Ch. 2 Limits and Continuity

Limits → what happens to the function $f(x)$ as its independent variable (x) approaches (NOT equal to) some value.

E.g: position function $s(t) = 16t^2$
"free fall object"
$t \to$ time

What's the object's velocity at time $t=2$?

Estimate the instantaneous velocity at $t=2$?

Solution:
average velocity over time interval $[t_1, t_2]$

$$\bar{v} = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} \left(\frac{\text{distance traveled}}{\text{elapsed time}} \right)$$

Compute the ave. velocity over smaller & smaller time intervals around 2sec.

for $[1.9, 2]$
$$\bar{v} = \frac{s(2) - s(1.9)}{2 - 1.9} \quad \boxed{s(t) = 16t^2}$$

$$= \frac{16 \cdot 2^2 - 16 \cdot (1.9)^2}{0.1} = 62.4$$

If we get closer to $t=2$, we should get better approximation to the instantaneous velocity at $t=2$.

What will we observe?

Is there a specific value the ave. velocity approaches to as the elapsed time gets smaller?

Time Interval	Elapsed time (Δt)	Average velocity ($\bar{v}$)
[1.99, 2]	0.01	63.84
[1.999, 2]	0.001	63.98
[2, 2.0001]	0.0001	64.0016
[2, 2.001]	0.001	64.016
[2, 2.01]	0.01	64.16

We observe that as Δt gets smaller, $\bar{v}$ gets closer to 64 (as long as one endpoint of the interval is $t=2$)

It's reasonable to expect the velocity at the instant $t=2$ to be 64.

Is there a way we can calculate $v(2)$ exactly?

Consider a very tiny number, h and setup the time interval as $[2, 2+h]$

The average velocity of the falling object over this tiny time interval $[2, 2+h]$ is:

$$\bar{v} = \frac{s(2+h) - s(2)}{2+h-2}$$

Recall $s(t) = 16t^2$

$$= \frac{16(2+h)^2 - 16 \cdot 2^2}{h}$$

$$= \frac{16(4+4h+h^2) - 16 \cdot 4}{h}$$

$$= \frac{\cancel{16 \cdot 4} + 16 \cdot 4h + 16 \cdot h^2 - \cancel{16 \cdot 4}}{h}$$

$$= \frac{64h + 16h^2}{h} = \frac{\cancel{h}(64+16h)}{\cancel{h}} = \boxed{64+16h}$$

$\bar{v} = 64 + 16h$

Average velocity has a limiting value of 64 as h tends to zero. $\left[\lim_{h \to 0} \bar{v} = 64\right]$

Informal Definition of Limit.

$$\lim_{x \to c} f(x) = L$$

"The limit of f(x) as x approaches c is L."

"f(x) can be made arbitrarily close to a unique number L by choosing x arbitrarily close to c (but not equal to c)"

Exp) Use a table of values to estimate

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{x^2+x-2}{x+2}$$

Solution: Domain of a rational function f(x) → check the denominator.

$$\left.\begin{aligned} x+2 &\neq 0 \\ x &\neq -2 \end{aligned}\right\} \quad f(-2) \text{ is undefined}$$

However, limit is concerned only when x approaches -2 NOT f(-2).

The table of values of $f(x)$ for x near -2:

x	-2.1	-2.05	-2.001	-2	-1.9997	-1.995
$f(x)$	-3.1	-3.05	-3.001	undefined	-2.9997	-2.995

Table suggests that: $f(x) \to -3$ as $x \to -2$;

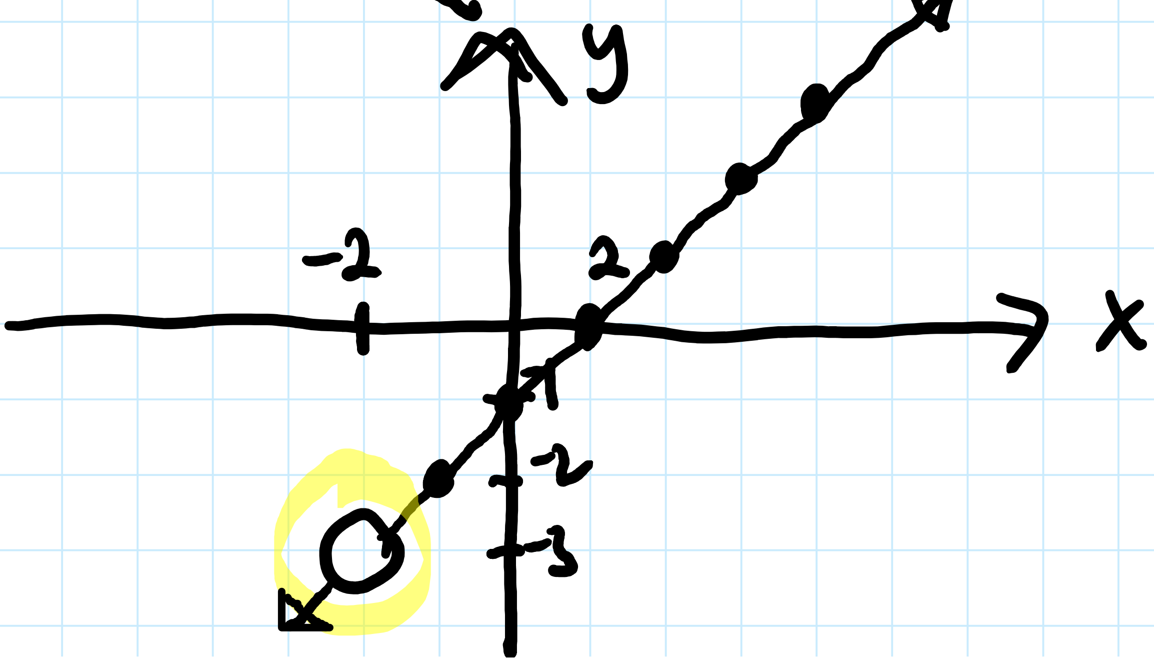
$$\lim_{x \to -2} \frac{x^2+x-2}{x+2} = -3$$

