Warm Up

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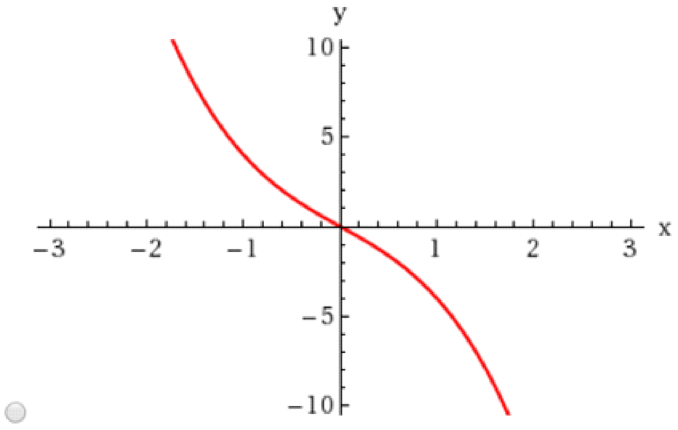
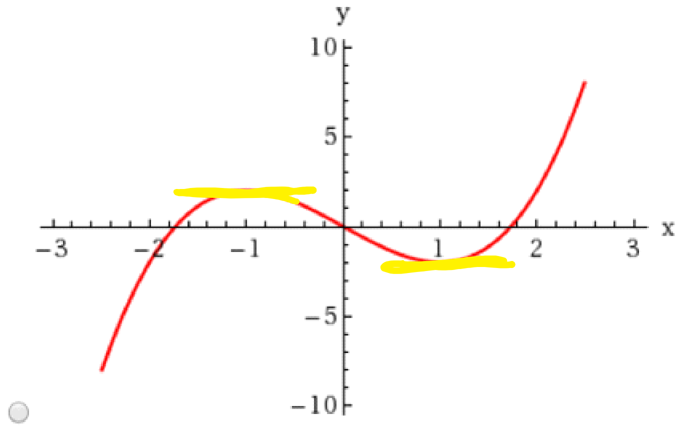
How many of the graphs below show functions with more than one horizontal tangent line?
Hint: What is the slope of a horizontal line?

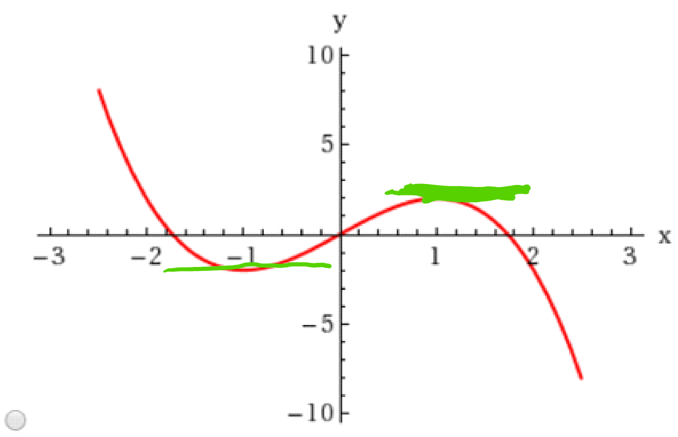
$\boxed{2}$

Graph I

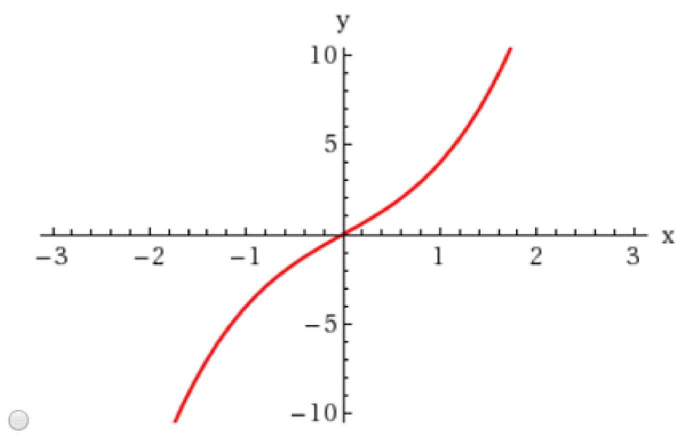


Graph II





Graph III



Graph IV

$f'(a)$

The Derivative

Computing the slope of the line tangent to the graph of a function f at a given point a gives us the instantaneous rate of change in f at a. This information about the behavior of a function is so important that it has its own name and notation.

> **DEFINITION The Derivative of a Function at a Point**
>
> The **derivative of f at a**, denoted $f'(a)$, is given by either of the two following limits, provided the limits exist and a is in the domain of f:
>
> $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad (1) \quad \text{or} \quad f'(a) = \lim_{h \to 0} \frac{f(\cancel{x} + h) - f(\cancel{x})}{h}. \quad (2)$$
>
> If $f'(a)$ exists, we say that f is **differentiable** at a.

