L'Hopital's Rule

Sunday, November 1, 2020 9:11 PM

Procedure: IF $\lim_{x \to c} \frac{f(x)}{g(x)}$ "DSP" $=$ "$\frac{0}{0}$" or "$\frac{\pm\infty}{\infty}$"

$\underbrace{\qquad\qquad}_{\text{(primary) indeterminate forms}}$

THEN $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$

where L is either a finite # or $\pm\infty$.

Indeterminate forms: are expressions that their value cannot be determined w/out further analysis. $\frac{0}{0}$, $\frac{\infty}{\infty}$

$\boxed{\overset{H}{=}}$ means L.R. is used

THEOREM 4.12 L'Hôpital's Rule

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies if $x \to a$ is replaced with $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

Try this! Looks familiar?

Sunday, November 1, 2020 9:22 PM

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Special Trig. Limit

Re-visit by using L.R.

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right) \overset{\text{"DSP"}}{=} \frac{\sin 0}{0} = \text{"}\frac{0}{0}\text{"} \to \text{indet. form} \Rightarrow \text{L.R}$$

$$\overset{H}{=} \lim_{x\to 0}\left(\frac{(\sin x)'}{x'}\right) = \lim_{x\to 0}\left(\frac{\cos x}{1}\right) \overset{\text{"DSP"}}{=} \cos 0 = 1$$

QUICK CHECK 1 Which of the following functions lead to an indeterminate form as $x \to 0$: $f(x) = x^2/(x + 2)$, $g(x) = (\tan 3x)/x$, or $h(x) = (1 - \cos x)/x^2$? ◄

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{x^2}{x+2}\right) \overset{\text{"DSP"}}{=} \frac{0}{2} \quad \text{X}$$

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}\left(\frac{\tan 3x}{x}\right) \overset{\text{"DSP"}}{=} \frac{\frac{\sin 0}{\cos 0}}{0} = \text{"}\frac{0}{0}\text{"} \checkmark$$

$$\lim_{x\to 0} h(x) = \lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right) \overset{\text{"DSP"}}{=} \frac{1-\cos 0}{0^2} = \text{"}\frac{0}{0}\text{"} \checkmark$$

0/0

Sunday, November 1, 2020 9:24 PM

"$\frac{0}{0}$, $\frac{\infty}{\infty}$"

$\frac{0}{5} \neq \frac{0}{0}$

L'Hôpital's Rule requires evaluating $\lim_{x\to a} f'(x)/g'(x)$. It may happen that this second limit is another indeterminate form to which l'Hôpital's Rule may again be applied.

EXAMPLE 2 **L'Hôpital's Rule repeated** Evaluate the following limits.

a. $\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$ **b.** $\lim_{x\to 2} \frac{x^3 - 3x^2 + 4}{x^4 - 4x^3 + 7x^2 - 12x + 12}$

a. $\lim_{x\to 0}\left(\frac{e^x - x - 1}{x^2}\right) \overset{\text{"DSP"}}{=} \frac{e^0 - 0 - 1}{0^2} = \text{"}\frac{0}{0}\text{"}$

Indeterminate form

use L.R.

$\overset{H}{=} \lim_{x\to 0}\left(\frac{(e^x - x - 1)'}{(x^2)'}\right) = \lim_{x\to 0}\left(\frac{e^x - 1}{2x}\right) \overset{\text{"DSP"}}{=} \frac{e^0 - 1}{0} = \text{"}\frac{0}{0}\text{"}$

L.R.

