2}\right) dt
\end{aligned}$$

$$
\begin{aligned}
&= \left[2\ln\left|\sin\left(\frac{t}{2}\right)\right|\right]_{\pi/3}^{\pi/2} \\
&= 2\left(\ln\left(\sin\left(\frac{\pi}{4}\right)\right) - \ln\left(\sin\left(\frac{\pi}{6}\right)\right)\right) \\
&= 2\left(\ln\left(\frac{\sqrt{2}}{2}\right) - \ln\left(\frac{1}{2}\right)\right) \\
&= \boxed{\ln(2)}.
\end{aligned}
$$

(m) $\displaystyle\int \cos^2(8x)\sin^2(8x)dx$

Solution: We use the double-angle formulas

$$
\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}, \quad \sin^2(\theta) = \frac{1-\cos(2\theta)}{2}.
$$

This gives

$$
\begin{aligned}
\int \cos^2(8x)\sin^2(8x)dx &= \int \frac{1+\cos(16x)}{2}\cdot\frac{1+\cos(16x)}{2}dx \\
&= \frac{1}{4}\int \left(1-\cos^2(16x)\right)dx \\
&= \frac{1}{4}\int \left(1-\frac{1+\cos(32x)}{2}\right)dx \\
&= \frac{1}{8}\int \left(1-\cos(32x)\right)dx \\
&= \boxed{\frac{1}{8}\left(x-\frac{\sin(32x)}{32}\right)+C}.
\end{aligned}
$$

(n) $\displaystyle\int_{\pi/2}^{\pi} \sqrt{1+\sin(\theta)}d\theta$

Solution: One trick here is to multiply numerator and denominator by $\sqrt{1-\sin(\theta)}$. This gives

$$
\begin{aligned}
\int_{\pi/2}^{\pi} \sqrt{1+\sin(\theta)}d\theta &= \int_{\pi/2}^{\pi} \sqrt{1+\sin(\theta)}\cdot\frac{\sqrt{1-\sin(\theta)}}{\sqrt{1-\sin(\theta)}}d\theta \\
&= \int_{\pi/2}^{\pi} \frac{\sqrt{(1+\sin(\theta))(1-\sin(\theta))}}{\sqrt{1-\sin(\theta)}}d\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\pi/2}^{\pi} \frac{\sqrt{1-\sin^2(\theta)}}{\sqrt{1-\sin(\theta)}} d\theta \\
&= \int_{\pi/2}^{\pi} \frac{\sqrt{\cos^2(\theta)}}{\sqrt{1-\sin(\theta)}} d\theta \\
&= \int_{\pi/2}^{\pi} \frac{|\cos(\theta)|}{\sqrt{1-\sin(\theta)}} d\theta \\
&= \int_{\pi/2}^{\pi} \frac{-\cos(\theta)}{\sqrt{1-\sin(\theta)}} d\theta \quad \left(\cos(\theta) \leqslant 0 \text{ on } \left[\frac{\pi}{2}, \pi\right]\right) \\
&= \int_0^1 \frac{du}{\sqrt{u}} \quad (u = 1-\sin(\theta), du = -\cos(\theta)d\theta) \\
&= \left[2\sqrt{u}\right]_0^1 \\
&= \boxed{2}.
\end{aligned}
$$

(o) $\displaystyle\int_{\pi}^{2\pi} \frac{\sqrt{1+\cos(x)}}{e^x} dx$

Solution: We start with a double-angle identity in the square root to write

$$
\sqrt{1+\cos(x)} = \sqrt{2\cos^2\left(\frac{x}{2}\right)} = \sqrt{2}\left|\cos\left(\frac{x}{2}\right)\right|.
$$

For $\pi \leqslant x \leqslant 2\pi$, we have $\cos\left(\frac{x}{2}\right) \leqslant 0$, so $\left|\cos\left(\frac{x}{2}\right)\right| = -\cos\left(\frac{x}{2}\right)$. It follows that

$$
\int_{\pi}^{2\pi} \frac{\sqrt{1+\cos(x)}}{e^x} dx = -\sqrt{2}\int_{\pi}^{2\pi} e^{-x}\cos\left(\frac{x}{2}\right) dx.
$$

This integral can be calculated using IBP twice and solving algebraically for the unknown integral. For the first IBP, we take

$$
\begin{aligned}
&u = e^{-x} \;\Rightarrow\; du = -e^{-x}dx, \\
&dv = \cos\left(\frac{x}{2}\right) dx \;\Rightarrow\; v = 2\sin\left(\frac{x}{2}\right).
\end{aligned}
$$

