Optimal Foreign Reserves and Central Bank Policy Under Financial Stress

By LUIS FELIPE CÉSPEDES AND ROBERTO CHANG*

Draft: October 17, 2023

We study foreign reserves accumulation and liquidity policy in an open economy under financial stress. Firms and households finance investment and consumption by borrowing from banks, which borrow from abroad. Binding financial constraints cause the domestic interest rate to rise over the world rate and the exchange rate to depreciate, implying inefficiently low investment and consumption. A role emerges for a central bank that accumulates reserves to provide international liquidity when financial frictions bind. Our analysis yields novel insights on the determinants of optimal reserves accumulation cum liquidity provision and their role vis a vis *capital flow management policies.*
JEL: F3, F4, E5

The period surrounding the global financial crisis of 2008-9 has motivated lively and important debates on macroeconomic policy. Two observations have attracted particular attention, especially in emerging economies. First, prior to the crisis, several central banks had accumulated significantly high amounts of foreign reserves. A justification for such accumulation, often mentioned by central bankers, was the need to build a "war chest" of international liquidity, to be available in case of sudden outflows of capital,

* Céspedes: Department of Economics, Universidad de Chile, and Central Bank of Chile (email: lfcespedes@fen.uchile.cl). Chang: Rutgers University and NBER (email: chang@econ.rutgers.edu). Simon Gilchrist was coeditor for this article. We thank Fernando Broner, Mario Crucini, Filippo Occhino, Jaume Ventura, Bernardo Pagnoncelli, Alep Simsek, and seminar participants at the BIS, CREI, PUC de Chile, Fordham University, Central Bank of Chile, NBER IFM Program Meeting, University of Central Florida, Universidad de Chile, Purdue University, and Universidad Católica del Perú for useful comments and suggestions. Céspedes acknowledges financial support from FONDECYT Grant No 1200865. The views expressed herein are exclusively those of the authors and do not necessarily reflect the position of the Central Bank of Chile.

similar to those experienced in the late 1990s.[1] Second, when the global crisis did erupt, those same central banks were in a good position to mitigate its impact, in particular by providing foreign currency liquidity that compensated for the external credit crunch, prevented the collapse of the financial sector, and calmed the markets.[2]

The first observation has motivated a literature that focuses on accounting for the observed levels of foreign reserves accumulation. That literature has served to identify variables, such as the probability of a "sudden stop" or the degree of financial development, that are key determinants of optimal levels of reserves and were ignored by older treatments which tied optimal reserve levels to international trade considerations. In turn, the second observation has spurred a host of studies on central bank responses to financial crunches and sudden stops, especially on the so called "unconventional policies" designed to increase the availability of credit in a financial crisis.

While these two literatures have delivered important and useful lessons, they have evolved mostly in parallel. It is not hard, however, to argue that the two issues, optimal foreign reserves accumulation and policy responses to crises, are intimately connected. For one thing, as mentioned, central bankers often say that they hoard foreign reserves *because* they plan to use them to finance international liquidity assistance in case of a crisis. For another, reserves accumulation may affect private sector behavior, inducing domestic agents to borrow more, and potentially placing the financial system in a more fragile situation. Finally, it may be the case that the probability of crisis itself may be affected by the level of international reserves and by the existence and nature of liquidity policies that the central bank implements during crises.

Accordingly, this paper analyzes the interaction between foreign reserves accumulation and central bank liquidity policies under financial duress, in a small open economy in which the probability of financial crisis is determined endogenously. This exercise yields several interesting lessons about the benefits and costs of reserves, and their link to the precise details of *ex post* central bank responses to crises and of financial intermediation.

[1] For example, the third survey on central banks' reserve management practices conducted by the World Bank found that self-insurance against external shocks is the primary justfication cited by central bankers for holding reserves.

[2] For a discussion of Latin American experiences, see Céspedes, Chang, and Velasco (2014).

We develop our arguments in a small open economy model that extends that of Céspedes, Chang, and Velasco (2017, hence CCV). In the model, domestic agents cannot borrow directly from international lenders, but instead they obtain credit from domestic financial intermediaries or banks, which in turn borrow from abroad. Because of incentive or enforcement problems, however, banks' foreign debts are limited to a multiple of their net worth. The borrowing constraint may or may not bind in equilibrium. If it does, there is a credit crunch that causes the domestic interest rate to rise above the world rate, the real exchange rate to depreciate, and investment to fall. In other words, when financial constraints bind the economy experiences a financial crisis.

The inclusion of domestic banks in the model is warranted for several reasons. First, it stresses that credit supply is determined by the way banks leverage net worth, a link that becomes central in crises. Second, having banks in the model allows us to show that the impact of foreign reserves policy depends on exactly how reserves are deployed to mitigate a crisis. In our model, as in CCV, the use of reserves to finance liquidity assistance in crises is more effective if liquidity is provided to banks instead of firms or households. The reason is that banks lever up credit from the central bank to secure additional credit in the world market. Third, the inclusion of financial intermediation adds realism to the model along several fronts. A notable one is that central bank credit assistance in recent crises episodes has been directed towards banks. Additionally, domestic banking sectors were the most important recipients of capital inflows to emerging market economies before the global financial crisis, and also the sectors that suffered the most significant fall in flows after the collapse of Lehman (Avdjiev et al. (2018)).

Our model departs from the one in CCV in two key respects. First, the severity of the collateral constraint can be affected by exogenous shocks, which can be thought of as exogenous "sudden stops" of international capital flows. Second, whether the collateral constraint binds or not also depends on inherited debt, which is determined in the model by the initial consumption and borrowing decisions of households, and is therefore endogenous.

As a consequence, under *laissez faire*, crises can occur with a probability that depends on the basic parameters of the economy. We characterize laissez faire equilibria and compare equilibrium outcomes against the *first best* allocation that would be chosen by a social planner having to respect only aggregate resource constraints. The comparison highlights the presence of inefficiencies both in initial consumption and investment. Both kinds of inefficiencies, however, reflect the main distortion: when financial constraints bind, domestic interest rates raise above the world interest rate.

Suboptimal consumption and investment motivate central questions: can foreign reserves policy improve efficiency and raise welfare, and how? For answers, we extend our setting to allow the central bank to accumulate international reserves to finance liquidity assistance in a sudden stop. Specifically, the central bank has the option of borrowing long term in order to acquire short term international assets. Such option can come at a cost, but enables the central bank to provide liquidity to domestic agents in case of need.

Taking advantage of the option requires the central bank to play an active role, one that would not be taken up by private agents. This is an important feature of our model: as we show, domestic banks would not find it individually optimal to issue long term debt to buy short term assets; this reflects that individual banks do not internalize that increasing their short term assets would result in a lower interest spread when financial constraints bind, raising efficiency. In this sense, a central bank policy of reserves accumulation is socially necessary.

We study in detail the *second best* problem associated with an optimal policy of accumulating reserves to provide credit to banks in crises. This problem differs from the first best one in that, if there is a premium for borrowing long term, holding reserves entails a financial cost and hence it reduces the economy's aggregate resources. More importantly, the second best problem features two restrictions other than resource constraints. One such restriction is the economywide borrowing constraint, written as a function of the allocation chosen by the planner in the spirit of Lorenzoni (2008) and Bianchi (2011). The second added restriction is the household's Euler condition, also as a function of the allocation, to account for the fact that the household's initial borrowing depends on the expected return to savings, which is in turn a function of the planner's allocation choice.

Analysis of the second best problem yields an intuitive characterization of the optimal reserves cum liquidity policy. If the financial cost of reserves is zero (i.e. if the term premium is nil), it is optimal to build a stock of reserves large enough to eliminate crises completely. This policy implements, in fact, the first best allocation, and is obviously best since reserves are costless. At the other extreme, if the financial cost of reserves exceeds a certain threshold, it is optimal to hold zero reserves. The intuition is that the benefits of liquidity provision in crises, while positive, are limited, so they cannot justify holding reserves if the financial cost is too large.

In the intermediate case of a positive but not too large term premium, the second best policy is to accumulate some reserves to be used to provide liquidity in crises. However, crises are not completely eliminated in this case. This implies that the second best policy reduces both consumption and investment inefficiencies, as well as the probability of crises, but does not eliminate them completely.

While our main focus is on foreign reserves and liquidity policy, our model can be readily modified to study other policy tools. There are, of course, numerous possible directions and extensions that lie outside the focus of this paper. However, we discuss how our analysis would change if the government could resort to a macroprudential or (using recent terminology proposed by the IMF) capital flow management (CFM) instrument, such as a proportional tax on initial debt. The inclusion of the CFM instrument in our model motivates the study of an alternative planning problem, one in which the planner chooses an allocation subject only to aggregate resource feasibility and the external borrowing constraint; the household's Euler condition is not restrictive in this case, since the CFM instrument can be adjusted to effectively control the initial debt. Analysis of the alternative planning problem illustrates key differences between a policy of reserves accumulation and liquidity provision vis a vis CFM policies. In particular, while a reserves cum liquidity policy can address at the same time both the consumption and the investment inefficiencies caused by financial frictions, as mentioned earlier, CFM policies can reduce one wedge only at the cost of exacerbating the other one.

To complement our analytical results, we explore a calibrated economy, emphasizing the optimal (second best) amount of reserves and its determinants. The optimal level of reserves turns out to depend on the economy's fundamentals and also, and as mentioned, on the details about the specific liquidity policy that the central bank implements in a crisis.

We show, for example, that the optimal level of reserves increases when their cost falls. This is as expected, of course, but underscores that the model is consistent with the view that one of the factors behind the observed reserves buildup has been the ample availability of international liquidity since the millennium. Also, we find that an increase in uncertainty easily implies that optimal reserves are higher. This is consistent with arguments often advanced by central bankers to justify reserves accumulation.

As for the impact of policy details, we show that the optimal level of foreign reserves depends on whether the central bank policy in a crisis takes the form of lending to banks (liquidity facilities, in the terminology of Gertler and Kiyotaki (2010)) or to firms and households (direct lending). This result follows because the benefits of reserves are given by their effectiveness in alleviating financial constraints when they become binding and, as demonstrated by CCV, direct lending is less effective than liquidity facilities because of leverage. On the other hand, we also find that, with direct lending, optimal reserves can be smaller or larger than under liquidity facilities. This reflects the fact that the optimal level of reserves is determined not by their net value (which is unambiguously smaller under direct lending) but by their marginal value (which can be smaller or larger, depending on fundamental elasticities).

Our paper is closely related to a large body of literature that focuses on the optimal amount of international reserves. A classic contribution is Heller (1966), in which shocks to the trade balance, such as a fall in foreign demand, are the main motive to hold external reserves. The optimal level of international reserves is given by the amount that minimizes the total cost of adjustment taking into consideration the cost of holding liquid international reserves and the probability that there will be a need for that level of reserves. Our paper is in the same spirit, but does not assign as big a role to trade factors. Instead, financial frictions take center stage.

The role of foreign reserves in mitigating the impact of financial shocks has been a focus of recent studies. A prominent and influential one is Jeanne and Ranciere (2011) which builds a model in which the accumulation of reserves is viewed as an insurance device against a sudden stop. They find that the optimal level of reserves then depends on the probability of the sudden stop, the consumer´s risk aversion, and the opportunity cost of holding reserves. Their model, however, takes several variables, such as consumption and savings, and the probability of a sudden stop, as either exogenous or ad hoc functions of reserves policy. In our model, consumption, savings, and investment are derived as equilibrium outcomes, which must then reflect expectations about central bank policy and reserves accumulation. In addition, the probability of crisis is endogenous, and interacts with private decisions, accounting in particular for the possibility of self insurance.[3]

Our paper is also related to recent studies on policy responses to financial crises, and especially to the literature on "unconventional" central bank policy. An early survey is Gertler and Kiyotaki (2010), which compares the different kinds of liquidity provision policies implemented by advanced country central banks in the midst of the global crisis. For emerging economies, CCV develops a similar comparison, emphasizing the importance of occasionally binding financial constraints. CCV also show the equivalence between sterilized foreign exchange intervention and liquidity provision, thus connecting the analysis of unconventional policy with an ongoing reconsideration of the implications of foreign exchange intervention.[4] Relative to this literature, our paper emphasizes that it is often the case, in emerging economies, that the ability of central banks to alleviate crises depends on their access to *international* liquidity, which they cannot themselves produce and, therefore, they must arrange for in advance, by hoarding reserves.

The present paper's perspective on the benefits of reserves accumulation contrasts with the one suggested by CCV and recently developed by Bocola and Lorenzoni (2020). In the models of CCV and Bocola and Lorenzoni, there may be multiple equilibria ex post.