$\overset{H}{=} \lim_{x\to 0}\left(\frac{(e^x - 1)'}{(2x)'}\right) = \lim_{x\to 0}\left(\frac{e^x}{2}\right) \overset{\text{"DSP"}}{=} \frac{e^0}{2} = \frac{1}{2}$

det. ✓ $\left\{ -\infty \atop +\infty \right.$ $\frac{\text{finite, non-zero } 0}{0} \rightarrow \frac{4}{0}$

$\rightarrow \frac{4}{0^-} = -\infty$

$\rightarrow \frac{4}{0^+} = +\infty$

Inf/inf

Sunday, November 1, 2020 9:25 PM

THEOREM 4.13 **L'Hôpital's Rule** (∞/∞)

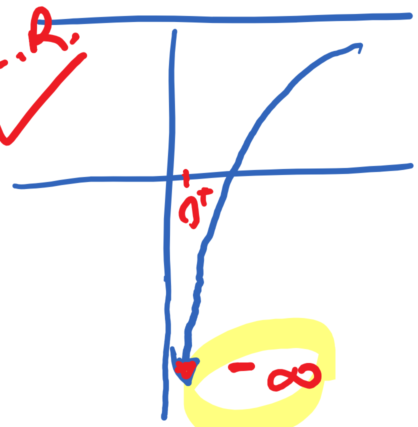
Suppose f and g are differentiable on an open interval I containing a, with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies for $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

L.R $\neq$ Quotient Rule

Recall:



$f(x) = \ln x$

EXAMPLE 3 **L'Hôpital's Rule for ∞/∞** Evaluate the following limits.

b. $\displaystyle \lim_{x \to 0^+} \frac{\ln x}{\csc x} \overset{\text{"DSP"}}{=} \frac{\ln(0^+)}{\csc(0^+)} = \frac{-\infty}{\infty}$ L.R. ✓

$$\overset{H}{=} \lim_{x \to 0^+} \left(\frac{(\ln x)'}{(\csc x)'} \right) = \lim_{x \to 0^+} \left(\frac{\frac{1}{x}}{-\csc x \cdot \cot x} \right)$$

$\csc x = \dfrac{1}{\sin x}$

re-write

$$= \lim_{x \to 0^+} \left(\frac{\frac{1}{x}}{-\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}} \right) = \lim_{x \to 0^+} \left(\frac{\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}} \right)$$

$\csc(0^+) = \dfrac{1}{\sin(0^+)} = \dfrac{1}{0^+} = +\infty$

"non-zero #" / 0

$$= \lim_{x \to 0^+} \left(\frac{1}{x} \cdot \frac{\sin^2 x}{-\cos x} \right) = \lim_{x \to 0^+} \left(\frac{\sin^2 x}{-x \cdot \cos x} \right) \overset{\text{"DSP"}}{=} \text{"}\frac{0}{0}\text{"}$$

↳ indet. form

$$\overset{H}{=} \lim_{x \to 0^+} \left(\frac{2 \cdot \sin x \cdot \cos x}{-1 \cdot \cos x - x \cdot (-\sin x)} \right) \overset{\text{"DSP"}}{=} \frac{0}{-1 + 0 \cdot 0} = \frac{0}{-1} = 0$$

use L.R.

final answer

Exp) Evaluate $\displaystyle \lim_{x \to 0} \left(\frac{1 - \cos x}{\sec x} \right)$

$$\lim_{x \to 0} \left(\frac{1 - \cos x}{\sec x} \right) \overset{\text{"DSP"}}{=} \frac{1 - \cos 0}{\sec 0} = \frac{1 - 1}{1} = \underbrace{\frac{0}{1}}_{\neq \frac{0}{0}} = 0$$

(not a indet. form)

(no need to use L.R)

$\cos 0 = 1$

$\sec 0 = \dfrac{1}{\cos 0} = 1$

1 ^(inf)

Sunday, November 1, 2020 9:36 PM

(Other/Secondary) Indeterminate Forms

$\underbrace{1^{\infty},\ 0^{0},\ \infty^{0}}_{\text{use ln properties}}$ $\quad \infty-\infty \quad 0\cdot\infty$

Procedure:

1) "DSP" to obtain an indeterminate form
2) Re-write "other indeterminate forms" into "primary indeterminate form" $\rightarrow \boxed{\frac{0}{0},\ \frac{\infty}{\infty}}$ L.R.
3) Use L.R.

