DEFINITION Concavity and Inflection Point

Let f be differentiable on an open interval I. If f' is increasing on I, then f is **concave up** on I. If f' is decreasing on I, then f is **concave down** on I.

If f is continuous at c and f changes concavity at c (from up to down, or vice versa), then f has an **inflection point** at c.

THEOREM 4.10 Test for Concavity

Suppose f'' exists on an open interval I.

- If $f'' > 0$ on I, then f is concave up on I.
- If $f'' < 0$ on I, then f is concave down on I.
- If c is a point of I at which f'' changes sign at c (from positive to negative, or vice versa), then f has an inflection point at c.

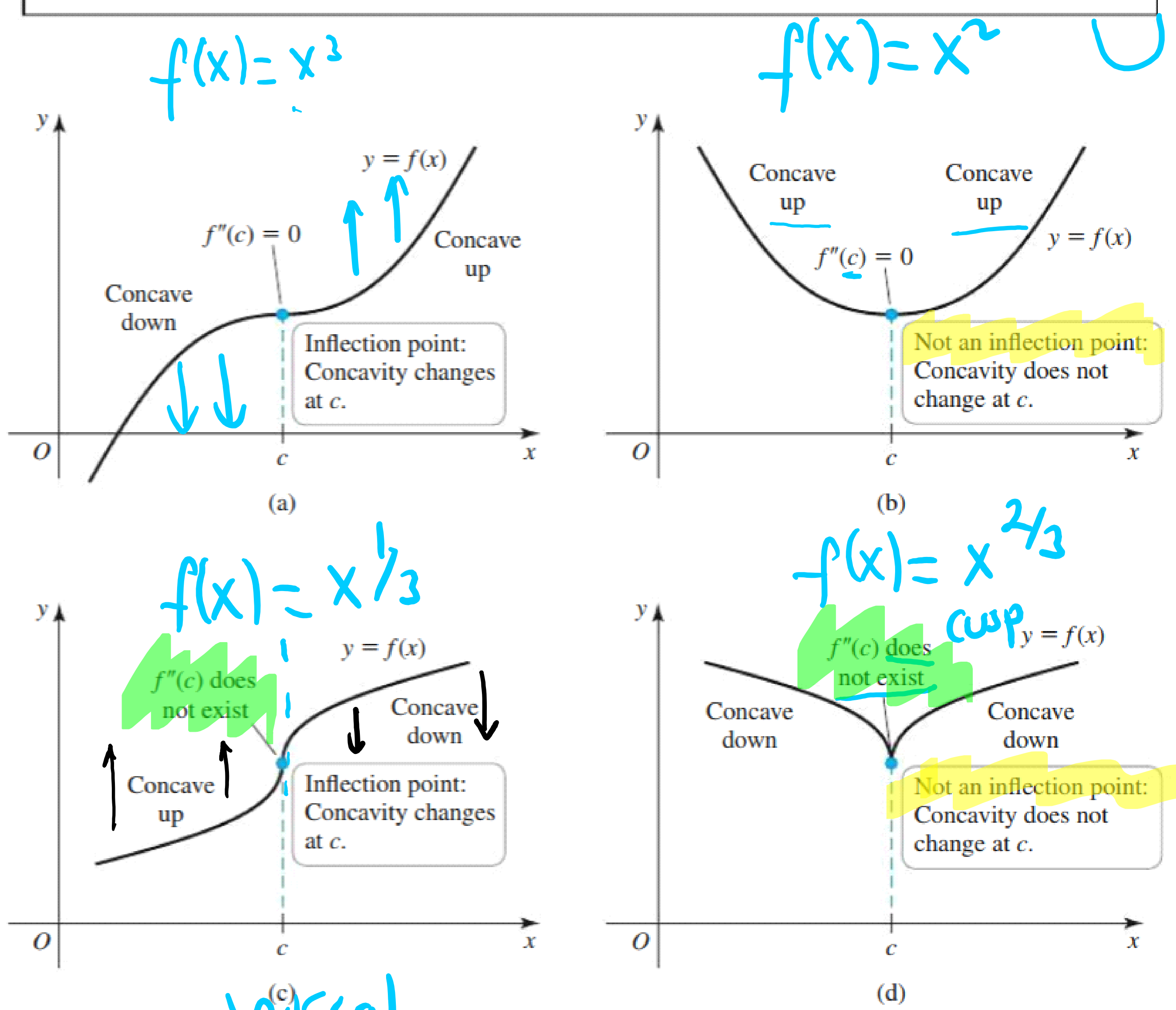

Figure 4.35

vertical tangent

$f''(c) \to 0$ ONE

second-order crit. P.