[3]Other noteworthy recent contributions include Caballero and Panageas (2004) and Durdu, Mendoza, and Terrones (2009).

[4]For further development, see Chang (2018).

The central bank can then eliminate bad equilibria by implementing an appropriate lending of last resort policy. But the latter is only credible if the central bank has accumulated a large enough stock of international reserves.

As discussed, in our paper foreign exchange reserves are seen as a "war chest" that central banks can use in case of a financial crunch. This perspective complements recent studies on *ex ante* taxes or subsidies on international borrowing, leverage constraints, foreign borrowing limits, and other CFM policies. Notable contributions include Lorenzoni (2008), Bianchi (2011), Bianchi and Mendoza (2018), and Korinek and Simsek (2016). For a useful discussion between ex ante and ex post policies, see Benigno et al. (2013) and Jeanne and Korinek (2020).

Section I describes the model that serves as the setting for the analysis. Section II derives equilibrium under laissez faire and compares it against the first best allocation. Reserves accumulation to finance liquidity provision is introduced in section III, which derives the associated second best policy problem, characterizes its solution, and discuss implications. Section III also discusses the implications of allowing for CFM policies. Section IV discusses the determinants of the optimal level of reserves in a calibrated version of the model. Final remarks are collected in section V. Some formal proofs are delayed to an Appendix.

I. The Model

To convey our main ideas, we study a simple small open economy whose main features are described in this section. There are three periods ($t = 0, 1, 2$) and two goods, one tradable and another nontradable. The real exchange rate is defined as the relative price of nontradables in terms of tradables. The economy is inhabited by a representative household that owns domestic financial intermediaries (banks) and firms. Initially, households borrow from the home banks in order to finance consumption; in turn, banks finance their loans by borrowing from international capital markets. In the second period, households roll over their debts and firms buy productive capital, borrowing from domestic banks to pay for investment. Banks then borrow further from world capital markets to repay their previous debt and finance their new loans to households and firms. Crucially, in the second period, banks' international debt is limited by a collateral constraint that depends on their own net worth. The constraint may or may not bind. If it does, the domestic interest rate rises above the world rate, and the exchange rate depreciates. In that case, investment turns to be inefficiently low. In addition, initial consumption and debt depend on the likelihood of binding financial constraints, as they determine the expected cost of credit to households.

A. Households

Households consume only tradables, and only at $t = 0$ and $t = 2$. Preferences are given by expected utility, given by $U(C_0) + \beta E(C_2)$, where C_t denotes consumption in period t, β is a discount factor, $U(.)$ is (for simplicity) a CRRA function, and $E(.)$ is the expectation operator.

To express budget constraints, we take tradables as the numeraire. The only source of income for households are profits from firms and banks, which are paid only at $t = 2$. Hence, to finance consumption at $t = 0$, households borrow from domestic banks an amount $L_0^H = C_0$ at a gross interest rate R_0. In period $t = 1$, households roll over their debts at interest rate R_1, borrowing $L_1^H = R_0 L_0^H$. Finally, in period $t = 2$ households receive profits from banks and firms (denoted by Π^B and Π^F), repay their debts, and consume, so $C_2 = \Pi^B + \Pi^F - R_1 L_1^H$.

Finally, because there are no frictions in intermediation in the initial period, in any equilibrium the domestic interest rate R_0 must equal the world rate, denoted by R_0^*. Combining these considerations, the household's budget constraint can be written as:

$$R_0^* R_1 C_0 + C_2 = \Pi^B + \Pi^F \quad (1)$$

The household takes as given the interest rates $R_0 = R_0^*$ and R_1 as well as profits Π^B and Π^F. In equilibrium, the interest rate at which the household will have to roll over its debt in period 1, R_1, will be random. Denoting its expectation by $E(R_1)$, the first order condition for initial consumption is:

$$U'(C_0) = \beta R_0^* E(R_1) \tag{2}$$

The interpretation is the usual one: the household chooses initial consumption and debt by equating the marginal utility of initial consumption to its discounted expected price. Here, the relevant price is given by the two period interest rate, $R_0^* R_1$. Anticipating future discussion, financial frictions imply that the domestic interest rate in period $t = 1$ may raise above the world rate R_1^*. Thus $E(R_1) > R_1^*$, leading to inefficiently low consumption and borrowing.

B. Firms

At $t = 2$, competitive domestic firms produce Y_2 tradable goods with capital K_2 via the Cobb Douglas function $Y_2 = AK_2^\alpha$, with $0 < \alpha \leq 1$.

For production at $t = 2$, a typical firm purchases capital at $t = 1$, financing the purchase with a loan from domestic banks at rate R_1. Letting Q_1 denote the price of capital, the firm must then borrow from domestic banks an amount given by $L_1^F = Q_1 K_2$.

At $t = 2$, firms repay their debts to domestic banks, and send profits to the household, so $\Pi^F = Y_2 - R_1 L_1^F = AK_2^\alpha - R_1 Q_1 K_2$. Hence the profit maximizing demand for capital is given by:

$$\alpha AK_2^{\alpha-1} = R_1 Q_1 \tag{3}$$

The interpretation is again standard: investment equates the marginal product of capital to its cost. The cost of capital includes not only its price Q_1 but also the associated financial cost, given by the domestic interest rate R_1.

C. Capital production

Capital is produced in period 1 through the Cobb Douglas aggregator function $K_2 = \kappa I_H^{\gamma} I_W^{1-\gamma}$, where I_H and I_W denote inputs of nontradables and tradables, γ is a constant in [0, 1], and $\kappa = 1/\gamma^{\gamma}(1-\gamma)^{1-\gamma}$. The assumption of a Cobb Douglas aggregator can be generalized without substantially altering our analysis, and indeed in our numerical explorations in section IV we assume a C.E.S. aggregator. On the other hand, alternative assumptions about the elasticity of substitution between tradables and nontradables in capital production may introduce equilibrium multiplicity issues that we prefer to leave for future research.

Recalling that tradables is the numeraire, the price of capital is then:

$$Q = X^{\gamma} \tag{4}$$

where X is the price of nontradables in terms of the tradable, which we will refer as the *real exchange rate*.

The optimal input of nontradables is then given by $I_H = \gamma (Q_1/X_1) K_2$, while the demand for tradables is given by:

$$I_W = (1-\gamma) Q_1 K_2 \tag{5}$$

D. Banks

As indicated before, in each period $t = 0, 1$, domestic banks borrow from world capital markets at rates R_t^*, and lend to domestic households and firms at rates R_t. We assume that banks are competitive. Financial intermediation is frictionless at $t = 0$, so equilibrium requires that $R_0 = R_0^*$, as mentioned. In contrast, banks are subject to a collateral constraint at $t = 1$. This problem was discussed at length in CCV: we summarize its implications here.

At $t = 0$, banks borrow some amount D_0 from world capital markets and lend to households. At $t = 1$, banks roll over loans to households, lend to firms, repay their previous debts, and borrow a further amount D_1 from the world market to finance domestic loans. Also, banks have endowments T and N of tradables and nontradables respectively.

Hence, the amount of loans that the bank can extend at $t = 1$ is given by:

$$\begin{aligned} L_1 &= T + X_1 N + D_1 + R_0^* L_0^H - R_0 D_0 \\ &= T + X_1 N + D_1 \end{aligned}$$

the last equality being warranted because, in any equilibrium, $L_0^H = D_0$ and $R_0 = R_0^*$.

Finally, at $t = 2$, banks collect debt repayments from households and firms, repay their own debts, and send profits to households, and so $\Pi^B = R_1 L_1 - R_1^* D_1$. At $t = 1$, also, banks face the financial constraint

$$R_1 L_1 - R_1^* D_1 \geq \theta R_1 L_1 \tag{6}$$

where θ is a *random variable* realized at period $t = 1$. This constraint can be justified in various ways. For example, one can assume that at the start of period $t = 2$ domestic bankers can default on their foreign debt and divert a fraction θ of the payments made to the bank by firms. International lenders will then only accept contracts that satisfy the above constraint.

The assumption that θ is random is a key departure from CCV. For concreteness, we assume that θ can take n values, denoted by θ_s, $s = 1, ...n$, each with probability $\pi_s > 0$, and that this is the only source of uncertainty in the model. We also impose $\theta_1 = \underline{\theta} > 0$ and $\theta_n = \bar{\theta} < 1$. (To alleviate notation, the superscript s will be omitted when it is not needed.) A high realization of θ may reflect an exogenous tightening of international financial conditions, which can be regarded as a *sudden stop*.

The collateral constraint may or may not bind in equilibrium. If it does, we will sometimes say that there is a *financial crisis*. As we will see, a financial crisis will be more likely when θ turns out to be high. But importantly, the financial constraint may not bind even if θ is high; in other words, a *sudden stop* does not necessarily lead to a *financial crisis* in this model.

For future reference, we note implications for the bank's supply of loans. If θ is such that the collateral constraint does not bind, the cost of borrowing R_1 must be equal to R_1^*. In that case, the incentive constraint reduces to $L_1 - D_1 \geq \theta L_1$, the bank makes zero profits, and lends any quantity less than or equal to the multiple $1/\theta$ of its net worth:

$$L_1 \in [0, \frac{1}{\theta}(T + X_1 N)]$$

If θ is high enough so that the collateral constraint binds, $R_1 > R_1^*$. Combining the budget constraint of the bank with the binding collateral constraint, the bank´s supply of loans is:

$$L_1 = \frac{R_1^*}{R_1^* - (1-\theta)R_1}(T + X_1 N)$$

Loan supply is then a multiple of the bank´s net worth: the leverage ratio $R_1^*/[R_1^* - (1-\theta)R_1]$ is greater than one and finite in equilibrium (under the assumption that $R_1^* > (1-\theta)R_1$,which is maintained hereon). The previous expression indicates that banks leverage their capital to finance loans.

As in CCV, a real exchange rate depreciation (a fall in X_1) reduces bank´s net worth and, more novel, an increase in the spread raises the leverage ratio. On the other hand, we will see in the next section that a real exchange rate depreciation increases the spread and therefore, the leverage ratio. Hence a depreciation has ambiguous effects on the supply of loans when financial frictions bind.

II. Laissez Faire Equilibrium

In this section we study equilibrium under laissez faire. As the only source of uncertainty is θ, it is convenient to start with the analysis of the continuation equilibrium, from $t = 1$ on, which depends not only on the realized value of θ but also on the economy's inherited debt D_0. It turns out that, given θ and D_0, the collateral constraint may or may not bind. Hence the distribution of continuation outcomes, and in particular of R_1, depends on D_0.

In turn, D_0 is determined by initial consumption and savings choices which, as we have seen, reflect expectations about R_1. Equilibrium is then determined by a fixed point problem. The solution has no closed form but is not hard to illustrate numerically, as done later in this section.

We close the section with a discussion of the economy's first best social planning problem. A comparison of the social optimum against the laissez faire outcome underscores that this economy displays inefficiencies in both consumption and production.

A. Continuation Equilibrium

Consider the economy from $t = 1$ on, after $\theta = \theta_s$ is realized. At this point, the economy has an initial level of debt $D_0 = C_0$. From then on, the continuation economy is essentially the same as in CCV, so their results apply here.

It is useful to note that, in any continuation equilibrium with $\theta = \theta_s$ (which will be indicated with an s subscript hereon), several variables are pinned down by the domestic interest rate R_{1s}. To see this, start with the investment demand for tradables:

$$I_{Ws} = (1-\gamma)L^F_{1s} \tag{7}$$

which is (5) in state s, together with $L^F_{1s} = Q_{1s}K_{2s}$, which is the value of investment expenditure.

Since the supply of nontradables is equal to the bank's endowment of this good, N, equilibrium in the nontradables market is given by $I_{Hs} = N$. The production function for capital then implies that capital must be

$$K_{2s} = \kappa N^{\gamma}((1-\gamma)L^F_{1s})^{\gamma} \tag{8}$$

In turn, the price of capital $Q_{1s} = L_{1s}^F / K_{2s}$ can be written as

$$Q_{1s} = \left(\frac{L_{1s}^F}{N/\gamma} \right)^{\gamma} \tag{9}$$

and, from (4), the exchange rate must be

$$X_{1s} = \gamma \frac{L_{1s}^F}{N} \tag{10}$$

So I_{Ws}, K_{2s}, Q_{1s}, and X_{1s} are all functions of investment expenditure L_{1s}^F. The latter variable is determined by (3), which becomes:

$$L_{1s}^F = (\alpha A (N/\gamma)^{\alpha\gamma} / R_{1s})^{1/(1-\alpha(1-\gamma))} \tag{11}$$

The intuition for the above equations is clear. A higher interest rate R_{1s} induces firms to reduce investment expenditures. This is partly accomplished via a fall in capital production, and a fall in the price of capital. The latter also implies a real exchange rate depreciation.