* The limits that define the derivative of a function at a point are exactly the same limits used to compute the slope of a tangent line and the instantaneous rate of change of a function at a point. When you compute a derivative, remember that you are also finding a rate of change and the slope of a tangent line.

EXAMPLE 4 Derivatives and tangent lines Let $f(x) = \sqrt{2x} + 1$. Compute $f'(2)$, the derivative of f at $x = 2$, and use the result to find an equation of the line tangent to the graph of f at $(2, 3)$.

$f'(2) \to m_{tan}\big|_{x=2}$

$a = 2 \qquad f(2) = 3 \to f(2) = \sqrt{2\cdot 2} + 1 = 2+1 = 3$

$$f'(2) = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2}\left(\frac{\sqrt{2x}+1-3}{x-2}\right)\cdot\left(\frac{\sqrt{2x}+2}{\sqrt{2x}+2}\right)$$

$(\sqrt{2x}-2)$

$(a-b)(a+b) = a^2 - b^2 \quad \Big\} \quad (\sqrt{2x}-2)(\sqrt{2x}+2) = (\sqrt{2x})^2 - 2^2$

$= 2x - 4$

$m_{tan}\big|_{x=2} = \frac{1}{2}, \quad (2,3)$

$y - 3 = \frac{1}{2}(x-2)$

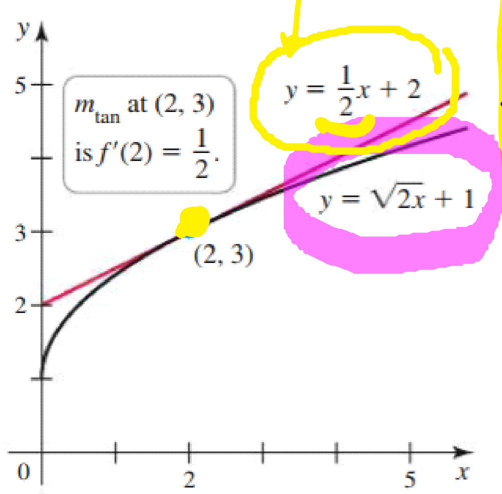


Figure 3.10

$$\boxed{f'(2)} = \lim_{x\to 2}\left(\frac{(2x-4)}{\cancel{(x-2)}(\sqrt{2x}+2)}\right)$$

$2\cancel{(x-2)}$

$$= \lim_{x \to 2}\left(\frac{2}{\sqrt{2x}+2}\right) \overset{DSP}{=} \frac{2}{\sqrt{2\cdot 2}+2} = \boxed{\frac{1}{2}}$$

3.2. The Derivative as a Function

0 slope | pos. slope | neg. slope

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The derivative of a function *f* **at a point** *a* is the slope of the line tangent to the graph of *f* that passes through (a, f(a)).

We now extend this concept of a derivative at a point to **all points in the domain of *f*** to create a new function called the derivative of *f*.

The tangent line changes along the curve of a function, therefore, the slope of the tangent line for *f* is itself a function, called the derivative of *f*.

$f'(x)$

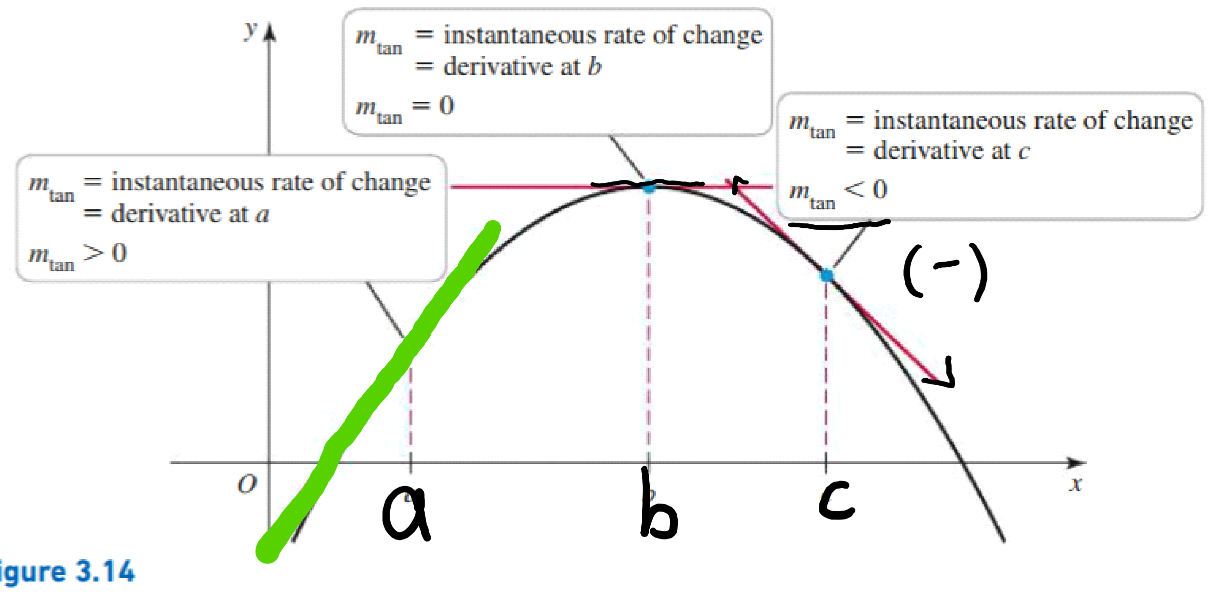


Figure 3.14

- To emphasize an important point, $f'(2)$ or $f'(-2)$ or $f'(a)$, for a real number a, are real numbers, whereas f' and $f'(x)$ refer to the derivative *function*.