We get

$$
\begin{aligned}
\int_{\pi}^{2\pi} e^{-x}\cos\left(\frac{x}{2}\right) dx &= \left[2e^{-x}\sin\left(\frac{x}{2}\right)\right]_{\pi}^{2\pi} - \int_{\pi}^{2\pi} 2\sin\left(\frac{x}{2}\right)(-e^{-x})dx \\
&= 2e^{-2\pi}\sin(\pi) - 2e^{-\pi}\sin\left(\frac{\pi}{2}\right) + 2\int_{\pi}^{2\pi} e^{-x}\sin\left(\frac{x}{2}\right) dx \\
&= -2e^{-\pi} + 2\int_{\pi}^{2\pi} e^{-x}\sin\left(\frac{x}{2}\right) dx.
\end{aligned}
$$

For the second IBP, take

$$u = e^{-x} \;\Rightarrow\; du = -e^{-x}dx,$$
$$dv = \sin\left(\frac{x}{2}\right)dx \;\Rightarrow\; v = -2\cos\left(\frac{x}{2}\right).$$

This gives

$$\begin{aligned}\int_{\pi}^{2\pi} e^{-x}\cos\left(\frac{x}{2}\right)dx &= -2e^{-\pi} + 2\left(\left[-2e^{-x}\cos\left(\frac{x}{2}\right)\right]_{\pi}^{2\pi} - \int_{\pi}^{2\pi}\left(-2\cos\left(\frac{x}{2}\right)\right)(-e^{-x})dx\right)\\ &= -2e^{-\pi} - 4e^{-2\pi}\cos(\pi) + 4e^{-\pi}\cos\left(\frac{\pi}{2}\right) - 4\int_{\pi}^{2\pi} e^{-x}\cos\left(\frac{x}{2}\right)dx\\ &= -2e^{-\pi} + 4e^{-2\pi} - 4\int_{\pi}^{2\pi} e^{-x}\cos\left(\frac{x}{2}\right)dx.\end{aligned}$$

We can now solve for the unknown integral. We get

$$\begin{aligned}5\int_{\pi}^{2\pi} e^{-x}\cos\left(\frac{x}{2}\right)dx &= -2e^{-\pi} + 4e^{-2\pi}\\ \int_{\pi}^{2\pi} e^{-x}\cos\left(\frac{x}{2}\right)dx &= \frac{-2e^{-\pi} + 4e^{-2\pi}}{5} = \frac{-2e^{-\pi}\left(1 - 2e^{-\pi}\right)}{5}.\end{aligned}$$

Going back to the original integral, we have found

$$\int_{\pi}^{2\pi} \frac{\sqrt{1+\cos(x)}}{e^x}dx = -\sqrt{2}\int_{\pi}^{2\pi} e^{-x}\cos\left(\frac{x}{2}\right)dx = \boxed{\frac{2\sqrt{2}e^{-\pi}\left(1-2e^{-\pi}\right)}{5}}.$$

#37. Express $\int \sin^n(3x)dx$ in terms of $\int \sin^{n-2}(3x)dx$.

Solution: We split off a factor $\sin(3x)$ and use IBP with parts

$$u = \sin^{n-1}(3x) \;\Rightarrow\; du = 3(n-1)\sin^{n-2}(3x)\cos(3x)dx,$$
$$dv = \sin(3x)dx \;\Rightarrow\; v = -\frac{\cos(3x)}{3}.$$

This gives

$$\begin{aligned}\int \sin^n(3x)dx &= \int \sin^{n-1}(3x)\sin(3x)dx\\ \int \sin^n(3x)dx &= -\frac{\sin^{n-1}(3x)\cos(3x)}{3} - \int 3(n-1)\sin^{n-2}(3x)\cos(3x)\frac{-\cos(3x)}{3}dx\end{aligned}$$

$$\int \sin^n(3x)dx = -\frac{\sin^{n-1}(3x)\cos(3x)}{3} + (n-1)\int \sin^{n-2}(3x)\cos^2(3x)dx.$$