Therefore, much of the continuation equilibrium is determined once we find the domestic rate R_{1s}. To solve for R_{1s} itself, we turn to the market for domestic loans. The demand for loans in period $t = 1$ is given by the value of investment plus the amount of debt that households must roll over:

$$\begin{aligned} L_{1s} &= L_{1s}^F + L_{1s}^H \\ &= (\alpha A (N/\gamma)^{\alpha\gamma} / R_{1s})^{1/(1-\alpha(1-\gamma))} + R_0^* C_0 \end{aligned}$$

As mentioned, a higher R_{1s} reduces investment expenditures and, therefore, loan demand in equilibrium. In addition, higher initial consumption in $t = 0$ increases loan demand in period $t = 1$, as households must roll over a higher debt.

The supply of loans depends on whether the credit constraint binds or not. If the credit constraint does not bind, the domestic interest rate must equal the world rate, i.e. $R_{1s} = R_1^*$, as observed earlier. Then the preceding equation gives the equilibrium quantity of loans, which we denote by L_{1f}:

$$L_{1f} = (\alpha A(N/\gamma)^{\alpha\gamma}/R_1^*)^{1/(1-\alpha(1-\gamma))} + R_0^* C_0$$

It is easy to see that, given C_0, L_{1f} is also the *frictionless* amount of credit, i.e., the one that would obtain in the absence of the collateral constraint (6). Likewise, capital, the exchange rate, the price of capital, and investment demand for tradables will all be at their frictionless equilibrium values, which we denote with an f subscript.

Our analysis now implies that, for the financial constraint *not* to bind, the frictionless credit amount L_{1f} must satisfy:

$$L_{1f} \in [0, \frac{1}{\theta_s}(T + X_{1f}N)]$$

Hence in equilibrium, the financial constraint does not bind if θ_s is less than or equal to a threshold $\vec{\theta}$ is given by:

$$\vec{\theta} = \frac{T + X_{1f}N}{R_0^* C_0 + Q_{1f}K_{2f}} \tag{12}$$

Note that the threshold $\vec{\theta}$ falls with C_0, reflecting that an increase in C_0 increases the amount that households must roll over and, hence, the economy's demand for external credit in $t = 1$. This is a crucial aspect of our model, which will imply that the probability of a crisis is endogenous.

Turn to the case of binding financial constraints. As noted in subsection I.D, when

$R_{1s} > R_1^*$ the bank´s supply of loans must satisfy the borrowing constraint with equality:

$$L_{1s} = \frac{R_1^*}{R_1^* - (1-\theta_s)R_{1s}}(T + X_{1s}N)$$

Regarding the real exchange rate X_{1s} as a function of R_{1s} (given by (10)-(11)) the preceding equation gives loan supply as a function of R_{1s}. It follows that, when financial constraints bind, the domestic interest must solve:

$$R_0^* C_0 + Q_{1s}K_{2s} = \frac{R_1^*}{R_1^* - (1-\theta_s)R_{1s}}(T + X_{1s}N) \tag{13}$$

For equilibrium, a solution R_{1s} of the preceding equation must be such that $R_{1s} \geq R_1^*$. We restrict our analysis to parameter values that satisfy that constraint. The condition also guarantees that the equilibrium is unique.[5]

Summarizing, if $\theta_s \leq \vec{\theta}$, the continuation equilibrium is the frictionless outcome given C_0. But if $\theta_s > \vec{\theta}$, investment, the domestic interest rate, and the exchange rate must adjust for the economy to satisfy the collateral constraint. Intuitively, when the collateral constraint binds, investment falls below its frictionless value, the interest rate spread widens, and the exchange rate depreciates. In other words, binding financial constraints look like a crisis.

B. Equilibrium and Implications

As mentioned, the analysis of the continuation equilibrium resembles that in CCV. But in this model, the continuation equilibrium depends on the realization of θ_s, which is exogenous, and also on initial consumption C_0, which is endogenous. In words,

$$\begin{aligned} R_{1s} &= R_1^* \text{ if } \theta_s \leq \vec{\theta} \\ &= \rho(C_0, \theta_s) \ \text{ if } \theta_s > \vec{\theta} \end{aligned}$$

[5] A proof is given in the Appendix.

where $\rho(C_0, \theta_s)$ is the value of R_{1s} that solves (13) and $\vec{\theta}$ is given by (12).

To complete the characterization of laissez faire equilibrium, we therefore turn to the determination of initial consumption and debt. C_0 must satisfy the Euler condition of the household, (2), which becomes:

$$U'(C_0) = \beta R_0^* \left[R_1^* F(\vec{\theta}) + \sum_{\theta_s > \vec{\theta}} \rho(C_0, \theta)\pi_s \right]$$

where, recalling our assumptions about the distribution of θ,

$$F(\vec{\theta}) = \sum_{\theta_s \leq \vec{\theta}} \pi_s$$

is the probability of no crisis.

Noting that $\vec{\theta}$ is a function of C_0 (by (12)), the expression can be seen as an equation in the single unknown C_0. Given a solution C_0, the continuation equilibrium is determined as in the preceding subsection. Laissez faire equilibrium is hence completely characterized.[6]

In particular, a solution C_0 determines $\vec{\theta}$ and the probability of crisis, which is therefore endogenous. Whether the probability of crisis is zero, one, or something in between, depends on the parameters of the model, and especially on the distribution of θ. If the distribution of θ is very favorable (for instance, if $\underline{\theta}, \overline{\theta}$ are close to zero), the collateral constraint never binds, and the continuation outcome is the frictionless equilibrium, regardless of the realization of θ. In other cases, the probability of crisis is positive, and it can be one if the distribution of θ is sufficiently adverse.

In order to give a flavor of the behavior of the model, we compute outcomes for par-

[6]It may be of interest to consider how the analysis would be modified if households had a nonzero initial endowment. It should be easy to see that nothing essential would change as long as the endowment is small enough (less than C_0). For larger endowments, households would become savers in the initial period. The natural assumption is that they then become bank creditors, thus obtaining the world rate of interest on their savings. The condition $U'(C_0) = \beta R_0^* R_1^*$ would then pin down C_0; as discussed later, this means that initial consumption would be at its frictionless level. On the other hand, investment would be below the frictionless level when the financial constraint binds and the domestic interest rate raises over the world rate. In fact, in this case, the continuation equilibrium would be exactly as in CCV.

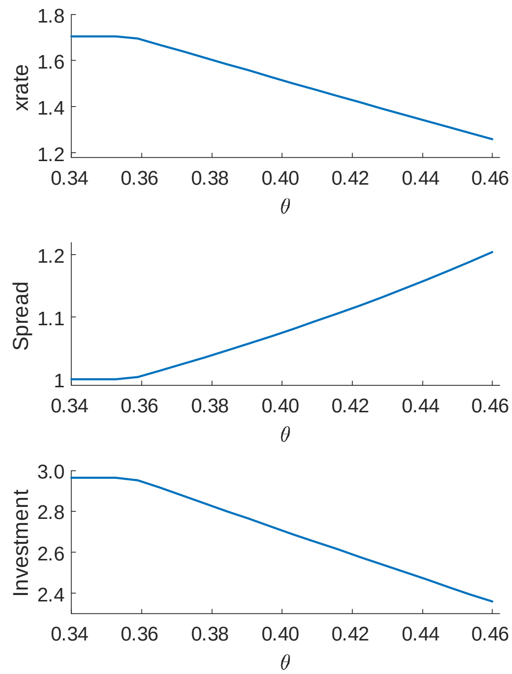

FIGURE 1. CONTINUATION EQUILIBRIUM

ticular parametrizations. We parametrize the model so that the frictionless C_0 is always equal to one. Other details of the parametrization are presented in section IV. In a baseline parametrization, C_0 in laissez faire is 0.9616, and aspects of the continuation equilibrium are depicted in Figure 1.

The figure shows that, in the continuation equilibrium, financial constraints do not bind if $\theta \leq \vec{\theta} = 0.3529$. In that case, the real exchange rate, the domestic interest rate, and investment are all at their frictionless values. For values of θ larger than $\vec{\theta}$, financial constraints bind. In that region, the exchange rate depreciates, the interest spread increases, and investment falls. As expected, these effects are stronger the larger is θ.

The laissez faire equilibrium depends on the parameters of the model in an intuitive way. To illustrate, Figure 2 describes the equilibrium crisis probability, the expected interest rate ($E(R_1)$) and initial debt and consumption, as functions of the expected value

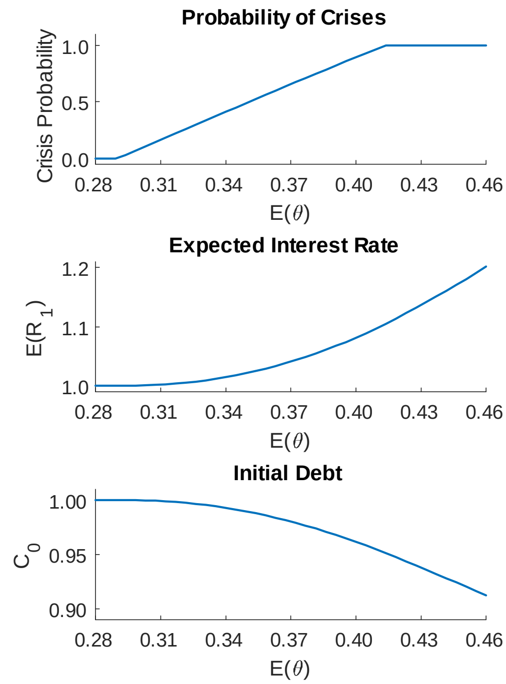

FIGURE 2. LAISSEZ FAIRE EQUILIBRIUM AND $E(\theta)$

of θ, keeping the dispersion of θ constant (i.e. taking $\underline{\theta} = E(\theta) - h, \bar{\theta} = E(\theta) + h$, the figure describes equilibrium outcomes as we vary $E(\theta)$, keeping h constant).

As expected, if the mean value of θ is small enough (less than 0.29 in the figure), financial constraints never bind. In this case, R_1 is always equal to $R_1^* = 1$, so that $E(R_1) = 1$. Initial debt is then equal to its frictionless value (one). As the mean value of θ rises, the probability of binding financial constraints goes up. Consequently, the equilibrium distribution of R_1 shifts to the right, and $E(R_1)$ goes up. Finally, initial consumption and debt go down, as expected future domestic interest rates increase.

Interestingly, an increase in ex ante uncertainty can result in an increase in the probability of crisis. This is depicted in Figure 3, which displays equilibrium outcomes as functions of h, keeping $E(\theta)$ constant at 0.34. For this parameterization, if uncertainty is sufficiently small (i.e. h is less than 0.01), financial constraints never bind, and crisis

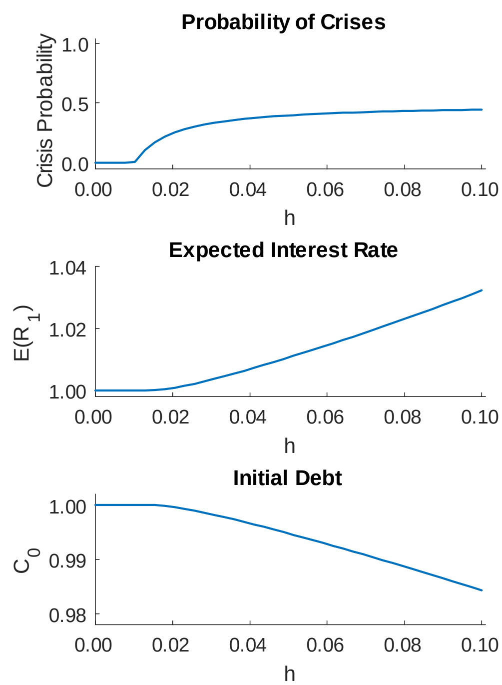

FIGURE 3. UNCERTAINTY AND LAISSEZ FAIRE OUTCOMES

do not occur. If, on the contrary, uncertainty is large enough, crises happen with positive probability. $E(R_1)$ goes up, reflecting this fact.

Notably, the figure shows that initial debt and consumption fall in response to an increase in uncertainty. This is in response, of course, to higher expected interest rates. A crucial fact is that the endogenous fall in initial debt is not sufficient, by itself, to eliminate the country's exposure to crises.

C. *First Best Versus Laissez Faire*

In previous subsections we discussed laissez faire equilibria. We turn now to the *first best* problem of a social planner that maximizes the expected discounted utility of the representative agent subject only to resource constraints. We write the latter as:

$$
\begin{aligned}
(14) \qquad C_2 &= AK_2^{\alpha} - R_0^* R_1^* C_0 - R_1^*(I_W - T) \\
K_2 &= \kappa N^{\gamma} I_W^{1-\gamma}
\end{aligned}
$$

These equations reflect that, in period 0, the planner finances initial consumption C_0 by borrowing in the world market, and repaying $R_0^* R_1^* C_0$ at $t = 2$. In addition, the imported component of investment in period 1, $I_W - T$, is borrowed abroad and requires repayment of $R_1^*(I_W - T)$ at $t = 2$. The resulting amount of capital is $K_2 = \kappa N^{\gamma} I_W^{1-\gamma}$, which determines the production of tradables $Y_2 = AK_2^{\alpha}$. In period 2, consumption C_2 is Y_2 minus debt repayments.

Note that uncertainty does not affect these constraints. This is because we have assumed that the only source of uncertainty is shocks to the external credit limit; but the first best problem assumes that the only constraints are resource constraints.

The first best choice of C_0, I_W, C_2, and K_2 maximizes $U(C_0) + \beta C_2$ subject to the above constraints. As the reader can check, the first order conditions with respect to C_0 and C_2 reduce to

$$U'(\hat{C}_0) = \beta R_0^* R_1^*$$

where we use carets ("^") to identify the first best allocation.

The interpretation is instructive. For the planner, the price of consumption at $t = 0$ in terms of consumption at $t = 2$ is $\beta R_0^* R_1^*$. The optimality condition $U'(C_0) = \beta R_0^* R_1^*$ is, therefore, the usual equality of the marginal rate of substitution of initial and final consumption with the relative price.

Recall now that, in laissez faire, initial consumption is given by $U'(C_0) = \beta R_0^* E(R_1)$, and that $E(R_1) > R_1^*$ if financial constraints bind with positive probability. In such a case, then, $U'(C_0) > U'(\hat{C}_0)$, or $C_0 < \hat{C}_0$. That is, initial consumption in laissez faire is inefficiently low.

The intuition is clear: if financial constraints bind in period $t = 1$, the interest cost associated with initial debt is higher than the relevant world interest rate, which is $R_0^* R_1^*$.

This discourages households from initial consumption financed with debt. In this sense, there is initial underborrowing in laissez faire, and the distortion is associated with the wedge $E(R_1) - R_1^*$.

Turning to investment, the first order conditions of the social planner's problem for K_2 and I_W lead to:

$$\alpha A\hat{K}_2^{\alpha-1} = R_1^*[1/(1-\gamma)\kappa N^\gamma \hat{I}_W^{-\gamma}]$$

Efficient investment is given by the preceding equation together with $\hat{K}_2 = \kappa N^\gamma \hat{I}_W^{1-\gamma}$.

The preceding optimality condition equates the marginal product of capital with its social marginal cost. The latter is given by the cost, inclusive of interest, of borrowing at $t = 1$ the amount of tradables necessary to produce a unit of capital. That amount is the inverse of the marginal product of tradables in the capital production function, $(1-\gamma)\kappa N^\gamma I_W^{-\gamma}$.

By comparison, recalling that the "s" subscript refers to continuation equilibrium when $\theta = \theta_s$, $\alpha AK_{2s}^{\alpha-1} = R_{1s}Q_s$ in a laissez faire equilibrium. But the price of capital is $Q_{1s} = 1/(1-\gamma)\kappa N^\gamma I_{Ws}^{-\gamma}$, so that investment in laissez faire is given by

$$\alpha AK_{2s}^{\alpha-1} = R_{1s}[1/(1-\gamma)\kappa N^\gamma I_{Ws}^{-\gamma}]$$

together with $K_{2s} = \kappa N^\gamma I_{W_s}^{1-\gamma}$

The two preceding equations show that the laissez faire investment condition and the first best investment condition are the same if and only if $R_{1s} = R_1^*$. This has three implications. First, while first best investment is independent of the state of nature, laissez faire investment may not be. Second, laissez faire investment coincides with first best investment if $R_{1s} = R_1^*$, that is, in states in which financial constraints do not bind. Third, if financial constraints bind, $R_{1s} > R_1^*$. In that case, one can show that $\alpha AK_{2s}^{\alpha-1} > \alpha A\hat{K}_2^{\alpha-1}$, that is, that the marginal product of capital is greater in laissez faire than optimal. Equivalently, of course, $K_{2s} < \hat{K}_2$: investment is suboptimally low. And the degree of inefficiency depends on the spread between R_{1s} and R_1^*.

Summarizing: because the domestic interest rate can raise over the world interest rate in laissez faire, the laissez faire allocation can exhibit inefficiently low consumption and investment. Along both dimensions, the extent of the inefficiency depends on the size of the interest rate spread.

III. Optimal Reserves and Liquidity Policy

In our model, inefficiencies in consumption and investment can be mitigated if the central bank provides liquidity when there is a financial crisis. However, in order to do so, the central bank must have ready access to the necessary international liquidity.

Arguably, considerations of this kind have been a main motivation for the accumulation of international reserves in emerging economies. Hence it is of interest to examine the implications of reserves accumulation in our model, and to ask about the determinants of the optimal quantity of reserves. We turn to these questions in the rest of the paper.

A. Reserves and the Role of the Central Bank

To allow for the accumulation of reserves, we assume from now on that the government or central bank has access to long term loans in tradables in the world market at $t = 0$. More precisely, if it borrows F dollars at $t = 0$, it must repay $(1 + \tau)R_0^* R_1^* F$ dollars at $t = 2$, where $\tau \geq 0$ is a *term premium*. (It will become clear that the more interesting case is $\tau > 0$).

The central bank can invest its F reserves in the world market and earn R_t^* in periods $t = 0$ and $t = 1$. Therefore, in our setup, the central bank invests its foreign reserves in liquid instruments, as central banks do in reality.

In this context, in period $t = 1$, the central bank will be able to use the reserves to enact a policy aimed at alleviating financial frictions. To simplify the discussion, we assume that, if the financial constraint binds at $t = 1$, the central bank can lend but not borrow at that point.

Before proceeding, we show that allowing commercial banks to accumulate reserves in the same way as the central bank would not affect the laissez faire equilibria examined so far. This is because individual banks take interest rates and the exchange rate as given, and therefore fail to internalize that an increase in available liquidity at $t = 1$ may help mitigating financial constraints. In contrast, the central bank recognizes that reserves are useful to relax constraints when they bind.

To see this point, suppose that domestic banks have access to the same international option just granted to the government. That is, any individual bank can borrow, say F', for two periods, at interest cost $(1 + \tau)R_0^* R_1^*$. The option would enable the bank to increase its loans in $t = 1$ by $R_0^* F'$. But on the other hand, its profits at $t = 2$ decrease by the service of the long term debt, so:

$$
\begin{aligned}
\Pi^b &= R_1 L_1 - R_1^* D_1 - (1+\tau)R_0^* R_1^* F' \\
&= R_1 L_1 - R_1^*(L_1 - (T + X_1 N + R_0^* F')) - (1+\tau)R_0^* R_1^* F' \\
&= (R_1 - R_1^*)L_1 + R_1^*(T + X_1 N) - \tau R_0^* R_1^* F'
\end{aligned}
$$

This expression makes it clear that borrowing F' cannot increase bank profits, and must reduce them if $\tau > 0$. This is clear in states of nature in which $R_1 = R_1^*$, as then profits reduce to $\Pi^b = R_1^*(T + X_1 N) - \tau R_0^* R_1^* F'$. If $R_1 > R_1^*$, there is an additional, negative impact, in that F' reduces the binding credit limit on L_1.[7]

We conclude that no individual bank would set $F' > 0$. The private banking sector would not accumulate liquidity in this model even if, as we will see, doing so may turn out to be collectively beneficial ex post. As we now show, only the central bank, which internalizes the impact of liquidity on financial constraints, can take advantage of the option to accumulate reserves to improve welfare.

[7]The collateral constraint becomes $R_1 L_1 - R_1^* D_1 - (1+\tau)R_0^* R_1^* F' \geq \theta R_1 L_1$, which leads to $L_1 \leq (1/(1-\phi(1-\theta))[T + X_1 N - \tau R_0^* F']$.

B. Competitive Equilibrium With Foreign Reserves

For the rest of the paper, we assume that financial constraints bind with positive probability under laissez faire. In that case, the central bank could attempt to take advantage of the option to accumulate reserves in order to provide liquidity in crises. Several questions then emerge. What are the implications of reserves accumulation for equilibrium? What is the optimal level of reserves, and what are its determinants? Interestingly, the answers depend on the class of policies that the central bank can implement at $t = 1$. In this subsection and the next we analyze the case in which the central bank borrows F at $t = 0$ and lends the value of its reserves, $R_0^* F$, to domestic banks at $t = 1$. For brevity, the policy combination just described will be called a *liquidity policy with F reserves* or, even more simply, an *F reserves policy.* [8]

As in CCV we assume that, under an F reserves policy, the central bank loans to domestic banks carry the world interest rate R_1^*, and that the repayment of these loans can be enforced perfectly. This means that the banks' collateral constraint changes to

$$R_1 L_1 - R_1^*(D_1 + R_0^* F) \geq \theta R_1 L_1 - R_0^* R_1^* F$$

which implies that loans' supply is now constrained by

$$L_1 \leq \frac{R_1^*}{R_1^* - (1-\theta) R_1}(T + X_1 N + R_0^* F) \tag{15}$$

The central bank repays $(1+\tau) R_0^* R_1^* F$ to foreign lenders at $t = 2$. Part of the repayment is covered from the amount $R_0^* R_1^* F$ collected from banks. We assume that the remainder, $\tau R_0^* R_1^* F$, is financed with a lump sum tax on households.[9]

Amending the analysis of competitive equilibrium under laissez faire in section II to the case with F reserves policy is straightforward. For convenience, we characterize the

[8] The term "liquidity policy with F reserves" is motivated by Gertler and Kiyotaki (2010), since such a policy entails the central bank providing *liquidity facilities*.

[9] The government budget constraint in period 2 is then, using Tax to denote the lump sum tax on households: $Tax = (1+\tau) R_0^* R_1^* F - R_0^* R_1^* F = \tau R_0^* R_1^* F$.

equilibrium in the following:

Definition: A *competitive equilibrium with an F reserves policy* is given by $C_0 \geq 0$ and, for all s, $R_{1s} \geq R_1^*$, and nonnegative K_{2s}, Q_{1s}, X_{1s}, I_{ws}, that satisfy :

$$U'(C_0) = \beta R_0^* \sum_s \pi_s R_{1s} \tag{16}$$

$$C_{2s} = AK_{2s}^{\alpha} - R_1^* I_{Ws} + R_1^* T - R_0^* R_1^* C_0 - \tau R_0^* R_1^* F \tag{17}$$

$$\alpha A K_{2s}^{\alpha-1} = R_{1s} Q_{1s} \tag{18}$$

$$K_{2s} = \kappa N^{\gamma} I_{Ws}^{1-\gamma} \tag{19}$$

$$Q_{1s} = X_{1s}^{\gamma} \tag{20}$$

$$I_{ws} = (1-\gamma) Q_{1s} K_{1s} \tag{21}$$

$$\frac{R_1^*}{R_1^* - (1-\theta_s) R_{1s}} \left[T + X_{1s} N + R_0^* F\right] - \left(R_0^* C_0 + Q_{1s} K_{1s}\right) \geq 0, \text{ = if } R_{1s} > R_1^* \tag{22}$$

Under a liquidity policy with F reserves, equation (17) expresses how F tightens the economy's intertemporal budget constraint, which is (14) in laissez faire, by the financial cost of reserves accumulation, $\tau R_0^* R_1^* F$. On the other hand, (22) shows that F relaxes the borrowing constraint in states s where the constraint binds.

The interpretation of the other equations is the same as before. Hence, under our maintained assumptions, the preceding paragraph summarizes the impact of an F reserves policy on the economy's equilibrium outcomes.

C. The Second Best Policy Problem

Having defined the class of F reserves policies and characterized their implications for equilibrium, it is obviously of interest to identify the optimal such policy. In this subsection we approach this problem by analyzing the *second best problem* of choosing a competitive equilibrium with an F reserves policy that maximizes the welfare of the representative household. As emphasized in recent literature, the study of the second best problem provides useful insight.[10] In addition, while the definition of competitive equilibrium leads to a seemingly intractable problem, it turns out that the second best problem can be reformulated in a simpler yet useful way.

Before discussing the solution to the second best problem, it is useful to notice that by varying F and implementing liquidity policy with F reserves, the central bank can effectively select a competitive equilibrium. In particular, in the Appendix we prove the following:

Lemma 1. There is $\bar{F}$ such that for any $F \geq \bar{F}$, financial frictions do not bind in the competitive equilibrium with an F reserves policy.

By Lemma 1, the central bank can always find an F reserves policy that completely eliminates crises. But this may be too costly if $\tau > 0$. In fact, it is easy to check that a liquidity policy with $F = \bar{F}$ reserves results in expected utility given by $EU_f - \tau R_0^* R_1^* \bar{F}$, where EU_f is expected utility in the *frictionless* equilibrium. This utility level falls with τ and $\bar{F}$, and suggests that other choices of $F < \bar{F}$ result in higher welfare. This is illustrated in Figure 4 for the baseline parametrization discussed in the next section.

We simplify the second best problem by using the fact that, given the initial consumption and debt C_0, and the amount of reserves F, the continuation equilibrium is largely

[10] For example, Lorenzoni (2008) and Schmitt-Grohé and Uribe (2017), chapter 12.

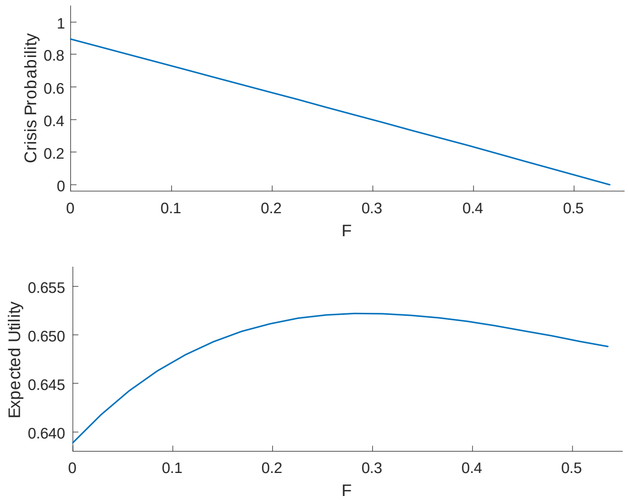

FIGURE 4. RESERVES, CRISIS PROBABILITY, AND EXPECTED UTILITY

pinned down by the interest rate R_{1s}. This is because our analysis of the continuation equilibrium in laissez faire extends to the present case:

Lemma 2. Given $R_{1s} \geq R_1^*$, there are unique $I_{ws}, K_{2s}, X_{1s}, Q_{1s}$,that satisfy the competitive equilibrium conditions (18)-(21).

Proof: The necessary expressions are given by (7), (8), (10), together with (11).■

Lemma 2 implies that, in any continuation equilibrium, I_{ws}, K_{2s}, X_{1s}, and Q_{1s} can be seen as functions of the interest rate R_{1s}, for $R_{1s} \geq R_1^*$. It also follows that if $R_{1s} = R_1^*$, I_{ws}, K_{2s}, X_{1s}, and Q_{1s} must equal their frictionless values, just like under laissez faire.

As an additional implication, Lemma 2 and (17) allow us to write C_{2s} as a function of R_{1s}, C_0, and F :

$$C_{2s} = C_2(R_{1s}, C_0, F) = AK_{2s}^{\alpha} - R_1^* I_{Ws} + R_1^* T - R_0^* R_1^* C_0 - \tau R_0^* R_1^* F \tag{23}$$

where, in the RHS, K_{2s} and I_{Ws} are seen as functions of R_{1s}, as given by the Lemma.

Finally, the borrowing constraint (22) can be written as:

$$\Psi(R_{1s}, C_0, F) \geq 0, \;\; = \text{ if } R_{1s} > R_1^* \tag{24}$$

where $\Psi(R_{1s}, C_0, F)$ is the term in the LHS of (22), with X_{1s} and $Q_{1s}K_{1s} = E_s$ regarded as the functions of R_{1s} given in Lemma 2.

The policy problem can be written as:

Second Best Problem. Choose $C_0 \geq 0, F \geq 0$, and $R_{1s} \geq R_1^*$, $s = 1, ...n$ to maximize

$$U(C_0) + \beta E(C_2) = U(C_0) + \beta \sum_s \pi_s C_2(R_{1s}, C_0, F) \tag{25}$$

subject to the Euler condition (16) and the borrowing constraint (24) ■

Our derivation highlights the crucial differences between this problem and the first best problem. One is that accumulating reserves entails an aggregate resource cost τF. This loss reduces final consumpion, and is captured in the definition of $C_2(R_{1s}, C_0, F)$ given by (23). Also, in the first best problem the choice of allocations is only constrained by aggregate resource feasibility. In the second best problem, in contrast, the Euler equation (16) appears as an additional constraint. This is because the social planner must recognize that an F reserves policy affects the return on savings, which affects the household's decisions on initial consumption and debt. Finally, the credit limit (24) is an additional restriction in the second best problem as well. The reason is essentially the same as in Lorenzoni (2008): under a liquidity policy with F reserves the social planner must still respect the external financial constraint.

The second best problem is analyzed in the Appendix. The most important result for our purposes is that the first order condition with respect to F can be written as

$$\sum_s \beta \pi_s \omega_s \left[\frac{R_1^*}{R_1^* - (1 - \theta_s) R_{1s}} \right] R_0^* \leq \tau R_0^* R_1^*, \;\; = \text{ if } F > 0 \tag{26}$$

where $\beta\pi_s\omega_s$ is the nonnegative Lagrange multiplier associated with the borrowing constraint (24). This expression has several implications which we collect in the following Proposition:

Proposition: (i) If $\tau = 0$, an optimal liquidity policy with reserves prescribes $F \geq \bar{F}$, so that crises do not occur.

(ii) Let Δ denote the value of the LHS of the inequality (26) under laissez faire. Then, $F = 0$ is optimal only if $\tau \geq \Delta / R_0^* R_1^*$

(iii) If $0 < \tau < \Delta / R_0^* R_1^*$, the optimal F reserves policy implies $0 < F < \bar{F}$, which allows crises to occur with positive probability.

Proof: For (i), observe that $\tau = 0$ implies $\omega_s = 0$ all s, and therefore the financial constraint never binds. If $F = 0$ then the outcome is the laissez faire equilibrium. Then the LHS of (26) is given by Δ, and (ii) follows. To prove (iii), observe that $F \geq \bar{F}$ implies that $\omega_s = 0$, all s; but this is ruled out by (26) if $\tau > 0$. This and (ii) imply $0 < F < \bar{F}$, and (iii) follows. ■

The Proposition is quite intuitive. If the term premium is zero, there is no financial cost for the economy to accumulate any amount of reserves, which can then be deployed to increase the supply of domestic credit sufficiently to keep the interest rate spread at zero for any possible value of θ_s. In this case, the second best policy entails an amount of reserves F equal to or greater than the value $\bar{F}$ of Lemma 1.

Part (ii) of the Proposition says that, if $\tau > 0$, it cannot be optimal to accumulate reserves if the term premium τ is too large. This is intuitive, but the condition tells us what is the precise meaning of "too large": if τ is greater than $\Delta / R_0^* R_1^*$, the benefit from a liquidity policy with reserves is not sufficient to justify the financial cost of the policy.

Hence, if $\tau > 0$ but sufficiently small, $F > 0$, that is, some reserves accumulation is optimal. But then $\omega_s > 0$ for at least some s. In other words, the second best policy prescribes some reserves accumulation, but not enough to eliminate crises completely. The above argument is similar to others in international trade and public finance. At $F = \bar{F}$, the benefit of reducing F by a marginal amount is positive, while the cost, in terms of allowing for a crisis to occur with small probability, is zero to first order.

In the case in which $F > 0$, the intuition for the first order condition (26) is that the optimal choice of F equates the marginal cost of reserves, given by $\tau R_0^* R_1^*$, to their marginal benefit, given by the LHS of (26). To understand the latter, note that a unit of reserves borrowed at $t = 0$ becomes R_0^* units of available tradables at $t = 1$, which can be leveraged up by banks to increase lending, and relax the state s borrowing constraint, by $\left[R_1^*/(R_1^* - (1-\theta_s)R_{1s})\right] R_0^*$. This quantity is valued via the Lagrange multiplier $\beta\pi_s\omega_s$, which is the increase in social welfare resulting from relaxing the borrowing constraint (24) by one unit.

Analysis of other optimality conditions gives additional insight. The first order condition for R_{1s} yields that, if $R_{1s} > R_1^*$,

$$\omega_s \frac{\partial \Psi(R_{1s}, C_0, F)}{\partial R_{1s}} = -\frac{\partial C_2(R_{1s}, C_0, F)}{\partial R_{1s}} + \lambda R_0^*$$

The intuition is as follows. As shown in the Appendix, the marginal increase in R_{1s} reduces the excess demand for loans or, equivalently, relaxes the financial constraint (22). This has a positive value to the planner, which is captured by the LHS of the equality.

The RHS turns out to be the cost of higher interest rates in terms of the investment distortion and the consumption distortion. To see that the first term represents the investment distortion, the Appendix shows that

$$\frac{\partial C_2(R_{1s}, C_0, F)}{\partial R_{1s}} = \frac{1-\gamma}{Q_{1s}}[\alpha A K_{2s}^{\alpha-1} - R_1^* Q_{1s}]\frac{dL_{1s}^F}{dR_{1s}}$$

which is negative because, from (11) and Lemma 2, $dL_s^F/dR_{1s} < 0$. The term in brackets is positive if $R_{1s} > R_1^*$, because in that case the marginal product of capital is greater than its first best value.

In turn, the last term in the RHS, λR_0^*, reflects that higher interest rates induce households to save more and consume less in the initial period. This is costly because, as we saw, C_0 is short of its first best level. Hence the term λR_0^* represents the impact of higher interest rates in terms of exacerbating the consumption distortion.

Finally, the first order condition with respect to C_0 is

$$U'(C_0) = \beta R_0^* R_1^* + \sum_s \beta \pi_s \omega_s - \lambda U''(C_0)$$

A marginal increase in C_0 increases utility directly by $U'(C_0)$. The RHS gathers three associated costs. First, final consumption falls by the cost of the associated debt, $R_0^* R_1^*$. A second cost is that the increase in initial debt resulting from a higher C_0 implies higher interest rates in states s where the financial constraint binds, with shadow price $\beta \pi_s \omega_s$. Finally, to induce households to increase C_0, the government must reduce expected interest rates in $t = 1$, which is feasible only if F increases. This imposes a cost associated with adherence to the Euler condition and summarized by the term $-\lambda U''(C_0)$.

That the second best policy balances distortions in investment and consumption can be stated more succintly as follows. Our discussion implies that a competitive equilibrium with a reserves policy is pinned down if F is given. The expected value of such an equilibrium is then a function of F, given by:

$$\begin{aligned} V(F) &= U(C_0) + \beta E(C_2) \\ &= U(C_0) - \beta R_0^* R_1^* C_0 + \beta \sum_s \pi_s \left[A K_{2s}^{\alpha} - R_1^* (I_{ws} - T) \right] - \beta \tau R_0^* R_1^* F \end{aligned}$$

where the last equality follows from the equilibrium condition (17).

If $\tau > 0$, the optimal value of F must satisfy $V'(F) = 0$ or, regarding C_0,K_{2s}, and I_{ws} as functions of F, can be expressed as

$$\left[U'(C_0) - \beta R_0^* R_1^* \right] \frac{dC_0}{dF} + \beta \sum_s \pi_s \left\{ \frac{d}{dI_{ws}} \left[A K_{2s}^{\alpha} - R_1^* I_{ws} \right] . \frac{dI_{ws}}{dF} \right\} - \beta \tau R_0^* R_1^* = 0$$

or, using that in any continuation equilibrium, $d(AK_{2s}^{\alpha})/dI_{ws} = R_{1s}$,

$$[U'(C_0) - \beta R_1^* R_0^*]\frac{dC_0}{dF} + \beta E\left\{(R_1 - R_1^*)I_w \frac{dI_w}{dF}\right\} = \beta\tau R_0^* R_1^*$$

The optimal F reserves policy equates the marginal financial cost of reserves, given by $\beta\tau R_0^* R_1^*$, to their marginal value, which is the LHS of the equality. A marginal increase in F helps reducing R_{1s} at times of crisis and, hence, induces an increase in C_0. The increase in welfare is given by dC_0/dF times the "consumption wedge" $U'(C_0) - \beta R_1^* R_0^*$. In addition, the increase in F allows for a fall in R_{1s} and to an increase in investment. The marginal benefit is given by dI_{ws}/dF multiplied by the "investment wedge", measured by the difference between the cost to firms of borrowing tradables for investment, equal to $R_{1s}I_{ws}$, and the world cost $R_1^* I_{ws}$.