$f' \rightarrow$ "f prime"

- The process of finding f' is called *differentiation*, and to *differentiate* f means to find f'.

Limit Def. of Derivative

DEFINITION The Derivative Function

The **derivative** of f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists and x is in the domain of f. If $f'(x)$ exists, we say that f is **differentiable** at x. If f is differentiable at every point of an open interval I, we say that f is differentiable on I.

NOT $f'(a)$

$f'(x)$

$\frac{x - \ldots}{x}$ "$\frac{0}{0}$"

EXAMPLE 1 Computing a derivative Find the derivative of $f(x) = -x^2 + 6x$.

$(x+h)(x+h) = x^2 + \underbrace{xh + xh}_{2xh} + h^2$

$f(x+h) = -(x+h)^2 + 6(x+h)$

$$f'(x) = \lim_{h \to 0} \left(\frac{-(x+h)^2 + 6(x+h) - (-x^2+6x)}{h} \right)$$

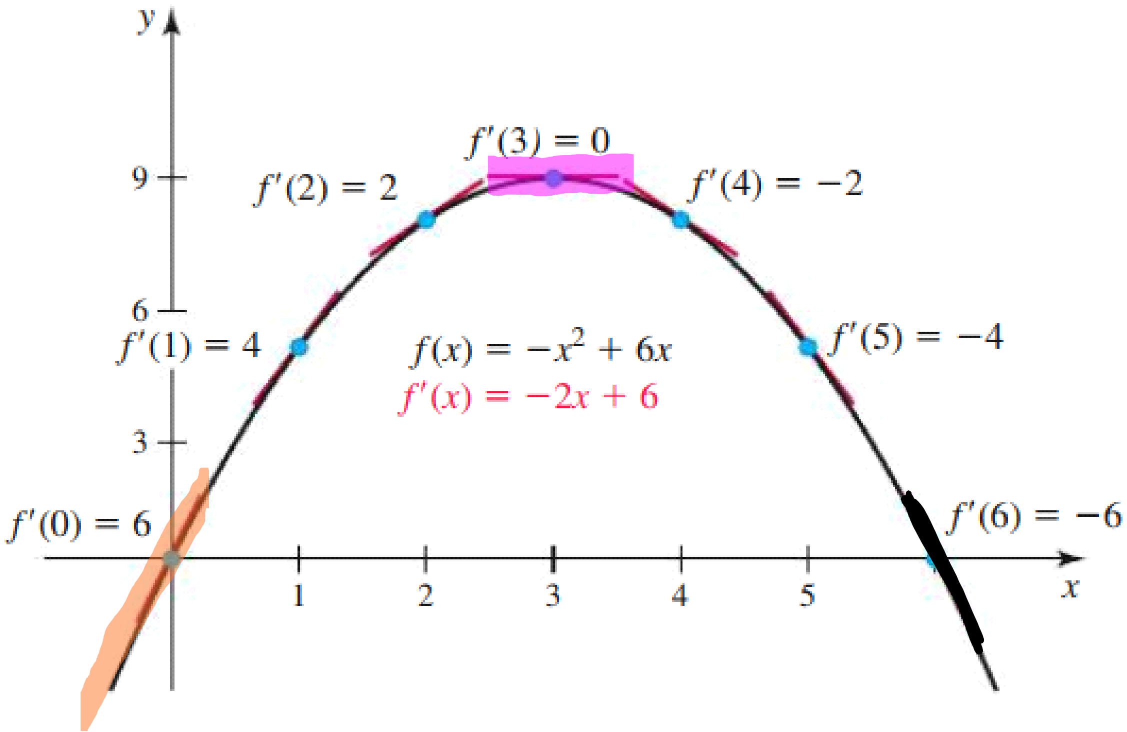
$$= \lim_{h \to 0} \left(\frac{-(x^2+2xh+h^2) + \cancel{6x} + 6h + \cancel{x^2} - \cancel{6x}}{h} \right) = \lim_{h \to 0} \left(\frac{\cancel{h}(-2x-h+6)}{\cancel{h}} \right)$$