In this last integral, we use the Phytagorean identity $\cos(3x)^2 = 1 - \sin(3x)^2$ to obtain

$$\int \sin^n(3x)dx = -\frac{\sin^{n-1}(3x)\cos(3x)}{3} + (n-1)\int \sin^{n-2}(3x)\left(1 - \sin^2(3x)\right)dx$$

$$\int \sin^n(3x)dx = -\frac{\sin^{n-1}(3x)\cos(3x)}{3} + (n-1)\int \sin^{n-2}(3x)dx - (n-1)\int \sin^n(3x)dx$$

We can now solve for the original integral by moving the term $-(n-1)\int \sin^n(3x)dx$ to the left-hand side.

$$\int \sin^n(3x)dx + (n-1)\int \sin^n(3x)dx = -\frac{\sin^{n-1}(3x)\cos(3x)}{3} + (n-1)\int \sin^{n-2}(3x)dx$$

$$n\int \sin^n(3x)dx = -\frac{\sin^{n-1}(3x)\cos(3x)}{3} + (n-1)\int \sin^{n-2}(3x)dx$$

We now divide by n to obtain the reduction formula

$$\boxed{\int \sin^n(3x)dx = -\frac{\sin^{n-1}(3x)\cos(3x)}{3n} + \frac{(n-1)}{n}\int \sin^{n-2}(3x)dx}.$$

#38. Calculate the volume of the solid obtained by revolving each region described below about (i) the x-axis and (ii) the y-axis.

(a) The region bounded by x-axis, the graph of $y = \cos^2(4x)$ on $0 \leqslant x \leqslant \frac{\pi}{8}$.

Solution: (i) We use the disk method. Revolving the vertical strip at x about the x-axis forms a disk with radius $r(x) = \cos^2(4x)$. Therefore the volume is

$$\begin{aligned} V &= \int_0^{\pi/8} \pi r(x)^2 dx \\ &= \int_0^{\pi/8} \pi \cos^4(4x)dx \\ &= \pi\int_0^{\pi/8} \left(\frac{1+\cos(8x)}{2}\right)^2 dx \\ &= \frac{\pi}{4}\int_0^{\pi/8} \left(1 + 2\cos(8x) + \cos^2(8x)\right)dx \end{aligned}$$

$$= \frac{\pi}{4}\int_0^{\pi/8}\left(1+2\cos(8x)+\frac{1+\cos(16x)}{2}\right)dx$$

$$= \frac{\pi}{8}\int_0^{\pi/8}\left(3+4\cos(8x)+\cos(16x)\right)dx$$

$$= \frac{\pi}{8}\left[3x+\frac{\sin(8x)}{2}+\frac{\sin(16x)}{16}\right]_0^{\pi/8}$$

$$= \boxed{\frac{3\pi}{64}\text{ cubic units}}.$$

(ii) We use the method of cylindrical shells. Revolving the vertical strip at x about the y-axis forms a shell with radius $r(x) = x$ and height $h(x) = \cos^2(4x)$. So the volume is

$$V = \int_0^{\pi/8} 2\pi r(x)h(x)dx$$

$$= \int_0^{\pi/8} 2\pi x\cos^(4x)dx$$

$$= \int_0^{\pi/8} 2\pi x\frac{1+\cos(8x)}{2}dx$$

$$= \pi\int_0^{\pi/8}\left(x+x\cos(8x)\right)dx$$

$$= \pi\left(\frac{\pi}{8}+\int_0^{\pi/8} x\cos(8x)dx\right).$$

This last integral can be computed using IBP with parts

$$u = x \;\Rightarrow\; du = dx,$$

$$dv = \cos(8x)dx \;\Rightarrow\; v = \frac{\sin(8x)}{8}.$$

This gives

$$V = \pi\left(\frac{\pi}{8}+\left[\frac{x\sin(8x)}{8}\right]_0^{\pi/8}-\int_0^{\pi/8}\frac{\sin(8x)}{x}dx\right)$$

$$= \pi\left(\frac{\pi}{8}+0-\left[-\frac{\cos(8x)}{64}\right]_0^{\pi/8}\right)$$

$$= \pi\left(\frac{\pi}{8}+\frac{\cos(\pi)-\cos(0)}{64}\right)$$

$$= \boxed{\pi\left(\frac{\pi}{8}-\frac{1}{32}\right)\text{ cubic units}}.$$

(b) The region bounded by the x-axis, the graph of $y = \sec^2(x)\tan(x)$ and the lines $x = 0$, $x = \frac{\pi}{4}$.