D. Foreign Reserves and Capital Flow Management

Our analysis of optimal reserves accumulation *cum* liquidity provision led to the second best problem stated in subsection III.C: to maximize the social objective function (25) subject to the Euler condition (16) and the financial constraint (24). An influential recent literature (e.g. Lorenzoni (2008) and Bianchi (2011)) has emphasized a related but slightly different problem, one in which the planner chooses an allocation subject to resource constraints and the financial constraint, but not the Euler condition. The study of that alternative planning problem has been useful to approach policies intended to manage the initial amount of borrowing, often called *macroprudential* or, following the more recent terminology proposed by the IMF, *Capital Flow Management (CFM)* policies. Although the combination of CFM policies with reserves accumulation and liquidity provision is an ample topic that is largely outside the scope of this paper, several interesting insights come out from allowing CFM tools in our model, which leads to the alternative planning problem alluded to. This is the subject of this subsection.

To see how the alternative planning problem emerges when CFM tools are available, let us assume that the government can subsidize (or impose a tax on) the household's borrowing at $t = 0$. The fiscal cost of subsidies is financed by a lump sum tax on the household in the same period.

As the reader can easily check, such a policy only alters the model's equilibrium conditions by modifying the Euler equation, which becomes

$$(1+z)U'(C_0) = \beta R_0^* E(R_1) \tag{27}$$

where z is the rate of the borrowing subsidy (tax, if negative).[11] In other words, the combination of a subsidy on borrowing and a lump sum tax is purely distorting, only affecting the consumption-savings decision in the initial period. Hence the government can adjust z to "manage" the initial amount of debt and consumption.

A main consequence is that if, in addition to a liquidity policy with reserves, the government can resort to the CFM just described, its decision problem is exactly like the one at the end of subsection III.C except for the change in the Euler condition. More precisely, the government's problem is to choose $C_0 \geq 0, F \geq 0, z,$and $R_{1s} \geq R_1^*$, $s = 1, ...n$ to maximize (25) subject to the external borrowing constraint (24) and the modified Euler condition (27).

Now, the subsidy rate z appears in this problem only via the modified Euler condition. Therefore the optimal values of C_0, F, and $\{R_{1s}\}$ can be obtained by maximizing (25) subject to (24). The optimal subsidy rate z is then pinned down by the modified Euler equation.

Therefore, in the rest of this subsection we focus on the following:

Alternative Planning Problem. Choose $C_0 \geq 0$, $F \geq 0, z,$and $R_{1s} \geq R_1^*$, $s = 1, ...n$ to maximize (25) subject only to the financial constraint (24). ■

The alternative planning problem is clearly of the same form as those studied by Lorenzoni (2008) and others, as mentioned at the beginning of this subsection. The only difference relative to the second best problem of subsection III.C is that the Euler condition

[11] The argument is standard. In particular, the household's budget constraint in $t = 0$ becomes $(1+z)L_0^H = C_0 + Tax$. But $Tax = zL_0^H$, so $L_0^H = C_0$, as we had earlier.

(16) is dropped from the constraints. This is legitimate, as mentioned, because when the CFM tool is available the planner effectively controls initial consumption and borrowing by adjusting z.

We analyze the alternative planning problem formally in the Appendix. For the case $\tau = 0$, key implications are summarized in the following:

Lemma 3. Assume $\tau = 0$. Then: (i) A liquidity policy with reserves of size $F \geq \bar{F}$ together with $z = 0$ are optimal and deliver the first best outcome; (ii) no policy with $F = 0$ can be optimal.

Corollary. If $\tau = 0$, the first best outcome is attainable with only a liquidity policy with F reserves. If only the CFM policy is available, the first best outcome cannot be attained.

The proof of Lemma 3 is given in the Appendix. Part (i) is obvious once we recall that $\bar{F}$ is the level of reserves that suffices to provide enough liquidity to prevent a crisis, implementing the first best outcome. The necessity of $z = 0$ the follows from (27) and the fact that, at the first best, $U'(\hat{C}_0) = \beta R_0^* R_1^* = \beta R_0^* E(R_1)$.

Part (ii) of the Lemma is not so obvious but is an important observation, since it highlights the limitations of CFM policy in our context. To see this point, recall that the CFM policy means that the government manages the initial amount of debt and consumption C_0 Then, to identify how CFM policy works, we examine the impact on welfare of a small change in C_0.

Adapting the same arguments as at the end of the last subsection, one can write the change in welfare resulting from an infinitesimal change dC_0 in initial consumption and debt as

$$\left\{ \left[U'(C_0) - \beta R_0^* R_1^* \right] + \beta \sum_s \pi_s (R_{1s} - R_1^*) I_{ws} \frac{dI_{ws}}{dC_0} \right\} dC_0$$

This is to say, a small increase in C_0 affects welfare via its impact on the *consumption wedge* $U'(C_0) - \beta R_0^* R_1^*$ and, recalling that $dI_{ws}/dC_0 < 0$, on the *investment wedge* $(R_{1s} - R_1^*) I_{ws}$. In our model, there is both underborrowing and underinvestment under laissez faire, i.e. $U'(C_0) > U'(\hat{C}_0) = \beta R_0^* R_1^*$ and $R_{1s} \geq R_1^*$, with strict inequality for some s. Hence, a small increase in C_0 from laissez faire more reduce the consumption distortion but, because $dI_{ws}/dC_0 < 0$, it increases the investment distortion at the same time. The intuition is, of course, that a subsidy to initial borrowing can bring initial consumption closer to its first best level, but the associated increase in foreign debt tightens financial constraints, pushing investment further away from its first best level when the constraints bind.

We then see that the CFM policy always involves a trade-off: by raising C_0 relative to laissez faire, it reduces the consumption distortion but increases the investment distortion. In contrast, by reducing interest rate spreads, a liquidity policy with reserves can mitigate the investment distortion and the consumption distortion simultaneously.

Finally, turn to the case $\tau > 0$. To state our main results, let C_0^o and (ω_s^o, R_s^o), $s = 1, ...n$ denote the solution of the planning problem of this subsection under the additional restriction $F = 0$. (Note that this allocation would be optimal if the planner had access to CFM policy only.) The corresponding optimality conditions then imply that $\{C_0^o, (\omega_s^o, R_s^o)_{s=1}^n\}$ must solve the system:

$$\frac{\partial C_2(R_{1s}^o, C_0^o, 0)}{\partial R_{1s}} + \omega_s^o \frac{\partial \Psi(R_{1s}^o, C_0^o, 0)}{\partial R_{1s}} \leq 0, = 0 \text{ if } R_{1s}^o > R_1^*, \; s = 1, ...n \tag{28}$$

$$U'(C_0^o) = \beta \sum_s \pi_s \left[R_0^* R_1^* + \omega_s^o R_0^* \right] \tag{29}$$

(28) is just the first order condition (26), after imposing $F = 0$. In turn, (29) is the first order condition for the optimal choice of initial consumption and debt. It implies that, if only the CFM policy were available, the government would raise C_0 over its laissez faire value, but leave it short of the first best (since $\sum_s \pi_s \omega_s^o > 0$, as we will see). The reason is the trade-off mentioned before: higher initial consumption helps in terms of the consumption distortion, but the resulting increase in debt tightens the financial constraint when it binds.

Given these definitions, the Appendix proves the following:

Lemma 4. In the case $\tau > 0$: (i) The first best outcome is not attainable. (ii) Defining $\tilde{\tau}$ by

$$\sum_s \beta\pi_s\omega_s^o \left[\frac{R_1^*}{R_1^* - (1-\theta_s)R_{1s}^o}\right] R_0^* = \tilde{\tau} R_0^* R_1^* \tag{30}$$

if $\tau \geq \tilde{\tau}$, the optimal policy is given by $F = 0, C_0 = C_0^o$, and $(\omega_s, R_s) = (\omega_s^o, R_s^o)$, $s = 1, ...n$. (iii) If $0 < \tau < \tilde{\tau}$, the optimal policy requires a combination of both an F reserves policy and a CFM policy.

The formal proof is in the Appendix, but the intuition should be clear. Since the first best outcome is not attainable when $\tau = 0$ and $F = 0$, it cannot be attained with positive τ and $F = 0$. With $\tau > 0$, both the reserves cum liquidity policy and the CFM policy are distorting, so the theory of the second best prescribes that both instruments should be employed, unless τ is too large. This is what (ii) and (iii) say, with the threshold value of τ being defined by (30).

Further exploration of reserves cum liquidity policies when CFM tools are available is an interesting endeavor but best left for further research. Such an extension would require, in particular, to admit that, just like reserves accumulation may involve the τ deadweight losses, the application of CFM measures typically involves deadweight costs as well.

But the discussion of this subsection highlights significant differences between a policy of reserves accumulation and liquidity provision *vis a vis* CFM policies. We have seen, in particular, that the former can address at the same time both the consumption distortion and the investment distortions caused by financial frictions. In contrast, CFM policies can reduce one wedge but at the cost of exacerbating the other one. As a consequence, the planner would not resort to CFM policies if the term premium τ were zero. With a positive τ both kinds of policies would be employed, unless τ were too large.

IV. Determinants of the Optimal Level of Reserves

This section explores some determinants of the optimal level of reserves, emphasizing those that have not been identified in previous work (such as Jeanne and Ranciere (2011)). For this part of the analysis we resort to numerical exercises. We parametrize the model in order to illustrate its qualitative implications, rather than providing quantitative lessons, for which this model is not suited.

We set most parameters at conventional values. For example, we choose a baseline value of one for the world real interest rate R_t^*, $t = 0, 1$. We assume that the capital production aggregator is C.E.S. with an elasticity of substitution between tradables and nontradables equal to 1.4, while the share of the nontradable good in the production of capital is set to 0.5. The capital share α in tradables production is assumed to be 0.8. In the baseline parametrization we assume that θ is distributed uniformly between 0.34 and 0.46 and τ, the term premium, is equal to 0.02. We assume that $U(C) = C^{1-\sigma}/(1-\sigma)$, with $\sigma = 2$. The nontradable endowment is equal to one while the tradables endowment is assumed to be zero. Choices for other parameters are discussed below.

A. The Term Premium

In this model, an increase in the term premium τ will generally lead to a fall in the optimal amount of reserves, and hence an increase in the probability of crises. This is illustrated by Figure 5.

In the figure, the solid line graphs expected utility as a function of reserves F in the absence of a term premium or, equivalently, assuming that $\tau = 0$. Naturally, the solid line is increasing till F reaches 0.45, which corresponds to $\bar{F}$, that is, the value of reserves that eliminates crises.

In turn, the two rays in the figure (blue dashed-dotted or red dashed) correspond to $\left\{V(0) + \beta\tau R_0^* R_1^* F\right\}$ under two different values for the term premium ($\tau = 0.02$ and $\tau = 0.04$, respectively). Therefore, if $\tau > 0$, total expected utility is just the vertical difference between the solid graph and the corresponding ray. The optimal level of reserves

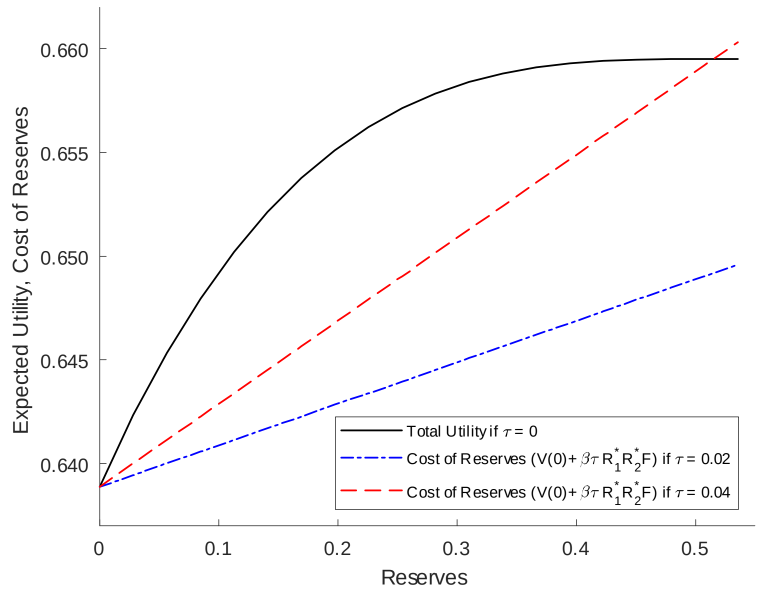

FIGURE 5. THE TERM PREMIUM

is that which maximizes that distance or, equivalently, given by the point on the solid line that has slope $\beta\tau R_0^* R_1^*$. It is then obvious that an increase in τ leads to a smaller value for the optimal F. This accords with intuition and the previous results of e.g. Jeanne and Ranciere (2011), where the opportunity cost of reserves corresponds to the interest rate spread on domestic agent's external debt.

Interestingly, during the period 2002-2008 there was a significant decrease in the term premium in global financial markets.[12] Our model then suggests that this decrease may have been a main factor underlying the observed increase in foreign reserves before the global crisis.

[12] This is an empirical regularity consistent with different proxies for the term premium. For example, the term premium on a 10 year zero coupon bond, retrieved from FRED, Federal Reserve Bank of St. Louis, decreased from 1.85% in March 2002 to a minimum of 0.27% in March 2007 and remained at low levels until early 2008.

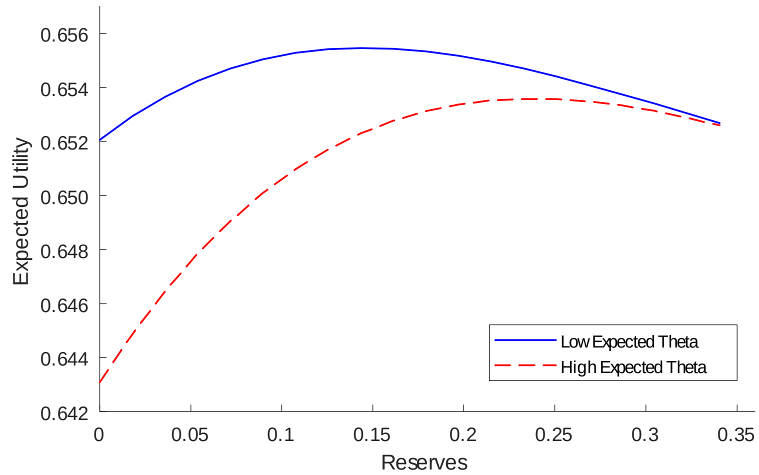

FIGURE 6. RESERVES AND $E(\theta)$

B. Mean Value of Shocks

An increase in the expected value of θ, keeping the dispersion of θ constant, typically increases the probability of binding financial constraints. One would then expect the benefits of liquidity provision in crises become stronger, and the optimal amount of reserves to increase.

Figure 6 provides an illustration. The solid (blue) line graphs $V(F)$ for $E(\theta) = 0.38$, while the dashed (red) line is $V(F)$ for $E(\theta) = 0.40$. As anticipated, for any F, expected utility is higher when $E(\theta)$ is lower. Also, for F sufficiently high, the two lines converge. This is because a sufficiently large F completely eliminates crises in both cases, and secures the same outcome (the frictionless one).

For this parametrization, the figure also tell us that optimal reserves are higher for higher $E(\theta)$. It should be noted, however, that this result may not be general since, as apparent from the figure, it depends on details about the curvature of $V(F)$ or, more precisely, on how $V'(F)$ changes with $E(\theta)$. This underscores the crucial point that the optimal level of reserves depends not on their total benefits, but on their *marginal* benefits, relative their marginal cost.

For a suggestive interpretation, one can take smaller $E(\theta)$ as a characteristic of more developed financial markets. In this sense, the model is consistent with the view that financial development justifies smaller international reserves accumulation. Evidence for such a view is presented by Dominguez (2010).

C. Impact of Ex Ante Uncertainty

Now consider the case of a mean preserving spread of θ. We have assumed that θ is uniformly distributed in $[\underline{\theta}, \overline{\theta}]$, with $\underline{\theta} = E(\theta) - h$, and $\overline{\theta} = E(\theta) + h$. Intuitively, for given $E(\theta)$, an increase in h must increase the probability of crisis, and justify higher reserves. An illustration is given by Figure 7, which fixes $E(\theta)$ (at 0.4) and plots $V(F)$ for $h = 0.04$ (solid line) and $h = 0.06$ (broken line). The analysis of the figure is the same as that of changes in $E(\theta)$. In this case, higher uncertainty justifies increased accumulation of reserves.

In our model, increased ex ante uncertainty raises the probability of crises. In response, it is optimal for the central bank to accumulate more reserves, so as to be ready to provide liquidity to domestic markets in case the crisis actually occurs. This is consistent with observed increases in international reserves in the period prior to the Lehman bankruptcy. Between May 2007 and the second quarter of 2008, the CBOE 10-Year Treasury Note Volatility Futures, an indicator that can be interpreted as a proxy for future financial conditions' volatility, increased steadily. This increase in uncertainty was prominently mentioned by central banks as a crucial element to accelerate foreign reserves accumulation. For example, in April 2008, the Central Bank of Chile announced a program of reserves purchases in response to increased financial risk.

D. Reserves accumulation and ex post tools

In our model, the reason for the central bank to accumulate reserves, even they are costly, is to alleviate financial constraints when they bind, using reserves to increase

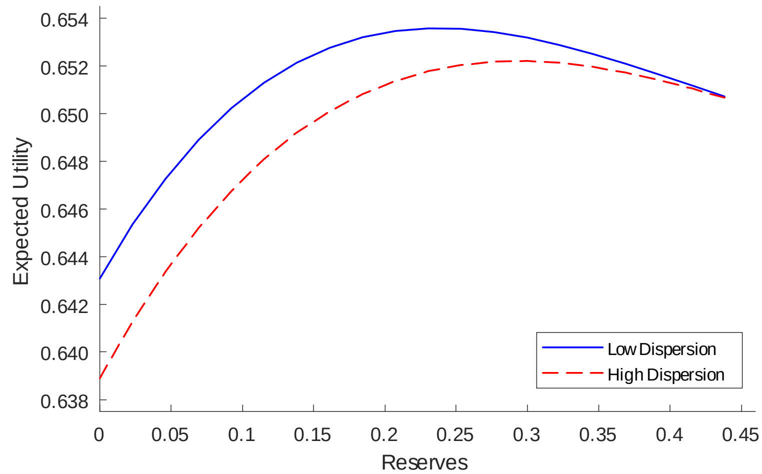

FIGURE 7. THE IMPACT OF UNCERTAINTY

credit to domestic agents. It follows that the optimal quantity of reserves should depend on the effectiveness of government intervention and the precise way the central bank intervenes.

In order to illustrate this point, consider the implications of changing the way the central bank provides liquidity in a crisis. So far we have assumed that, when the collateral constraint binds, the central bank extends credit to domestic banks (liquidity facilities). Instead, assume a policy of *direct lending*: in a crisis, the central bank lends directly to households and firms.

In any continuation equilibrium, the analysis of CCV applies, and implies that direct lending is less effective than liquidity facilities. The argument is as follows: we assume that at $t = 1$, the central bank increases the domestic supply of credit by its reserves $R_0^* F$; in period $t = 2$, the government collects $R_1 F$ in debt repayments and pays its foreign debt. Extending the arguments of previous sections, it can be shown that if $R_1 = R_1^*$, total loan supply is given by

$$L_1 \in [0, \frac{1}{\theta}(T + X_{1f}N) + R_0^* F]$$

while, if $R_1 > R_1^*$, loan supply is the sum of domestic bank loans plus government loans:

$$L_1 = \frac{R_0^*}{R_0^* - (1-\theta)R_1}(T + X_1 N) + R_0^* F$$

Under direct lending, each dollar of reserves is used at $t = 1$ to increase loan supply by one unit. In contrast, with liquidity facilities, loan supply in a crisis is $(T + X_1 N + R_0^* F)(R_0^*/[R_0^* - (1-\theta)R_1])$, which is larger than under direct lending for any given F. This reflects leverage: under liquidity facilities, each dollar in reserves is lent to banks, which then leverage it to borrow $(R_0^*/[R_0^* - (1-\theta)R_1]) > 1$ dollars in the world market. Hence direct lending is less effective at reducing interest spreads, and raising welfare, than liquidity facilities, for any given amount of reserves.

The implications for the optimal amount of reserves, however, are ambiguous. This is (again) because, while welfare is lower under direct lending than under liquidity facilities for any F, the difference in the optimal level of reserves depends on the comparison of the *marginal* benefit of reserves under each policy, relative their cost.

This is illustrated by Figures 8 and 9. Figure 8 plots $V(F)$ under liquidity facilities (solid line) and direct lending (broken line), under the assumption $\tau = 0.02$. As implied by our analysis, the two policies deliver the same expected welfare for $F = 0$ (i.e. no intervention) and F large enough (since crises are eliminated in both cases). For intermediate values of F, liquidity facilities deliver higher welfare than direct lending. Also, in that case, the optimal level of reserves is higher with a direct lending policy.

In comparison, Figure 9 assumes a higher $\tau = 0.04$. One can check that, for any F, the height of each curve in that panel is lower than the corresponding one in the upper panel. The key observation, however, is that the optimal level of reserves is now smaller under direct lending. The intuition is that, if τ is higher, reserves under both policies must fall because of the higher cost. However, the fall is more pronounced under direct lending, which is the less effective policy.

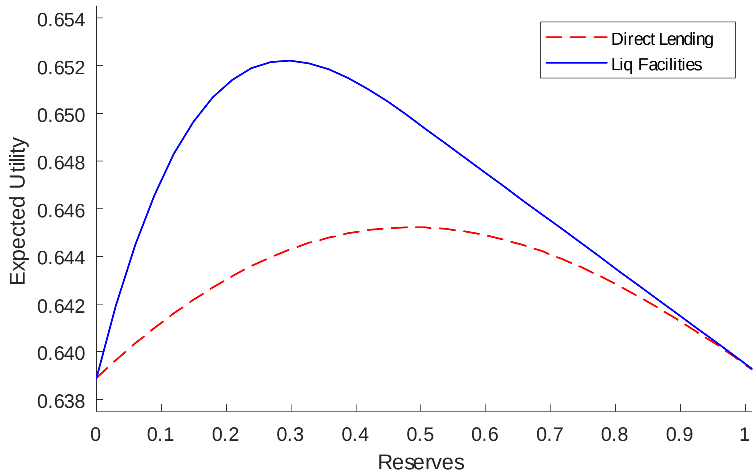

FIGURE 8. OPTIMAL RESERVES AND EX POST POLICIES, $\tau = 0.02$

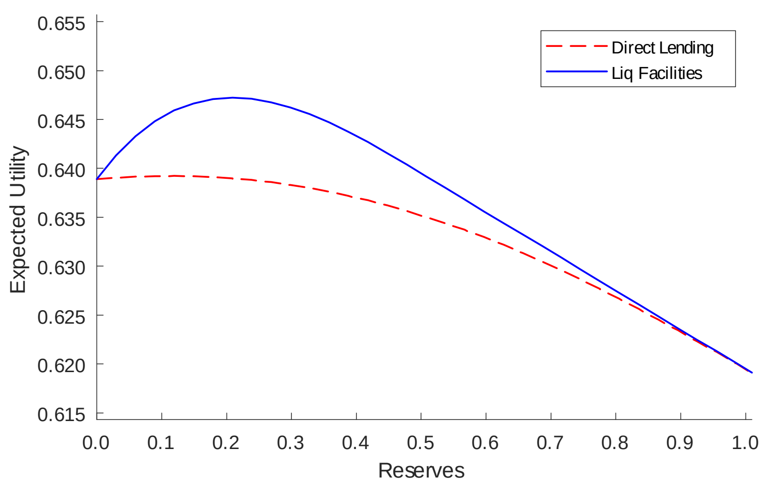

FIGURE 9. OPTIMAL RESERVES AND EX POST POLICIES, $\tau = 0.04$

V. Final Remarks

Our analysis can be extended in several interesting and potentially fruitful directions. One extension would be to allow for endogenous currency mismatches. In our model we could, for example, assume that in the initial period the household can borrow in either tradables or nontradables. In such a situation, real exchange movements would redistribute wealth between domestic banks and households, altering equilibrium outcomes when financial frictions bind. This extension is straightforward but the solution is quite involved, so we leave it for future work.

A further direction would be to develop a multiperiod version of our model and examine implications for dynamics. Such an extension would also allow the model to be calibrated or estimated. This would also be a substantial project.

We must mention also that the introduction of collateral constraints in contexts like the one developed here may result in multiplicity of equilibria, as discussed by CCV, Schmitt-Grohé and Uribe (2021), Jeanne and Korinek (2019), Bocola and Lorenzoni (2020), and others. The study of macroprudential policies and their interaction with foreign reserves accumulation in the presence of multiplicity of equilibria raises a host of issues that deserve a separate treatment.