$-\cancel{x^2} - 2xh - h^2$

$$= \lim_{h \to 0} \left(\frac{-2x-h+6}{1} \right) \overset{DSP}{=} -2x+6$$

$$\lim_{h\to 0}\left(\frac{\quad}{1}\right)$$

horizontal tangent $\Rightarrow m_{tan}\big|_{x=a} = 0$



$f'(x) = -2x+6$

$f'(6) = -2 \cdot 6 + 6 = -6$

neg. slope

$f'(0) = 6$ pos. slope

Recall from Algebra:

$m = 0$

Derivative Notation

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In addition to the notation $f'(x)$ and $\frac{dy}{dx}$, other common ways of writing the derivative include

$$\frac{df}{dx}, \qquad \frac{d}{dx}(f(x)), \qquad D_x(f(x)), \qquad \text{and} \qquad y'(x).$$

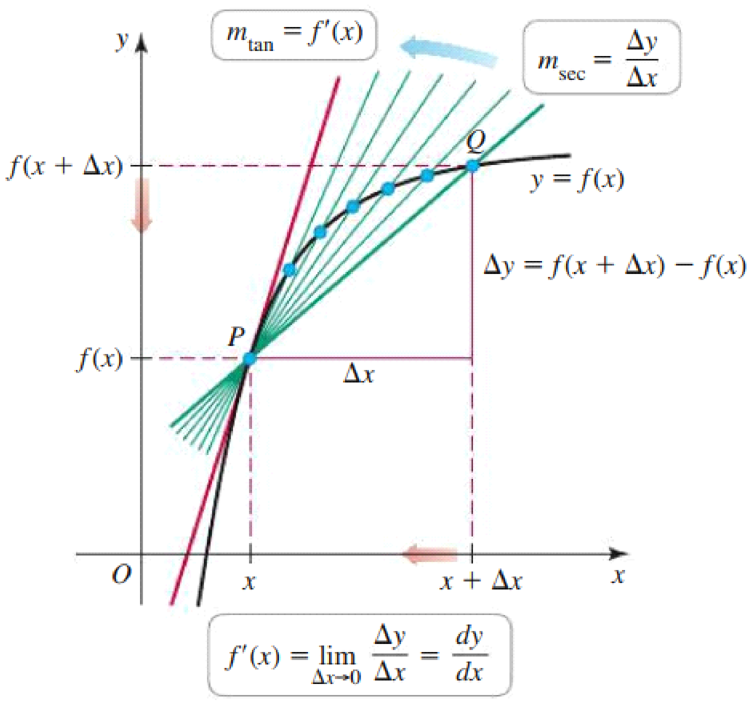
The following notations represent the derivative of f evaluated at a.

$$f'(a), \qquad y'(a), \qquad \left.\frac{df}{dx}\right|_{x=a}, \qquad \text{and} \qquad \left.\frac{dy}{dx}\right|_{x=a}$$

$y = f(x)$

$\left.\frac{dy}{dx}\right|_{x=a}$ } Leibniz Notation

$f'(x)$ } Lagrange Notation



➤ The notation $\frac{dy}{dx}$ is read *the derivative of y with respect to x or dy dx*. It does not mean dy divided by dx, but it is a reminder of the limit of the quotient $\frac{\Delta y}{\Delta x}$.

➤ The derivative notation dy/dx was introduced by Gottfried Wilhelm von Leibniz (1646–1716), one of the coinventors of calculus. His notation is used today in its original form. The notation used by Sir Isaac Newton (1642–1727), the other coinventor of calculus, is rarely used.

Interpret Derivative of a Function

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EXAMPLE 2 **A derivative calculation** Let $y = f(x) = \sqrt{x}$.

a. Compute $\frac{dy}{dx}$.

b. Find an equation of the line tangent to the graph of f at $(4, 2)$.