Is it a PoI?

check for concavity change!

EXAMPLE 7 **Detecting concavity** Identify the intervals on which the following functions are concave up or concave down. Then locate the inflection points.

a. $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$

$D: (-\infty, \infty)$

$f'(x) = 12x^3 - 12x^2 - 12x + 12$

$f''(x) = 36x^2 - 24x - 12 = 12(3x^2 - 2x - 1)$

CMSA

$3x \quad 1$

$x \quad -1$

$-3x + x = -2x$ ✓

$= 12(3x+1)(x-1)$

Second order crit. P: $\underbrace{f''(x) = 0}$ or ~~DNE~~ polynomial

$f''(x) = 0 = \underbrace{12(3x+1)(x-1)}_{x = -1/3,\ x = 1}$ } candidates for PoI

$f''(x) = 12(3x+1)(x-1)$

test points

$f''(-6) \to \ominus \cdot \ominus$ $\oplus$

$f''(0) \to \oplus \ominus$ $\ominus$

$f''(6) \to \oplus$

sign chart on $f''(x)$

	$-\infty$ … -6 …	$-1/3$	… 0 …	1	… 6 … $+\infty$
sign of $f''(x)$	$+$		$-$		$\oplus$
behavior of $f(x)$	∪	$\boxed{\text{PoI}}$	∩	$\boxed{\text{PoI}}$	∪

Point of Inflections at : $x = -1/3$, $x = 1$

$\underline{f}$ is Concave up on $(-\infty, -1/3)$, $(1, \infty)$

Concave down on $(-1/3, 1)$

Second Derivative Test It is now a short step to a test that uses the second derivative to identify local maxima and minima.

THEOREM 4.11 Second Derivative Test for Local Extrema
Suppose f'' is continuous on an open interval containing c with $f'(c) = 0$.

- If $f''(c) > 0$, then f has a local minimum at c (Figure 4.40a).
- If $f''(c) < 0$, then f has a local maximum at c (Figure 4.40b).
- If $f''(c) = 0$, then the test is inconclusive; f may have a local maximum, a local minimum, or neither at c.

➤ In the inconclusive case of Theorem 4.11 in which $f''(c) = 0$, it is usually best to use the First Derivative Test.

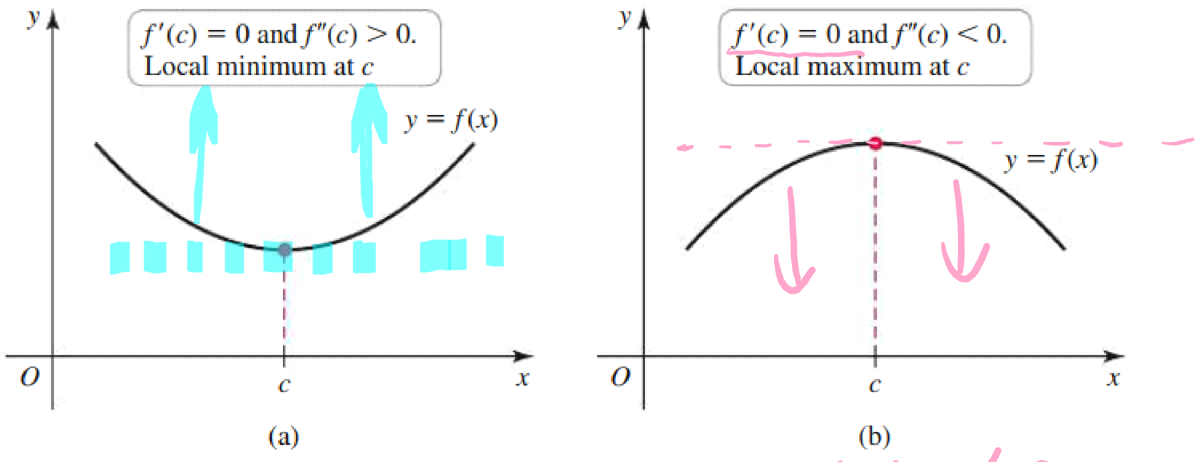

Figure 4.40

$f''(c) > 0$ $f''(c) < 0$

EXAMPLE 8 **The Second Derivative Test** Use the Second Derivative Test to locate the local extrema of the following functions.

a. $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$ on $[-2, 2]$

$$f'(x) = 12x^3 - 12x^2 - 12x + 12 = 12x^2(x-1) - 12(x-1)$$
$$= (x-1)(12x^2 - 12) = (x-1)\cdot 12(x^2-1)$$
$$= (x-1)\cdot 12(x-1)(x+1) = 12(x-1)^2\cdot(x+1)$$

First order crit. P. $f'(x) = 0$ or ~~DNE~~ poly.