Exp) Evaluate $\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}} \to y$

Step 1) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} \overset{\text{"DSP"}}{=} \left(1+\frac{1}{\infty}\right)^{\infty} = 1^{\infty} \to$ informal

$\ln y = \ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right)$

$2 = \frac{1}{\frac{1}{2}} = 2\checkmark$

$x = \frac{1}{\frac{1}{x}}$

$\ln y = \lim_{x\to\infty}\left(\ln\left(1+\frac{1}{x}\right)^{x}\right) = \lim_{x\to\infty}\left(x\cdot\ln\left(1+\frac{1}{x}\right)\right)$

$= \lim_{x\to\infty}\left(\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\right) \overset{\text{"DSP"}}{=} \frac{\ln\left(1+\frac{1}{\infty}\right)}{\frac{1}{\infty}} = \frac{\ln 1}{0} = \text{"}\frac{0}{0}\text{"}$ L.R.

"0 $\leftarrow$ $\frac{\text{non-zero finite \#}}{\text{inf.}}$"

$$\ln y = \lim_{x\to\infty}\left(\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\right) \overset{\text{"DSP"}}{=} \text{"}\frac{0}{0}\text{"} \to \text{L.R.}$$

side work

$$\left[\ln\left(1+x^{-1}\right)\right]' = \frac{-x^{-2}}{1+x^{-1}}$$

$$\left(\frac{1}{x}\right)' = \left(x^{-1}\right)' = -1\cdot x^{-2}$$

$$\overset{H}{=} \lim_{x\to\infty}\left(\frac{\frac{\cancel{-x^{-2}}}{1+x^{-1}}}{\cancel{-x^{-2}}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{1}{1+x^{-1}}\right) \overset{\text{"DSP"}}{=} \frac{1}{1+\underbrace{\frac{1}{x}}_{\infty}} = \frac{1}{1} = 1 = \ln y$$

0

Q: $y = ?$

$$\ln y = 1 \Rightarrow y = e^{1} = e \;\Big\}\; \text{final answer}$$

be alert!

Exp) Evaluate $\lim\limits_{x\to0^+}(x^x) \overset{\text{"DSP"}}{=} \boxed{\text{"}0^0\text{"}}$ other indet. for

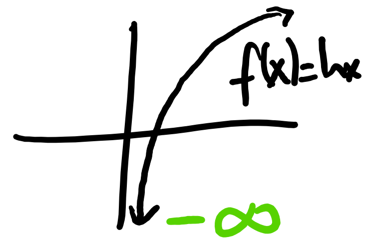
goal ⇒ re-write to get $\text{"}\frac{0}{0}\text{"}, \text{"}\frac{\infty}{\infty}\text{"}$

$$\lim_{x\to0^+}(x^x) \overset{\text{"DSP"}}{=} 0^0$$

Set $y = \lim\limits_{x\to0^+}(x^x)$

Use ln prop. $\ln y = \ln \lim\limits_{x\to0^+}(x^x)$

$$= \lim_{x\to0^+} \ln(x^x)$$



$$= \lim_{x\to0^+}(x\cdot\ln x) \qquad x = \frac{1}{\frac{1}{x}}$$

$$= \lim_{x\to0^+}\left(\frac{\ln x}{\frac{1}{x}}\right) \overset{\text{"DSP"}}{=} \left(\frac{\overbrace{\ln 0^+}^{-\infty}}{\underbrace{\frac{1}{0^+}}_{\infty}}\right) = \frac{-\infty}{\infty} \to \text{L.R}$$