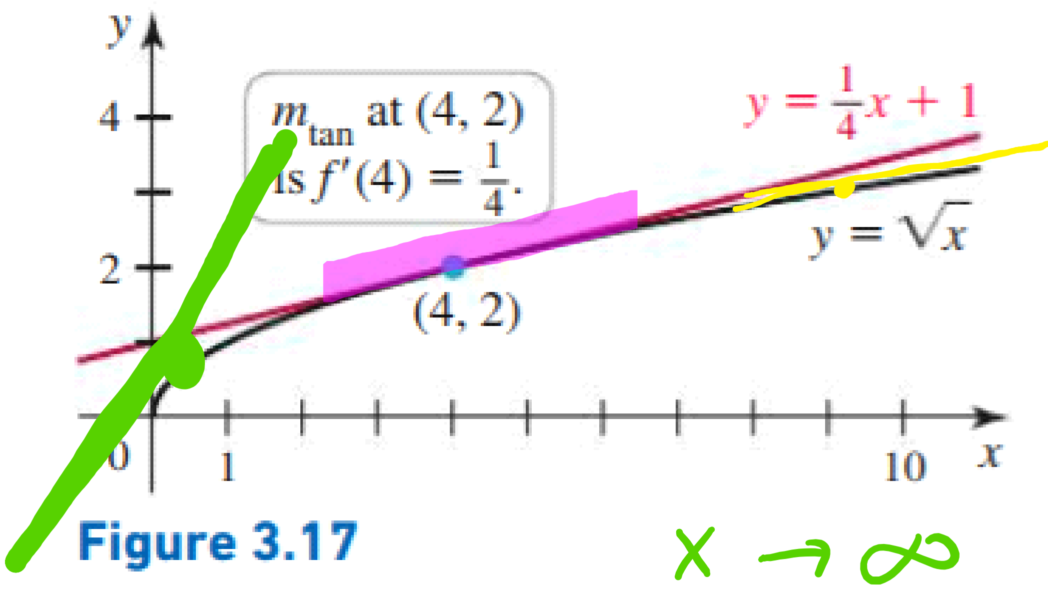

Figure 3.17

$x \to \infty$

QUICK CHECK 3 In Example 2, do the slopes of the tangent lines increase or decrease as x increases? Explain. ◀

Graphs of Derivatives

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The function f' is called the derivative of f because it is *derived* from f.

EXAMPLE 4 **Graph of the derivative** Sketch the graph of f' from the graph of f (Figure 3.18).

$m_{tan} = \dfrac{4-2}{0-(-2)} = \dfrac{2}{2} = 1$

$1.5 - 4 = -2.5$

$m_{tan} = \dfrac{\frac{3}{2}-4}{5-0} = -\dfrac{1}{2}$

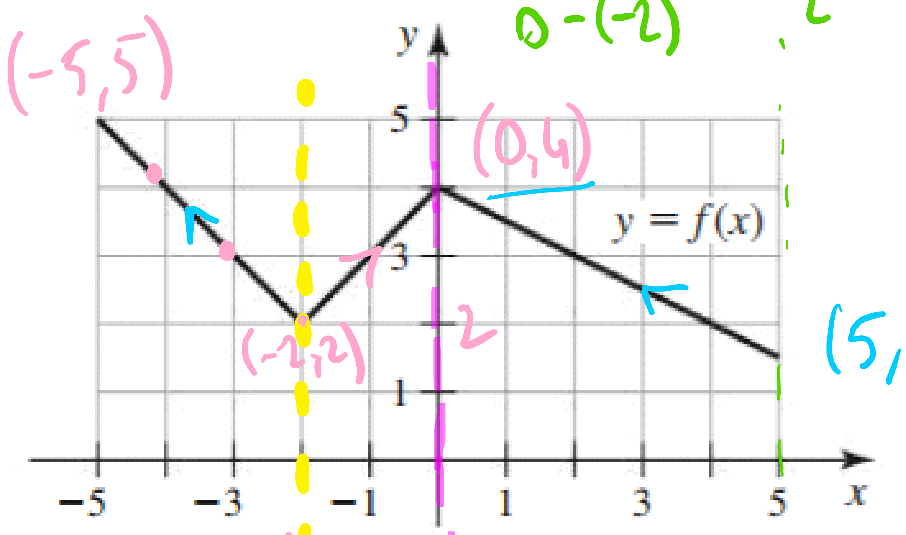
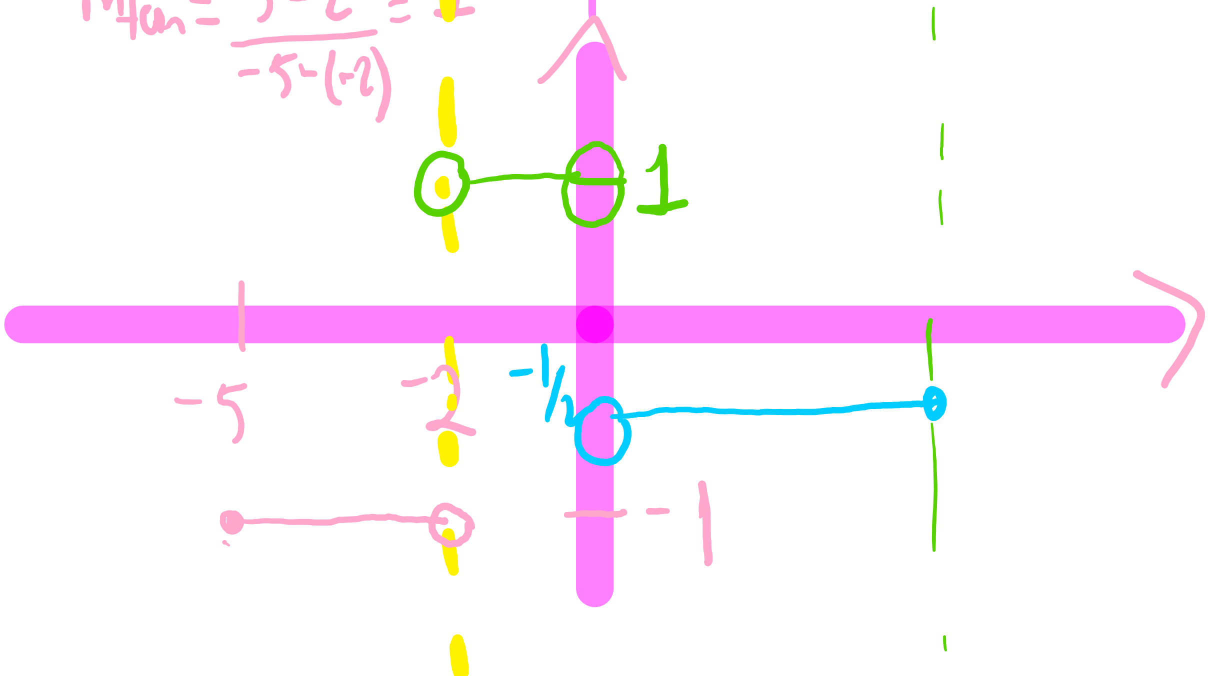


Figure 3.18

$m_{tan} = \dfrac{5-2}{-5-(-2)} = -1$



$f'(x)$

$x = 0,\ x = -2$

$\boxed{f'(x)}$

$f'(0)$, $f'(-2)$ undefined

THEOREM 3.1 Differentiable Implies Continuous
If f is differentiable at a, then f is continuous at a.

THEOREM 3.1 (ALTERNATIVE VERSION) Not Continuous Implies Not Differentiable
If f is not continuous at a, then f is not differentiable at a.

When Is a Function Not Differentiable at a Point?

A function f is *not* differentiable at a if at least one of the following conditions holds:

a. f is not continuous at a (Figure 3.24).
b. f has a corner at a (Figure 3.25).
c. f has a vertical tangent at a (Figure 3.26).

$f(x) = |x|$

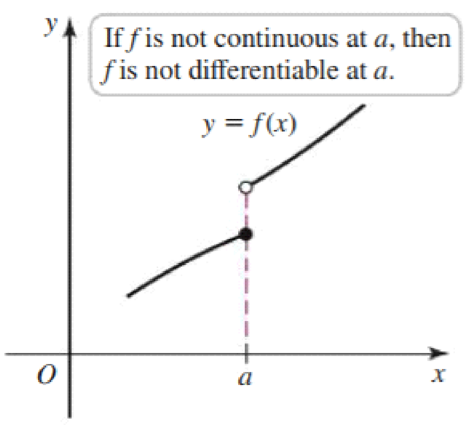


Figure 3.24

Function's "**Point of discontinuity**"

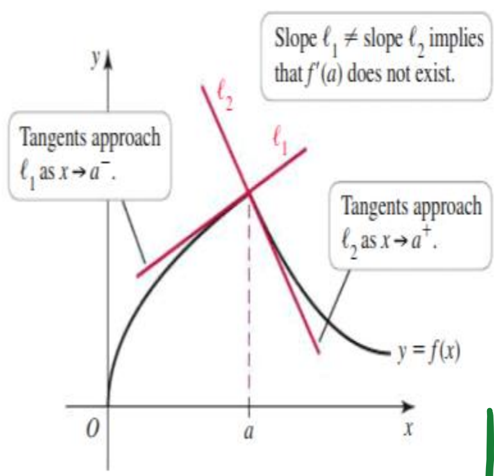
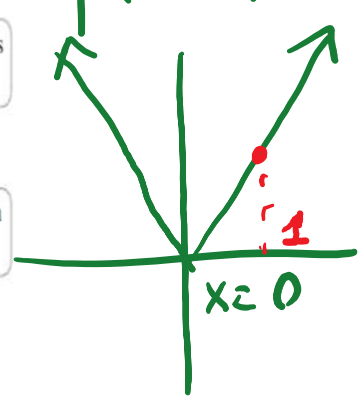


Figure 3.25

Function at its "**corner**"