$f'(x) = 12(x-1)^2\cdot(x+1) = 0$ } $x = 1, -1$ candidate for local ext.

$f''(x) > 0$ ∪ local min

$f''(x) < 0$ ∩ local max

$f''(x) = 0$ Inconclusive, use First Der. Test

$$f''(x) = 36x^2 - 24x - 12 = 12(3x^2 - 2x - 1) = \underbrace{12(3x+1)(x-1)}$$

$3x$ / x × 1 / -1

$f''(1) = 0$ Inconclusive → $f'(1)$ right/left if sign change in $f'(1)$

$f''(-1) \to \ominus\cdot\ominus = \oplus$ $f''(-1) > 0$ ∪

local min at $x = -1$

$$f'(x) = 12\underbrace{(x-1)^2}_{\text{pos.}} \cdot (x+1)$$

	$-\infty$	-2	-1	0	1	2	$+\infty$
f'		$-$		$+$		$+$	
f		$\searrow$	local min	$\nearrow$		$\nearrow$	

Since there's no sign change at 1^-, 1^+ for $f'(x)$

therefore; $x=1$ is a crit P. (NOT a local min/max)

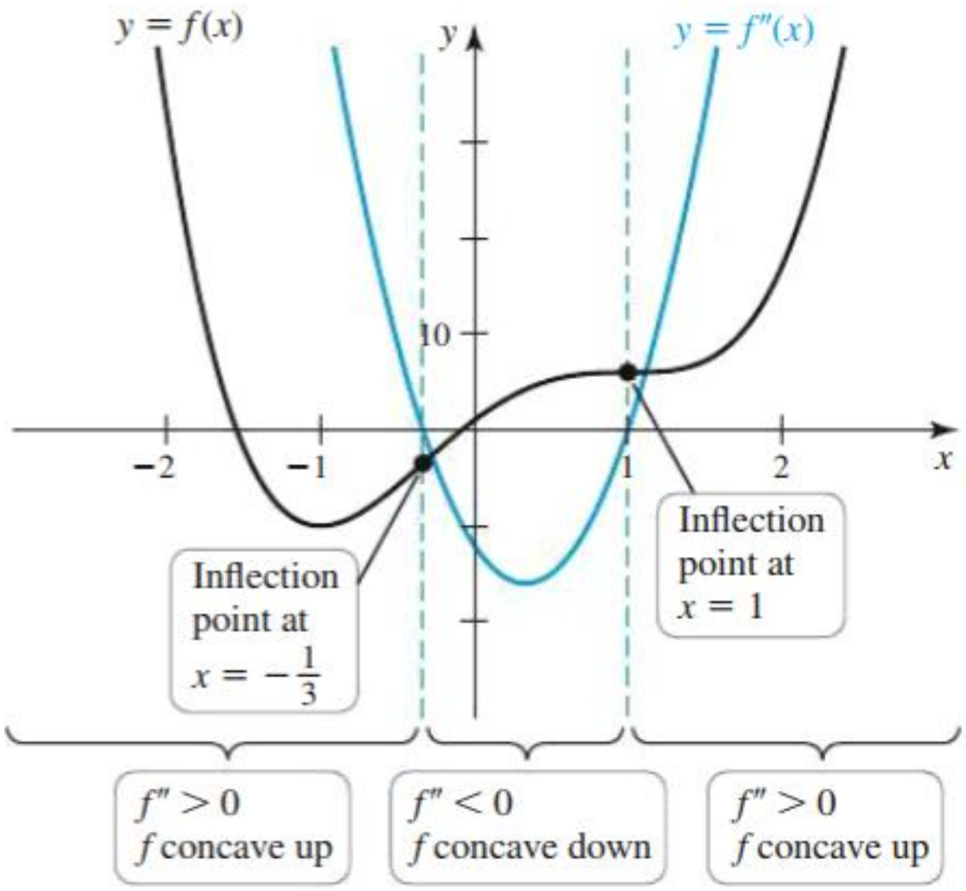

Figure 4.38

Ch 4 -

Wednesday, November 11, 2020 10:36 PM

Recap of Derivative Properties

This section has demonstrated that the first and second derivatives of a function provide valuable information about its graph. The relationships among a function's derivatives and its extreme values and concavity are summarized in Figure 4.43.