Since $f(x)$ is undefined at $x=-2$, there's a hole at $(-2,-3)$.

$$f(x) = \frac{x^2+x-2}{x+2} = \frac{\cancel{(x+2)}(x-1)}{\cancel{x+2}} = x-1; \text{ if } x \neq -2$$

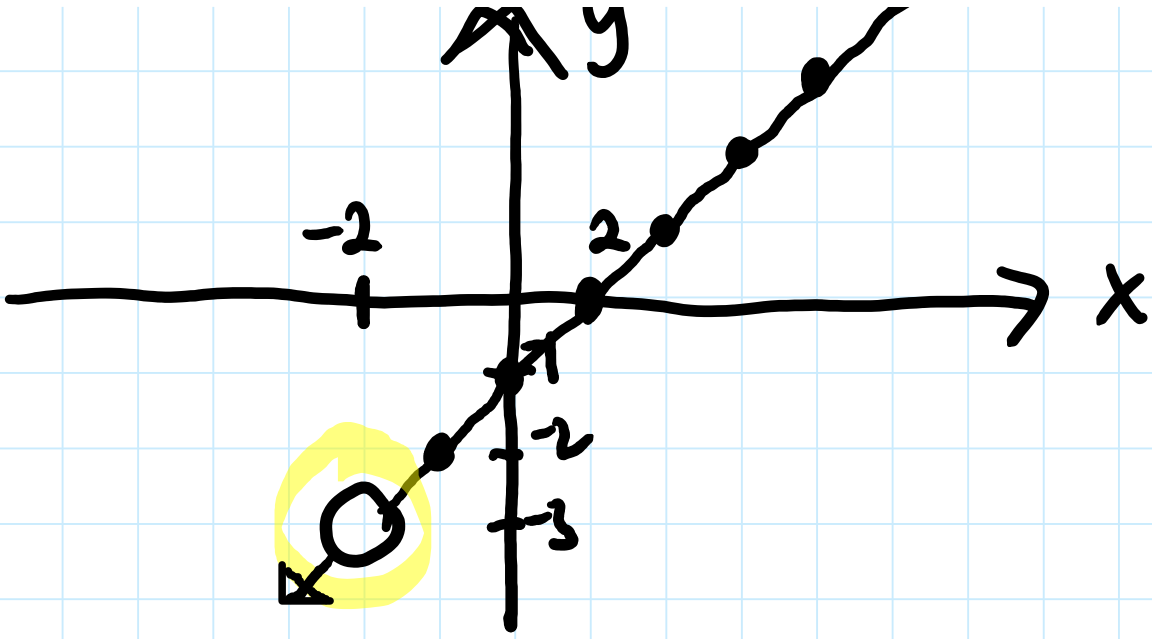
We can re-write $f(x)$ as a piece-wise function:

$$f(x) = \begin{cases} x-1, & \text{if } x \neq -2 \\ \text{undefined}, & \text{if } x = -2 \end{cases}$$



Graph of

$$f(x) = \frac{x^2+x-2}{x+2}$$



Graph of

$$f(x) = \frac{x^2+x-2}{x+2}$$

One-Sided Limits

Goal: Investigate the "limiting" behavior of a function from one-side;

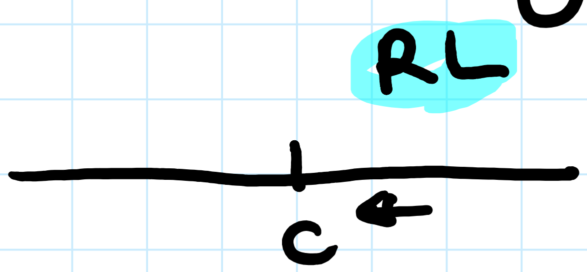
What's the limit of $f(x)$ as x approaches (gets arbitrarily close) "c" from left or right.