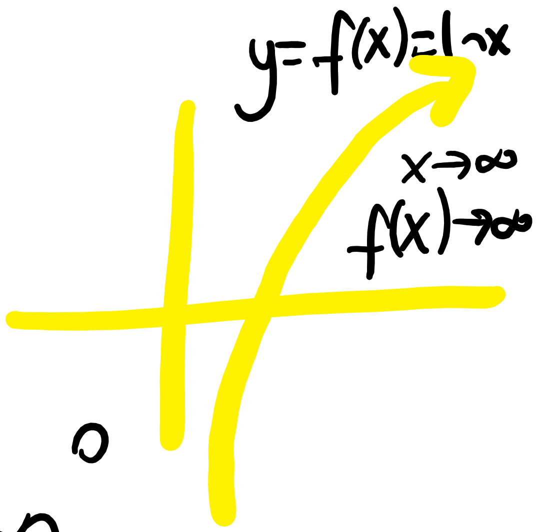
$$\ln y \overset{H}{=} \lim_{x\to0^+}\left(\frac{(\ln x)'}{\left(\frac{1}{x}\right)'}\right) = \lim_{x\to0^+}\left(\frac{1/x}{-x^{-2}}\right)$$

$\left(\frac{1}{x}\right) \to x^{-1}$ $\qquad -x^{-2} \to -\frac{1}{x^2}$

$$= \lim_{x\to0^+}\left(\frac{1}{\cancel{x}}\cdot\frac{-x^{\cancel{2}}}{1}\right) = \lim_{x\to0^+}(-x)$$

$$\overset{\text{"DSP"}}{=} 0$$

Since $\ln y = 0 \Rightarrow y = e^0 = 1$ → final answer



Exp) Evaluate $\lim\limits_{x\to\infty} (\ln x)^{1/x}$

$$\lim_{x\to\infty} (\ln x)^{1/x} \overset{\text{"DSP"}}{=} \underbrace{\ln(\infty)}_{\infty}^{1/\infty} = \infty^{0}$$

∞^0 other indet. form → ln prop.

"$\frac{0}{0}$, $\frac{\infty}{\infty}$" L.R.

Set $y = \lim\limits_{x\to\infty} (\ln x)^{1/x}$

$$\ln y = \ln \lim_{x\to\infty} (\ln x)^{1/x}$$

$$= \lim_{x\to\infty} \ln(\ln x)^{1/x} = \lim_{x\to\infty} \left(\frac{1}{x} \cdot \ln(\ln x) \right)$$

$$\overset{\text{"DSP"}}{=} \frac{\ln(\ln\infty)}{\infty} = \frac{\ln(\infty)}{\infty} = \text{"}\frac{\infty}{\infty}\text{"} \to \text{primary indet. form L.R.}$$

$$\overset{H}{=} \lim_{x\to\infty} \left(\frac{[\ln(\ln x)]'}{x'} \right)$$

$$[\ln(\ln x)]' = \frac{(\ln x)'}{\ln x} = \frac{\frac{1}{x}}{\ln x}$$

$$= \lim_{x\to\infty} \left(\frac{\frac{\frac{1}{x}}{\ln x}}{1} \right)$$

$$= \lim_{x\to\infty} \left(\frac{1}{x \cdot \ln x} \right) \overset{\text{"DSP"}}{=} \frac{1}{\underbrace{\infty \cdot \ln\infty}_{\infty\cdot\infty}} = 0$$

$$\ln y = 0 \Rightarrow y = e^0 = 1 \to \text{final answer}$$

Inf - inf

Sunday, November 1, 2020 9:37 PM

Exp) Evaluate $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right) \overset{\text{"DSP"}}{=} \underbrace{\frac{1}{0^+}}_{\infty} - \frac{1}{\sin 0^+} = \underbrace{\infty-\infty}_{\text{indet. form}}$$