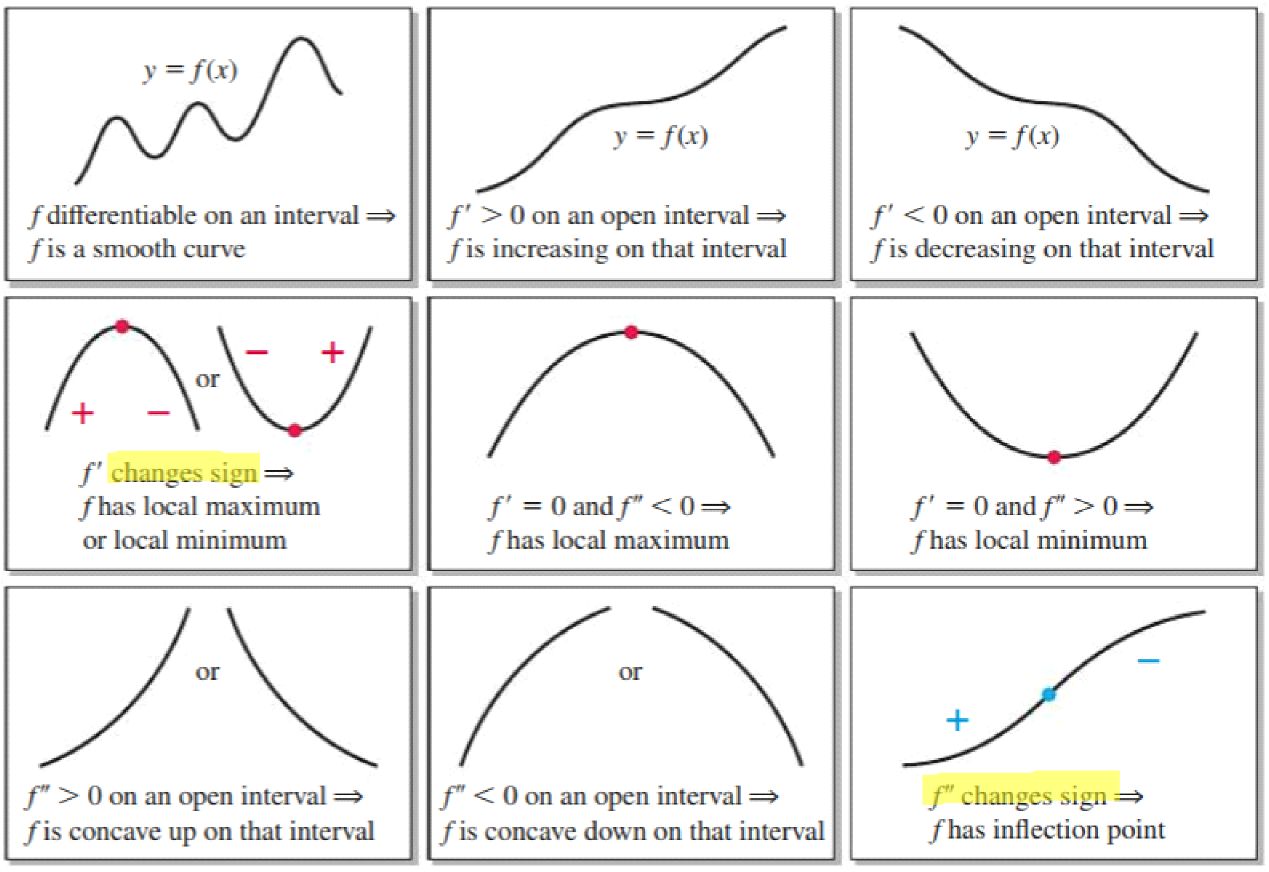

Figure 4.43

First-order crit. P.

$f'(x) = 0$ or DNE

in Domain

Interval

(, [ )

< , >   < , >