Solution: (i) We use the disk method. Revolving the vertical strip at x about the x-axis forms a disk with radius $r(x) = \sec^2(x)\tan(x)$. Therefore the volume is

$$\begin{aligned} V &= \int_0^{\pi/4} \pi r(x)^2 dx \\ &= \pi \int_0^{\pi/4} \sec^4(x)\tan^2(x) dx. \end{aligned}$$

The exponent of secant is even, so we can split off a factor $\sec^2(x)$, rewrite the rest of the integrand in terms of $\tan(x)$ using the Pythagorean identity $\sec^2(x) = \tan^2(x)+1$, and use the substitution $u = \tan(x)$, $du = \sec^2(x)dx$. The bounds become

$$\begin{aligned} x = 0 &\Rightarrow u = \tan(0) = 0, \\ x = \frac{\pi}{4} &\Rightarrow u = \tan\left(\frac{\pi}{4}\right) = 1. \end{aligned}$$

This gives

$$\begin{aligned} V &= \pi \int_0^{\pi/4} \sec^2(x)\tan^2(x)\sec^2(x) dx \\ &= \pi \int_0^{\pi/4} \left(\tan^2(x)+1\right)\tan^2(x)\sec(x)^2 dx \\ &= \pi \int_0^1 \left(u^2+1\right) u^2 du \\ &= \pi \int_0^1 \left(u^4+u^2\right) du \\ &= \pi \left[\frac{u^5}{5} + \frac{u^3}{3}\right]_0^1 \\ &= \pi \left(\frac{1}{5} + \frac{1}{3}\right) \\ &= \boxed{\frac{8\pi}{15} \text{ cubic units}}. \end{aligned}$$

(ii) We use the method of cylindrical shells. Revolving the vertical strip at x about the y-axis forms a shell with radius $r(x) = x$ and height $h(x) = \sec^2(x)\tan(x)$. So the volume is

$$\begin{aligned} V &= \int_0^{\pi/4} 2\pi r(x)h(x) \\ &= 2\pi \int_0^{\pi/4} x\sec^2(x)\tan(x) dx. \end{aligned}$$

We can compute this integral with an IBP taking the parts

$$u = x \;\Rightarrow\; du = dx,$$
$$dv = \sec^2(x)\tan(x)dx \;\Rightarrow\; v = \frac{\tan^2(x)}{2}.$$

We get

$$\begin{aligned} V &= 2\pi\left(\left[\frac{x\tan^2(x)}{2}\right]_0^{\pi/4} - \int_0^{\pi/4}\frac{\tan^2(x)}{2}dx\right) \\ &= 2\pi\left(\frac{\pi}{8} - \frac{1}{2}\int_0^{\pi/4}\left(\sec^2(x)-1\right)dx\right) \\ &= 2\pi\left(\frac{\pi}{8} - \frac{1}{2}\left[\tan(x)-x\right]_0^{\pi/4}\right) \\ &= 2\pi\left(\frac{\pi}{8} - \frac{1}{2}\left(1-\frac{\pi}{4}\right)\right) \\ &= \boxed{\frac{\pi(\pi-2)}{2}\text{ cubic units}}. \end{aligned}$$

#39. Evaluate $\int \sec^3(\theta)d\theta$ and $\int \sec(\theta)\tan^2(\theta)d\theta$.

Solution: We can evaluate $\int \sec^2(\theta)d\theta$ with an IBP and solving for the unknown integral when it reappears on the right-hand side. For the IBP we use the parts

$$u = \sec(\theta) \;\Rightarrow\; du = \sec(\theta)\tan(\theta)d\theta,$$
$$dv = \sec^2(\theta)d\theta \;\Rightarrow\; v = \tan(\theta).$$

We get

$$\begin{aligned} \int \sec^3(\theta)d\theta &= \int \sec^2(\theta)\sec(\theta)d\theta \\ \int \sec^3(\theta)d\theta &= \tan(\theta)\sec(\theta) - \int \tan(\theta)\sec(\theta)\tan(\theta)d\theta \\ \int \sec^3(\theta)d\theta &= \tan(\theta)\sec(\theta) - \int \tan^2(\theta)\sec(\theta)d\theta \end{aligned}$$

We will use the Pythagorean identity $\tan^2(\theta) = \sec(\theta)^2 - 1$ to see the original integral reappear on the right-hand side.