$\sin 0 = 0$

$(\sin x)$ (x)

$$\lim_{x\to 0^+}\left(\frac{\sin x - x}{x\cdot \sin x}\right) \overset{\text{"DSP"}}{=} \text{"}\frac{0}{0}\text{"}$$

$$\underbrace{\text{"}\frac{0}{0}\text{"} \quad \text{"}\frac{\infty}{\infty}\text{"}}_{\text{L.R.}}$$

$$\overset{H}{=} \lim_{x\to 0^+}\left(\frac{\cos x - 1}{1\cdot \sin x + x\cdot \cos x}\right) \overset{\text{DSP}}{=} \frac{\cos 0 - 1}{\underbrace{\sin 0}_{0} + 0\cdot \underbrace{\cos 0}_{1}} = \text{"}\frac{0}{0}\text{"}$$

$$\overset{H}{=} \lim_{x\to 0^+}\left(\frac{-\sin x}{\cos x + \underbrace{1\cdot\cos x + x\cdot(-\sin x)}_{(x\cdot \cos x)'}}\right) \overset{\text{"DSP"}}{=} \frac{0}{1+1+0} = \frac{0}{2}$$

$$= 0$$

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right) = 0$$

EXAMPLE 4 **L'Hôpital's Rule for $0 \cdot \infty$** Evaluate $\lim_{x\to\infty} x^2 \sin\left(\frac{1}{4x^2}\right)$.

$$\lim_{x\to\infty}\left(x^2 \cdot \sin\left(\frac{1}{4x^2}\right)\right) \overset{\text{"DSP"}}{=} \infty \cdot \sin\underbrace{\left(\frac{1}{\infty}\right)}_{} \quad \frac{1}{\infty} \to 0$$

$$= \infty \cdot 0 \quad \text{other indet. form} \quad \to \text{"}\frac{\infty}{\infty}\text{"}, \ \text{"}\frac{0}{0}\text{"} \quad \text{pr. indet. form}$$

Re-write:

$$\lim_{x\to\infty}\left(\frac{\sin\left(\frac{1}{4x^2}\right)}{\frac{1}{x^2}}\right) \overset{\text{"DSP"}}{=} \text{"}\frac{0}{\frac{1}{\infty}}\text{"} = \text{"}\frac{0}{0}\text{"}$$

$$2 = \frac{1}{\frac{1}{2}} \ ; \quad \underbrace{x^2 = \frac{1}{\frac{1}{x^2}}}$$

$$\overset{H}{=} \lim_{x\to\infty}\left(\frac{\cos\left(\frac{1}{4x^2}\right)\cdot\left[(4x^2)^{-1}\right]'}{(x^{-2})'}\right)$$

$$= \lim_{x\to\infty}\left(\frac{\cos\left(\frac{1}{4x^2}\right)\cdot(-1)\cdot(4x^2)^{-2}\cdot 8x}{-2\cdot x^{-3}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{\cos\left(\frac{1}{4x^2}\right)\cdot\frac{1}{\cancel{16}\,2\,x^{\cancel{4}\,3}}\cdot\cancel{8x}\,1}{\frac{2}{x^3}}\right) = \lim_{x\to\infty}\left(\frac{\cos\left(\frac{1}{4x^2}\right)\cdot\frac{1}{2\cancel{x^3}}}{\frac{2}{\cancel{x^3}}}\right)$$

$$= \lim_{x\to\infty}\left(\cos\left(\frac{1}{4x^2}\right)\cdot\frac{1}{4}\right) \overset{\text{"DSP"}}{=} \cos\left(\cancel{\frac{1}{\infty}}\,0\right)\cdot\frac{1}{4} = \frac{1}{4}$$

$$\frac{\frac{1}{2}}{\frac{2}{1}} = \frac{1}{4}$$

Summary

Sunday, November 1, 2020 9:40 PM

Important Notes

* NOT a quotient rule (differentiate $f(x)$, $g(x)$ separately)

* LR can be applied repeatedly
* justify the use of L.R. then use it!
* LR applies to one-sided limits and limits at ∞ as well ($x \to c^-$, $x \to c^+$, $x \to \mp\infty$)