There is also an empirical dimension worth exploring. The increase in foreign reserves in emerging markets accelerated significantly since 2002, a period during which many of the determinants emphasized in this paper experienced noticeable changes. Hence a formal empirical analysis of the links between foreign reserves accumulation and the determinants discussed here is also an attractive topic for future work

Finally, it may be of interest to introduce nominal rigidities in the model, so one can examine the interaction of reserves accumulation not only with unconventional policy but also with conventional monetary policy. It may be the case, for example, that reserves are instrumental in giving the central bank additional policy tools if the conventional tool, the policy interest rate, falls to its lower bound of zero.

APPENDIX

This appendix provides formal proofs for key results of the paper.

A. *Uniqueness of continuation equilibrium*

For the proof that the continuation equilibrium in subsection II.A is unique, define

$$\Psi_s = V_s \left[T + \gamma\, L_{1s}^F + R_0^* F\right] - (R_0^* C_0 + L_{1s}^F)$$

where the leverage coefficient is written as

$$V_s = \frac{R_1^*}{R_1^* - (1-\theta_s) R_{1s}} = \frac{1}{1-(1-\theta_s)\phi_s}$$

with $\phi_s = R_{1s}/R_1^*$ a measure of the interest rate spread.

Given (11) in the text we regard Ψ_s as a function of R_{1s}, C_0, and F, and wrote the collateral constraint as

$$\Psi_s = \Psi(R_{1s}; C_0, F) \geq 0 \tag{A1}$$

With this notation, the partial derivative of $\Psi(R_{1s}; C_0, F)$ with respect to R_{1s} is

$$\frac{\partial \Psi(R_{1s}; C_0, F)}{\partial R_{1s}} = \frac{d\Psi_s}{dL_{1s}^F}\frac{dL_{1s}^F}{dR_{1s}}$$

where $dL_{1s}^F/dR_{1s} < 0$ is obtained from (11), and $d\Psi_s/dL_{1s}^F$ is

$$\frac{d\Psi_s}{dL_{1s}^F} = \frac{\partial \Psi_s}{\partial L_{1s}^F} + \frac{\partial \Psi_s}{\partial V_s}\frac{dV_s}{\partial L_{1s}^F}$$

We now prove that an increase in investment expenditure L_{1s}^F must tighten the borrowing constraint Ψ:

Claim. If L_{1s}^F is such that $\Psi_s = 0$, $\frac{d\Psi_s}{dL_{1s}^F} < 0$.

Proof: Omitting the s subscripts for brevity:

$$\begin{aligned}
\frac{d\Psi}{dL_1^F} &= -1 + \gamma V + V^2 \left[T + \gamma L_1^F + R_0^* F\right](1-\theta)\frac{d\phi}{dL_1^F} \\
&= -1 + \gamma V - V(R_0^* C_0 + L_1^F)\frac{(1-\theta)\phi}{L_1^F}(1-\alpha(1-\gamma)) \\
&= -V R_0^* C_0 \frac{(1-\theta)\phi}{L_1^F}(1-\alpha(1-\gamma)) - 1 + \gamma V - V(1-\alpha(1-\gamma))(1-\theta)\phi
\end{aligned}$$

where we have used that $\Psi = 0$ implies $V\left[T + \gamma L_1^F + R_0^* F\right] = (R_0^* C_0 + L_1^F)$ and

$$\frac{L_1^F}{\phi}\frac{d\phi}{dL_1^F} = -(1-\alpha(1-\gamma))$$

Now the last term in the RHS is

$$\begin{aligned}
& -1 + \gamma V - V(1-\alpha(1-\gamma))(1-\theta)\phi \\
= & -V\left[1 - \gamma - (1-\theta)\phi + (1-\alpha(1-\gamma))(1-\theta)\phi\right] \\
= & -V(1-\gamma)\left[1 - \alpha(1-\theta)\phi\right]
\end{aligned}$$

Now, we know that $1 - \alpha(1-\theta)\phi > 1 - (1-\theta)\phi > 0$, so the term in brackets is positive. The claim now follows.

Now note that Ψ_s is a continuous function of L_{1s}^F and that, in any continuation equilibrium, L_{1s}^F must be such that $\Psi_s = 0$. The claim now means that Ψ_s is a decreasing function of L_{1s}^F, which ensures that a continuation equilibrium is unique.

B. *Proof of Lemma 1*

If crises never occur, $C_0 = C_{0f}$, that is, the initial consumption and debt will be at their frictionless level. Since investment and capital will also be at their frictionless levels, it follows that loan demand in period 1 will be $R_0^* C_{0f} + Q_{1f} K_{2f}$, and the exchange rate will be X_{1f}. The minimum F that eliminates crises, denoted by $\bar{F}$, must then be such that the collateral constraint just binds when θ is at its highest possible value, which we have denoted by $\bar{\theta}$. This is equivalent to

$$R_0^* \bar{F} = \bar{\theta}(R_0^* C_{0f} + Q_{1f} K_{2f}) - (T + X_{1f} N)$$

Given $\bar{F}$ thus defined, Lemma 1 then follows.

C. *Second Best Problem*

For the analysis of second best problem in subsection III.C, write the Lagrangian as:

$$\begin{aligned}
&U(C_0) + \beta \sum_s \pi_s C_2(R_{1s}, C_0, F) \\
&+\beta \sum_s \pi_s \omega_s \Psi(R_{1s}; C_0, F) \\
&+\lambda \left[U'(C_0) - \beta R_0^* \sum_s \pi_s R_{1s} \right]
\end{aligned}$$

where $\beta \pi_s \omega_s$ is the multiplier associated with (A1) and λ the multiplier associated with (16). Note that ω_s is nonnegative. We shall see that λ is nonnegative as well.

The first order condition for F now implies (26), after noting that

$$\frac{\partial C_2(R_{1s}, C_0, F)}{\partial F} = -\tau R_0^* R_1^*$$

and

$$\frac{\partial \Psi(R_{1s}, C_0, F)}{\partial F} = V_s R_0^*$$

The first order condition with respect to R_{1s} is:

$$\frac{\partial C_2(R_{1s}, C_0, F)}{\partial R_{1s}} + \omega_s \frac{\partial \Psi(R_{1s}, C_0, F)}{\partial R_{1s}} - \lambda R_0^* \leq 0, = 0 \text{ if } R_{1s} > R_1^*$$

where, from our previous discussion,

$$\begin{aligned}
\frac{\partial C_2(R_{1s}, C_0, F)}{\partial R_{1s}} &= \frac{d}{dR_{1s}}(AK_{2s}^{\alpha} - R_1^* I_{Ws}) \\
&= \frac{d}{dL_{1s}^F}(AK_{2s}^{\alpha} - R_1^* I_{Ws})\frac{dL_{1s}^F}{dR_{1s}} \\
&= \frac{1-\gamma}{Q_{1s}}[\alpha AK_{2s}^{\alpha-1} - R_1^* Q_{1s}]\frac{dL_{1s}^F}{dR_{1s}} \qquad \text{(A2)}
\end{aligned}$$

For any $R_{1s} \geq R_1^*$, $\alpha AK_{2s}^{\alpha-1} = R_{1s}Q_{1s} \geq R_1^* Q_{1s}$. So the term inside brackets in the last expression is nonnegative, implying that the derivative is nonpositive. Observe also that the derivative is zero at $R_{1s} = R_1^*$.

Finally, we prove that $\lambda \geq 0$. This follows from the fact that if s is such that the financial constraint does not bind, $R_{1s} = R_1^*$ and $\omega_s = 0$. The first order condition with respect to R_{1s} then reduces to $\lambda \geq 0$. This confirms an assertion in the text.

D. *Alternative Planning Problem*

For the alternative planning problem in subsection III.D the Lagrangian is

$$\begin{aligned}
&U(C_0) + \beta \sum_s \pi_s C_2(R_{1s}, C_0, F) \\
&+\beta \sum_s \pi_s \omega_s \Psi(R_{1s}; C_0, F)
\end{aligned}$$

where $\beta\pi_s\omega_s$ is the nonnegative multiplier associated with (A1).

Proceeding in the same way as before, we find that the first order condition with respect to F is still (26), reproduced here for convenience:

$$\text{(A3)} \qquad \sum_s \beta\pi_s\omega_s \left[\frac{R_1^*}{R_1^* - (1-\theta_s)R_{1s}}\right] R_0^* \leq \tau R_0^* R_1^*, \;\; = \text{ if } F > 0$$

while the first order condition with respect to R_{1s} is now:

$$\frac{\partial C_2(R_{1s}, C_0, F)}{\partial R_{1s}} + \omega_s \frac{\partial \Psi(R_{1s}, C_0, F)}{\partial R_{1s}} \leq 0, = 0 \text{ if } R_{1s} > R_1^* \tag{31}$$

Consider the case $\tau = 0$. Recall that a reserves and liquidity policy of size $F \geq \bar{F}$ implements the first best allocation, with $R_{1s} = R_1^*$ and $C_0 = \hat{C}_0$. Such a policy satisfies (A3) and (31) since $\omega_s = 0$, and $\partial C_2(R_1^*, \hat{C}_0, F)/\partial R_{1s} = 0$.[13] Also, because the first best inicial consumption level satisfies $U'(\hat{C}_0) = \beta R_0^* R_1^*$, the modified Euler condition (27) requires $z = 0$. This proves (i) in Lemma 3.

To show that any policy with $F = 0$ cannot be optimal if $\tau = 0$, suppose the contrary. Then $z = 0$ cannot be optimal, as it is strictly dominated by the policy of (i) in Lemma 3.

Given that z must be different from zero, (27) implies $E(R_1) = \sum_s \pi_s R_{1s} \neq R_1^*$. Hence $R_{1s} > R_1^*$ for at least some s, and so the corresponding multiplier ω_s must be strictly positive. But then the LHS of (A3) must be strictly positive, which cannot be the case if $\tau = 0$. This proves (ii) in Lemma 3.

Turn now to the case $\tau > 0$. We show that the first best allocation cannot be attained. This follows because, if the first best allocation were attainable, ω_s would be zero for all s, i.e. the financial constraint would not bind in any state. Then the the LHS of (A3) must be zero, so F cannot be strictly positive if $\tau > 0$. So F must be zero. Also, (27) would require $z = 0$. But the first best outcome would be attained in laissez faire, which has been ruled out by assumption. This proves (i) in Lemma 4.

Suppose for the moment that the planner were forced to set $F = 0$. The optimal CFL policy is then derived from the first order conditions with respect to R_{1s} would become

$$\frac{\partial C_2(R_{1s}, C_0, 0)}{\partial R_{1s}} + \omega_s \frac{\partial \Psi(R_{1s}, C_0, 0)}{\partial R_{1s}} \leq 0, = 0 \text{ if } R_{1s} > R_1^* \tag{A4}$$

[13] This follows from (A2), after noting that the policy implies $\alpha A K_{2s}^{\alpha-1} - R_1^* Q_{1s} = \alpha A \hat{K}_2^{\alpha-1} - R_1^* \hat{Q}_1 = 0$.

and the first order optimality condition for C_0, which is

$$U^{'}(C_0) = \beta \sum_s \pi_s \left[R_0^* R_1^* + \omega_s R_0^* \right] \tag{A5}$$

where we have used the definitions of $C_2(R_{1s}, C_0, F)$ and $\Psi(R_{1s}, C_0, F)$ and set $F = 0$. The system of equations and complementarity conditions given by (A4) and (A5) yield the solution for C_0, R_{1s}, and ω_s, $s = 1, ...n$. Denote the solution be denoted by C_0^o, R_{1s}^o, and ω_s^o.

Finally, we prove (ii) and (iii) of Lemma 4. Let $\tilde{\tau}$ be the minimum value of τ such that (A3) is satisfied, i.e.

$$\sum_s \beta \pi_s \omega_s^o \left[\frac{R_1^*}{R_1^* - (1-\theta_s) R_{1s}^o} \right] R_0^* = \tilde{\tau} R_0^* R_1^*$$

Given that definition, for any $\tau \geq \tilde{\tau}$ the first order condition (A3) is satisfied when $F = 0$, and C_0, R_{1s}, ω_s are equal to C_0^o, R_{1s}^o, and ω_s^o. By construction (A4) and (A5) are also satisfied, and hence $(F, C_0, R_{1s}, \omega_s) = (0, C_0^o, R_{1s}^o, \omega_s^o)$ solve the alternative maximization problem. In contrast, if $0 < \tau < \tilde{\tau}$, that solution violates (A3), and as a consequence the constrained optimal policy prescribes $F > 0$. Also, z cannot be zero, since otherwise an optimal F reserves policy would by itself eliminate crises completely, which was ruled out earlier.

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