$$\int \sec^3(\theta)d\theta = \tan(\theta)\sec(\theta) - \int \left(\sec^2(\theta)-1\right)\sec(\theta)d\theta$$

$$\int \sec^3(\theta)d\theta = \tan(\theta)\sec(\theta) - \int \sec^3(\theta)d\theta + \int \sec(\theta)d\theta$$
$$\int \sec^3(\theta)d\theta = \tan(\theta)\sec(\theta) - \int \sec^3(\theta)d\theta + \ln|\sec(\theta)+\tan(\theta)|$$

We can now move the term $-\int \sec^3(\theta)d\theta$ to the left hand side and finish solving

$$2\int \sec^3(\theta)d\theta = \tan(\theta)\sec(\theta) + \ln|\sec(\theta)+\tan(\theta)|$$
$$\Rightarrow \int \sec^3(\theta)d\theta = \boxed{\frac{1}{2}\left(\tan(\theta)\sec(\theta) + \ln|\sec(\theta)+\tan(\theta)|\right) + C}.$$

For the other integral, we can use the Pythagorean identity and the integral of $\sec^3$ that we just computed as follows:

$$\begin{aligned}\int \tan^2(\theta)\sec(\theta)d\theta &= \int \left(\sec^2(\theta)-1\right)\sec(\theta)d\theta \\ &= \int \sec^3(\theta)d\theta - \int \sec(\theta)d\theta \\ &= \frac{1}{2}\left(\tan(\theta)\sec(\theta) + \ln|\sec(\theta)+\tan(\theta)|\right) - \ln|\sec(\theta)+\tan(\theta)| \\ &= \boxed{\frac{1}{2}\left(\tan(\theta)\sec(\theta) - \ln|\sec(\theta)+\tan(\theta)|\right) + C}.\end{aligned}$$

#40. Calculate the arc length of the curve $y = x + \cos(x)\sin(x) - \frac{1}{8}\tan(x)$, $0 \leqslant x \leqslant \frac{\pi}{4}$.

Solution: We have

$$\begin{aligned}\frac{dy}{dx} &= 1 - \sin^2(x) + \cos^2(x) - \frac{1}{8}\sec^2(x) \\ &= \cos^2(x) + \cos^2(x) - \frac{1}{8}\sec^2(x) \\ &= 2\cos^2(x) - \frac{1}{8}\sec^2(x).\end{aligned}$$

So

$$\begin{aligned}1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \left(2\cos^2(x) - \frac{1}{8}\sec^2(x)\right) \\ &= 1 + 4\cos^4(x) + \frac{1}{64}\sec^4(x) - 2\cdot 2\cos^2(x)\cdot\frac{1}{8}\sec^2(x) \\ &= 1 + 4\cos^4(x) + \frac{1}{64}\sec^4(x) - \frac{1}{2}\end{aligned}$$

$$= 4\cos^4(x) + \frac{1}{64}\sec^4(x) + \frac{1}{2}$$
$$= 4\cos^4(x) + \frac{1}{64}\sec^4(x) + 2 \cdot 2\cos^2(x) \cdot \frac{1}{8}\sec^2(x)$$
$$= \left(2\cos^2(x) + \frac{1}{8}\sec^2(x)\right)^2.$$

Therefore the arc length is given by

$$L = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$= \int_0^{\pi/4} \sqrt{\left(2\cos^2(x) + \frac{1}{8}\sec^2(x)\right)^2}\, dx$$
$$= \int_0^{\pi/4} \left|2\cos^2(x) + \frac{1}{8}\sec^2(x)\right| dx$$
$$= \int_0^{\pi/4} \left(2\cos^2(x) + \frac{1}{8}\sec^2(x)\right) dx$$
$$= \int_0^{\pi/4} \left(2\frac{1+\cos(2x)}{2} + \frac{1}{8}\sec^2(x)\right) dx$$
$$= \int_0^{\pi/4} \left(1 + \cos(2x) + \frac{1}{8}\sec^2(x)\right) dx$$
$$= \left[x + \frac{1}{2}\sin(2x) + \frac{1}{8}\tan(x)\right]_0^{\pi/4}$$
$$= \left(\frac{\pi}{4} + \frac{1}{2}\sin\left(\frac{\pi}{2}\right) + \frac{1}{8}\tan\left(\frac{\pi}{4}\right)\right) - 0$$
$$= \boxed{\frac{\pi}{4} + \frac{5}{8}\ \text{units}}.$$