One-Sided Limits:

Left-hand limit: If we can make $f(x)$ as close to L as we please by choosing x arbitrarily close to c immediately to the left of c.

$$\lim_{x \to c^-} f(x) = L$$

Right-hand limit: If we can make $f(x)$ as close to L as we please by choosing x arbitrarily close to c immediately to the right of c.



$$\lim_{x \to c^+} f(x) = L$$

What if $RL \neq LL$ (informally speaking)

One-sided limit theorem:

The two-sided limit $\lim_{x \to c} f(x)$ exists

If and only if the two-sided limits

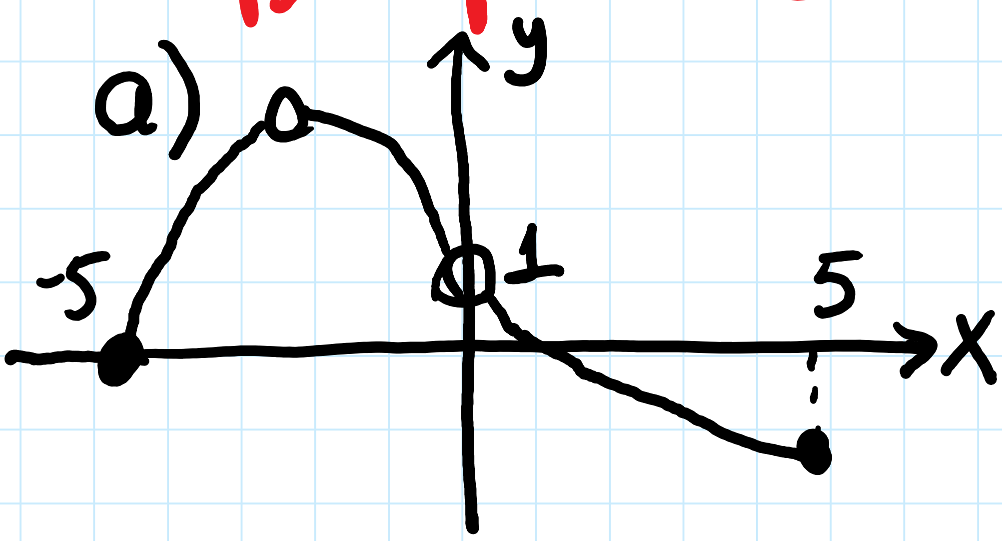
$\lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^+} f(x) = L$

both exist AND are equal.

If $\lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)$

Then $\lim_{x \to c} f(x) = L$.

Exp) Inspect the limits:

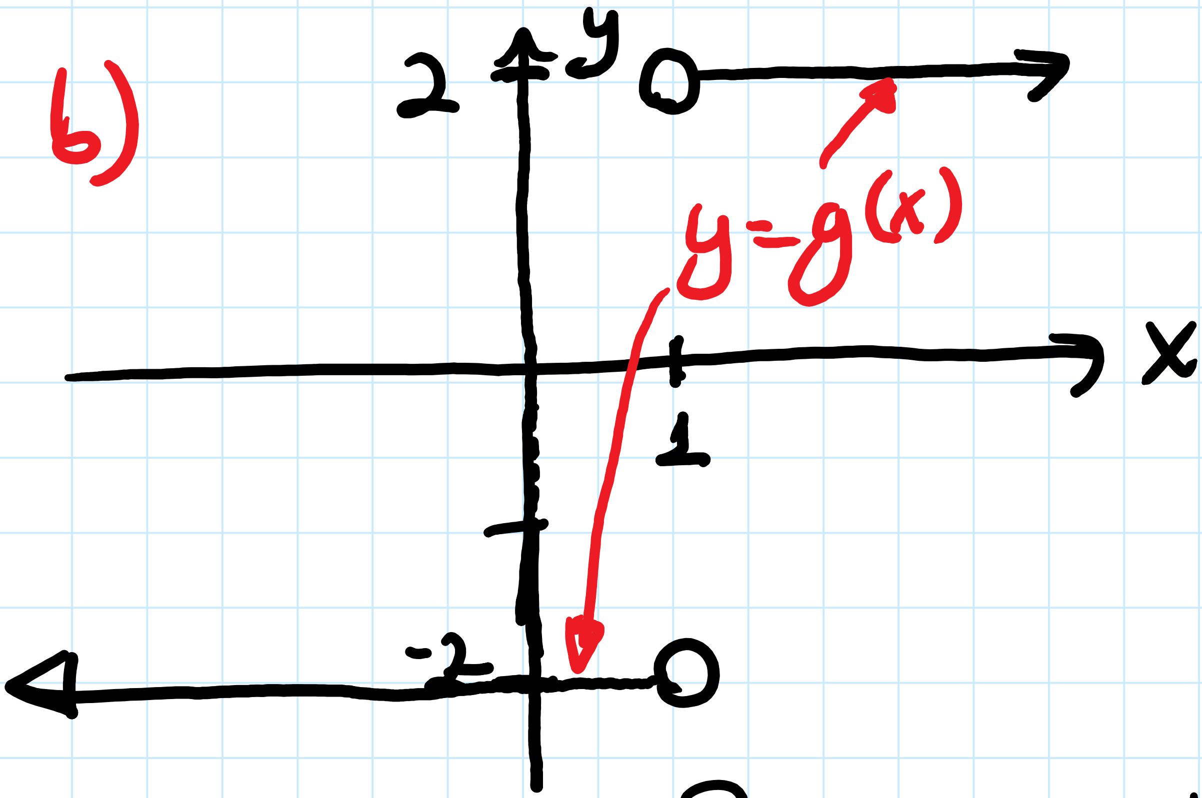
a) 