$|x|$ is NOT diff. at $x=0$

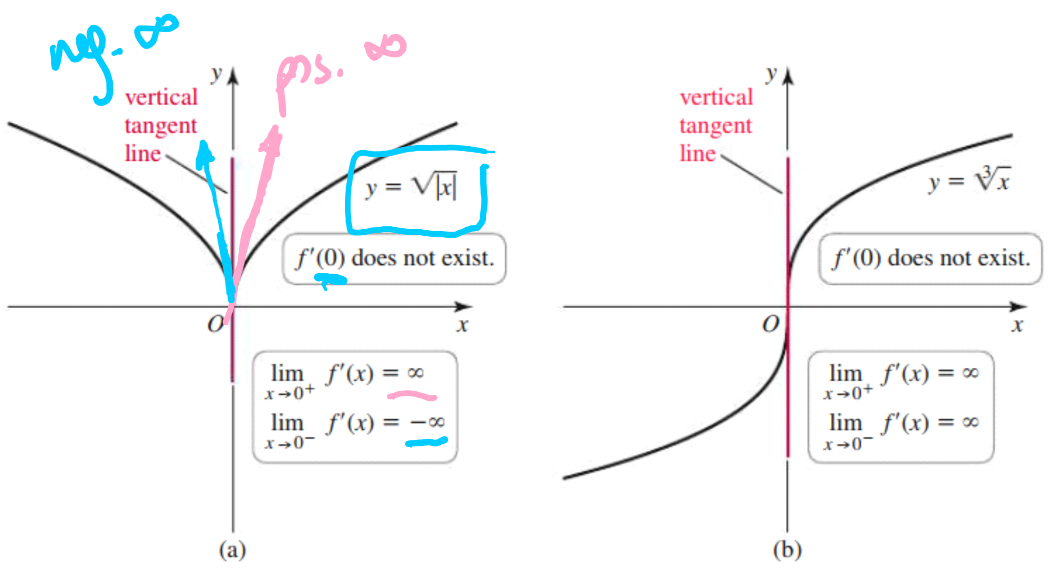


Figure 3.26

"Vertical Tangent" Line ***with*** "cusp" "Vertical Tangent" Line ***without*** a "cusp"

∃ cont.
1) $f(a)$
2) $\lim_{x \to a} f(x)$
3) 1) = 2)

EXAMPLE 7 **Continuous and differentiable** Consider the graph of g in Figure 3.27.

a. Find the values of x in the interval $(-4, 4)$ at which g is not continuous.

b. Find the values of x in the interval $(-4, 4)$ at which g is not differentiable.

c. Sketch a graph of the derivative of g.

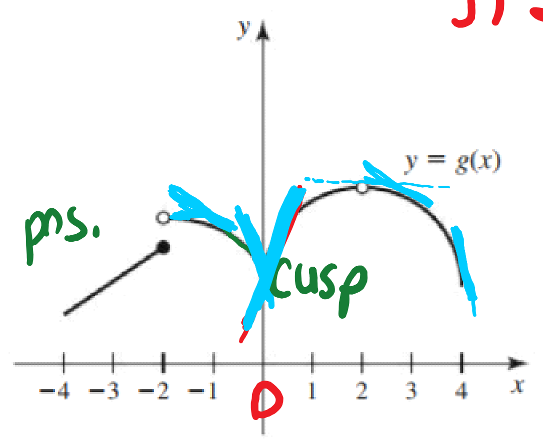
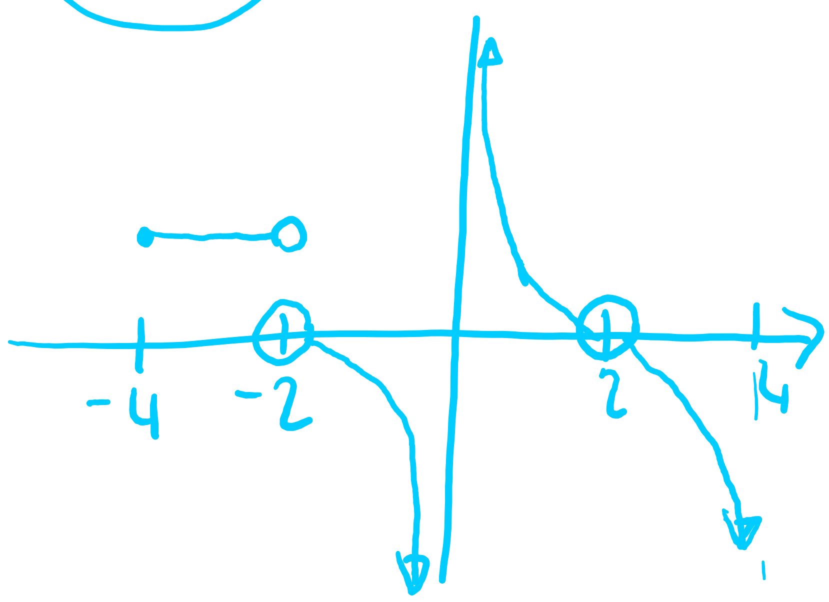


Figure 3.27

a. g is NOT cont. at $x = -2, 2$

b. g is NOT diff. $x = -2, 2, 0$

$g'(x)$



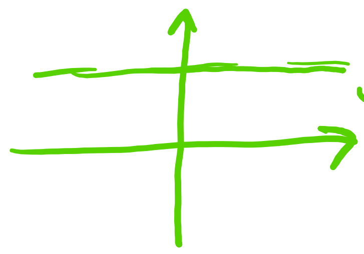
$g'(x)$

3.3 Rules of Differentiation

If you always had to use limits to evaluate derivatives, as we did in Section 3.2, calculus would be a tedious affair. The goal of this chapter is to establish rules and formulas for quickly evaluating derivatives—not just for individual functions but for entire families of functions. By the end of the chapter, you will have learned many time-saving rules and formulas, all of which are listed in the endpapers of the text.

DIFFERENTIATION RULES

1. *Constant Rule:* If $f(x) = c$ (c constant), then $f'(x) = \underline{0}$.
2. *Power Rule:* If r is a real number, $\frac{d}{dx}x^r = rx^{r-1}$
3. *Constant Multiple Rule:* $\frac{d}{dx}(c \cdot f(x)) = c \cdot f'(x)$
4. *Sum Rule:* $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
5. *Product Rule:* $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$
6. *Quotient Rule:* $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$



$y=2$, $\underline{m=0}$

$f(x) = x^3$, $f'(x) = 3 \cdot x^2$

$g(x) = 5x^4 + \underline{1}$

$g'(x) = 5 \cdot 4 \cdot x^3 + 0$

$= 20x^3$

EXAMPLE 2 Derivatives of constant multiples of functions Evaluate the following derivatives.

a. $\frac{d}{dx}\left(-\frac{7x^{11}}{8}\right)$ **b.** $\frac{d}{dt}\left(\frac{3}{8}\sqrt{t}\right)$

a. $\frac{d}{dx}\left(\frac{-7x^{11}}{8}\right) = \frac{-7}{8} \cdot \frac{d}{dx}\left(x^{11}\right) = \frac{-7}{8} \cdot 11 \cdot x^{10}$

$= \frac{-77}{8}x^{10}$

b. $\frac{d}{dt}\left(\frac{3}{8} \cdot t^{1/2}\right) = \frac{3}{8} \cdot \frac{1}{2} \cdot t^{-1/2} = \frac{3}{16} \cdot t^{-1/2}$

THEOREM 3.6 The Derivative of e^x

The function $f(x) = e^x$ is differentiable for all real numbers x, and

$$\frac{d}{dx}(e^x) = e^x.$$

➤ The Power Rule *cannot* be applied to exponential functions; that is, $\frac{d}{dx}(e^x) \neq xe^{x-1}$. Also note that $\frac{d}{dx}(e^{10}) \neq e^{10}$. Instead, $\frac{d}{dx}(e^c) = 0$, for any real number c, because e^c is a constant.

EXAMPLE 4 Finding tangent lines

a. Write an equation of the line tangent to the graph of $f(x) = 2x - \frac{e^x}{2}$ at the point $\left(0, -\frac{1}{2}\right)$.

b. Find the point(s) on the graph of f at which the tangent line is horizontal.

a. $\boxed{f'(x) = 2 - \frac{1}{2} \cdot e^x}$ $\quad m_{tan}\Big|_{x=a} = 0$

$a = ?$

$P(a, f(a))$

$f'(0) = 2 - \frac{1}{2} \cdot \cancel{e^0}^{1} = 2 - \frac{1}{2} = \frac{3}{2} = m_{tan}\Big|_{x=0}$

Eq. : $y - \left(-\frac{1}{2}\right) = \frac{3}{2}(x - 0) \Rightarrow y + \frac{1}{2} = \frac{3}{2}x$

b. $f'(x)\Big|_{x=a} = 0 \Rightarrow 2 - \frac{1}{2}\cdot e^{a} = 0$

$\boxed{2 = \frac{1}{2}\cdot e^{a}} \Rightarrow \ln 4 = \ln e^{a}$

$\ln 4 = a \cdot \cancel{\ln e}^{\,1} \Rightarrow \boxed{a = \ln 4}$

Recall $\boxed{f(x) = 2x - \frac{e^{x}}{2}}$

$f(a) \Rightarrow f(\ln 4) = 2\cdot \ln 4 - \frac{e^{\ln 4}}{2} = 2\cdot \ln 4 - 2$

4

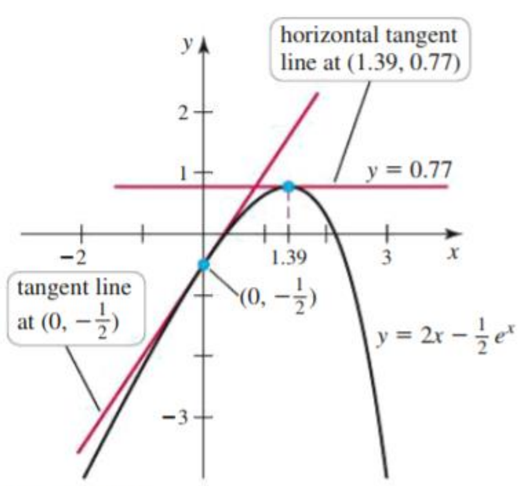


Figure 3.32

$(\ln 4, 2\ln 4 - 2)$ is the point at which the tangent line is horizontal

(x, y) $(a, f(a))$

QUICK CHECK 5 Determine the point(s) at which $f(x) = x^3 - 12x$ has a horizontal tangent line. ◄

$f'(x) = 0$

$$f'(x) = 3x^2 - 12 = 0$$
$$= 3(x^2 - 4) = 0$$
$$= 3(x-2)(x+2)$$

$3(x-2)(x+2) = 0$

$\underline{2} \quad \underline{-2}$

$3 \quad 3$

$(x-2)(x+2) = 0$

$\boxed{x = 2, -2}$

Points are: $(2, f(2)), (-2, f(-2))$.