$\lim_{x \to 0} f(x)$

Solution: $\lim_{x \to 0^-} f(x) = 1$ $\lim_{x \to 0^+} f(x) = 1$

Since $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 1$ therefore $\lim_{x \to 0} f(x) = 1$

Exp) b)



$\lim_{x \to 1} g(x) = ?$

$\lim_{x \to 1^-} g(x) = -2$

$\lim_{x \to 1^+} g(x) = 2$

Since $\lim_{x \to 1^-} g(x) \neq \lim_{x \to 1^+} g(x)$

$\lim_{x \to 1} g(x)$ DNE
"doesn't exist"

When does the limit not exist?

1) One-sided limits not equal.

① $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$

OR

② At least 1 one-sided limit DNE

$\lim_{x \to c} f(x)$ DNE.

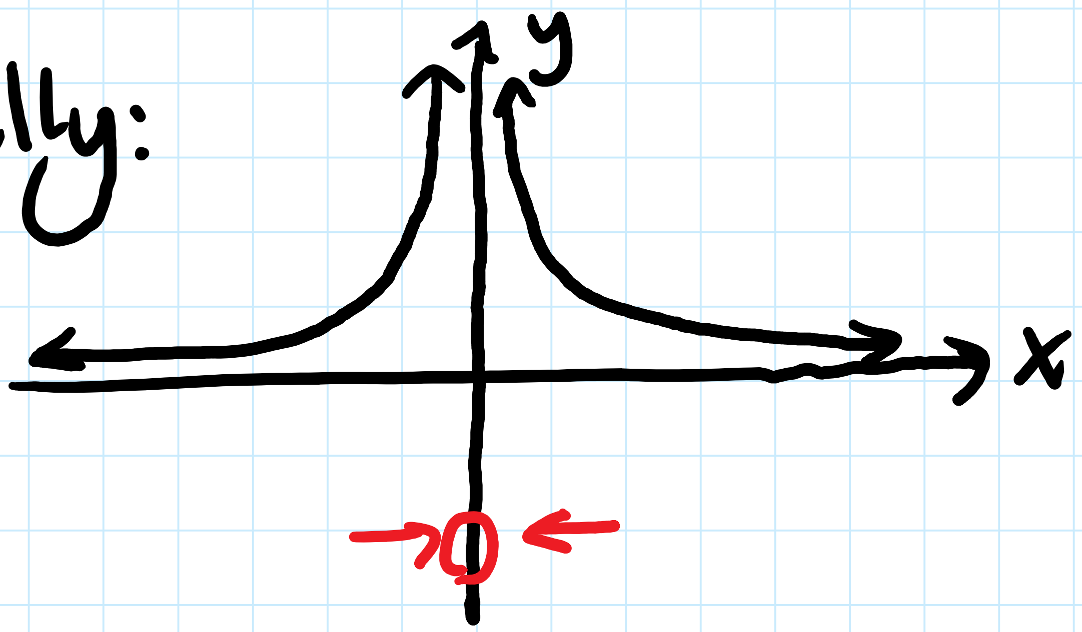
2) Infinite Limits

If $\lim_{x \to c} f(x) = \infty$ (or $-\infty$)

Then technically $\lim_{x \to c} f(x)$ DNE.

Exp) Evaluate $\lim_{x \to 0} \left(\frac{1}{x^2}\right)$

Graphically:



$\lim_{x \to 0} \frac{1}{x^2} = \infty$; graph of $f(x) = y = \frac{1}{x^2}$ rises without bound as $x \to 0$.

(we will distinguish infinite limits vs limits DNE later)

3) Infinite Oscillation

Exp: Evaluate $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

Graphically: