Ch 4.1 Maxima and Minima

Thursday, October 29, 2020 8:20 AM

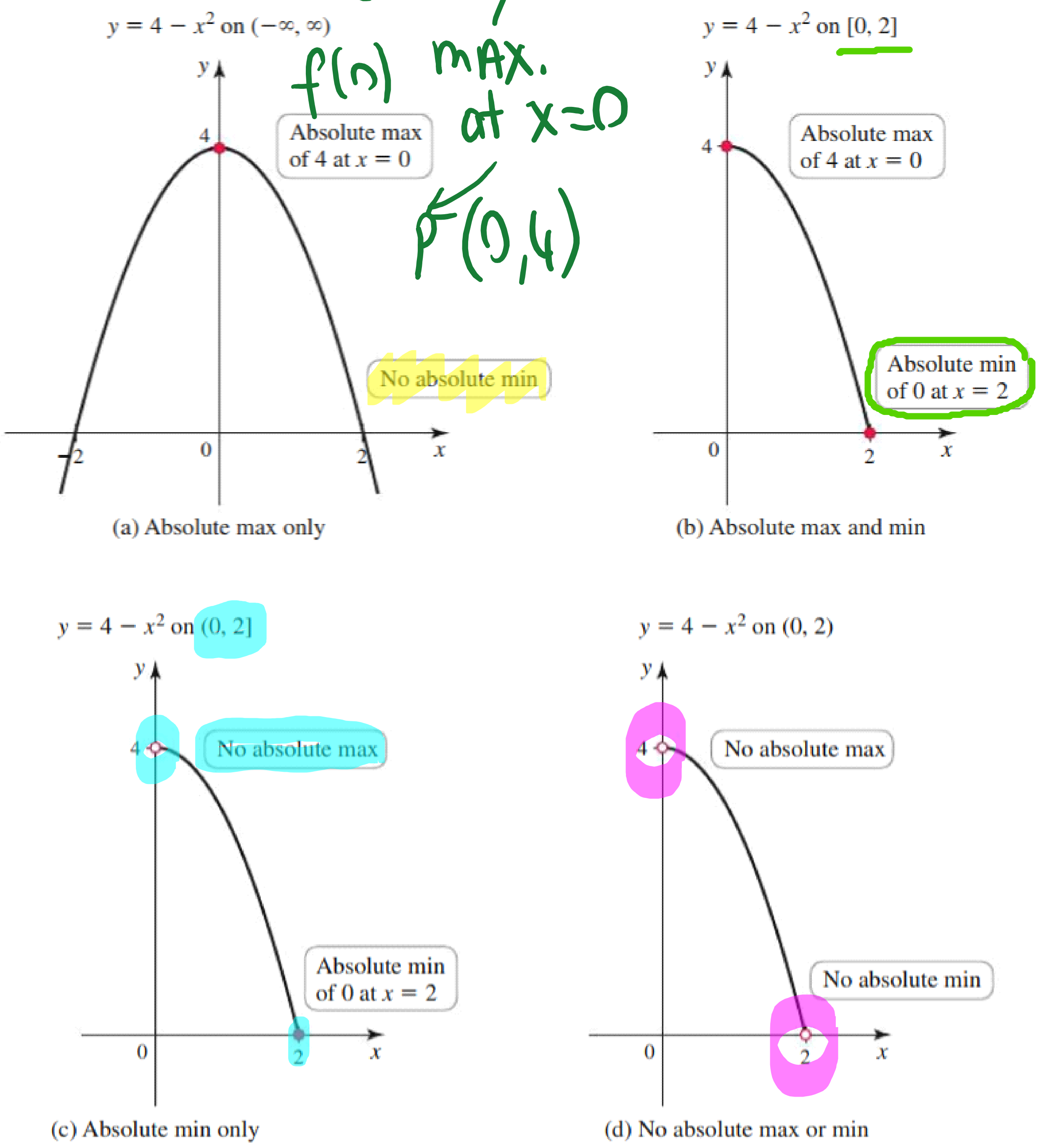
Global

DEFINITION Absolute Maximum and Minimum

Let f be defined on a set D containing c. If $f(c) \geq f(x)$ for every x in D, then $f(c)$ is an **absolute maximum** value of f on D. If $f(c) \leq f(x)$ for every x in D, then $f(c)$ is an **absolute minimum** value of f on D. An **absolute extreme value** is either an absolute maximum value or an absolute minimum value.

The existence and location of absolute extreme values depend on both the function and the interval of interest. Figure 4.2 shows various cases for the function $f(x) = 4 - x^2$. Notice that if the interval of interest is not closed, a function might not attain absolute extreme values (Figure 4.2a, c, and d).

Global / Abs.



$f(0)$ max. at $x=0$

$(0,4)$

(a) Absolute max only

(b) Absolute max and min

(c) Absolute min only

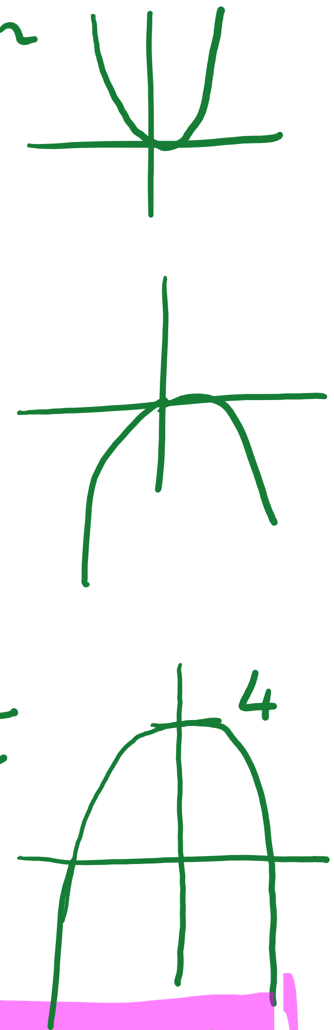
(d) No absolute max or min

$f(x) = x^2$

$-x^2$

$-x^2+4$

$4-x^2$



f, I *

global = abs.

However, defining a function on a closed interval is not enough to guarantee the existence of absolute extreme values. Both functions in Figure 4.3 are defined at every point of a closed interval, but neither function attains an absolute maximum—the discontinuity in each function prevents it from happening.

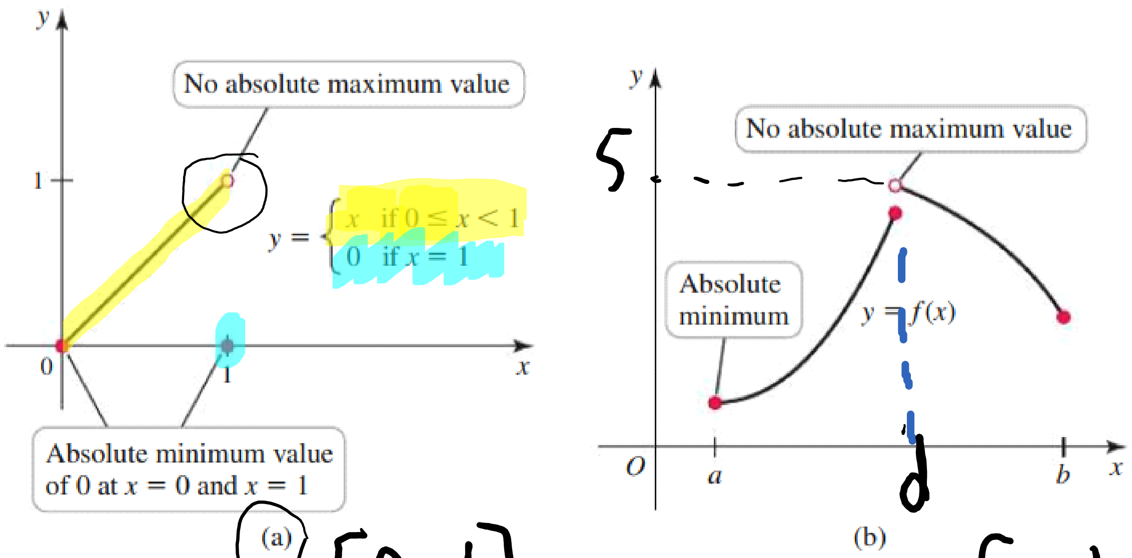


Figure 4.3

[0, 1]

[a,b]

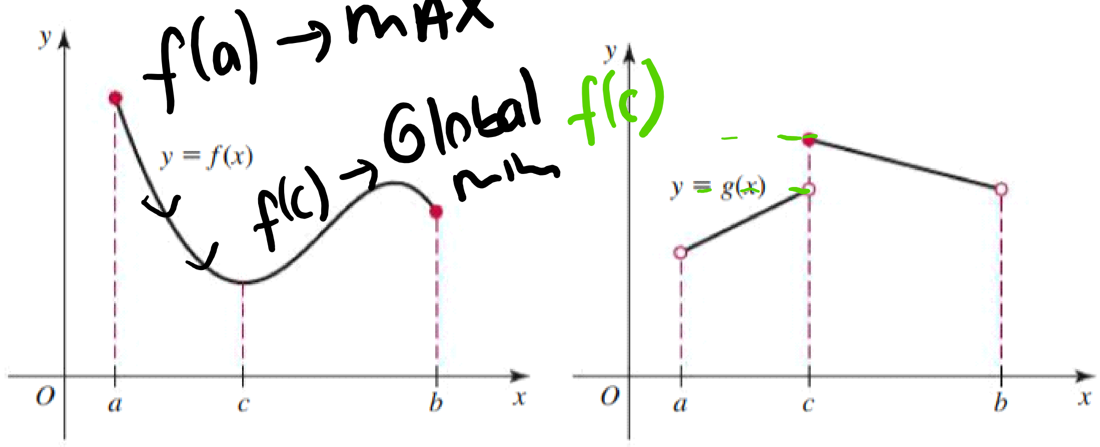
NOT def. at (d,5)

It turns out that *two* conditions ensure the existence of absolute maximum and minimum values on an interval: The function must be continuous on the interval, and the interval must be closed and bounded.

(EVT) Existence

> **THEOREM 4.1 Extreme Value Theorem**
> A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.

EXAMPLE 1 Locating absolute maximum and minimum values For the functions in **Figure 4.4**, identify the location of the absolute maximum value and the absolute minimum value on the interval $[a, b]$. Do the functions meet the conditions of the Extreme Value Theorem?



(a) (b)

(a, b)

Figure 4.4

a) $f(x)$ is cont. on $[a,b]$ a, b are endpoints

EVT guarantees that an ABS. MAX occurs at $x=a$ $(a, f(a))$ and ABS. MIN occurs at c $(c, f(c))$

b) $g(x)$ does NOT satisfy EVT ($g(x)$ is NOT cont. and is defined on an OPEN interval (a,b).

$g(x)$ does NOT have a ABS. MIN. value.

However, $g(c)$ is the ABS. MAX value

(ABS. MAX. occurs at $x=c$)

Local Maxima and Minima

(Relative max. and min.)

Figure 4.5 shows a function f defined on the interval $[a, b]$. It has an absolute minimum at the endpoint b and an absolute maximum at the interior point s. In addition, the function has

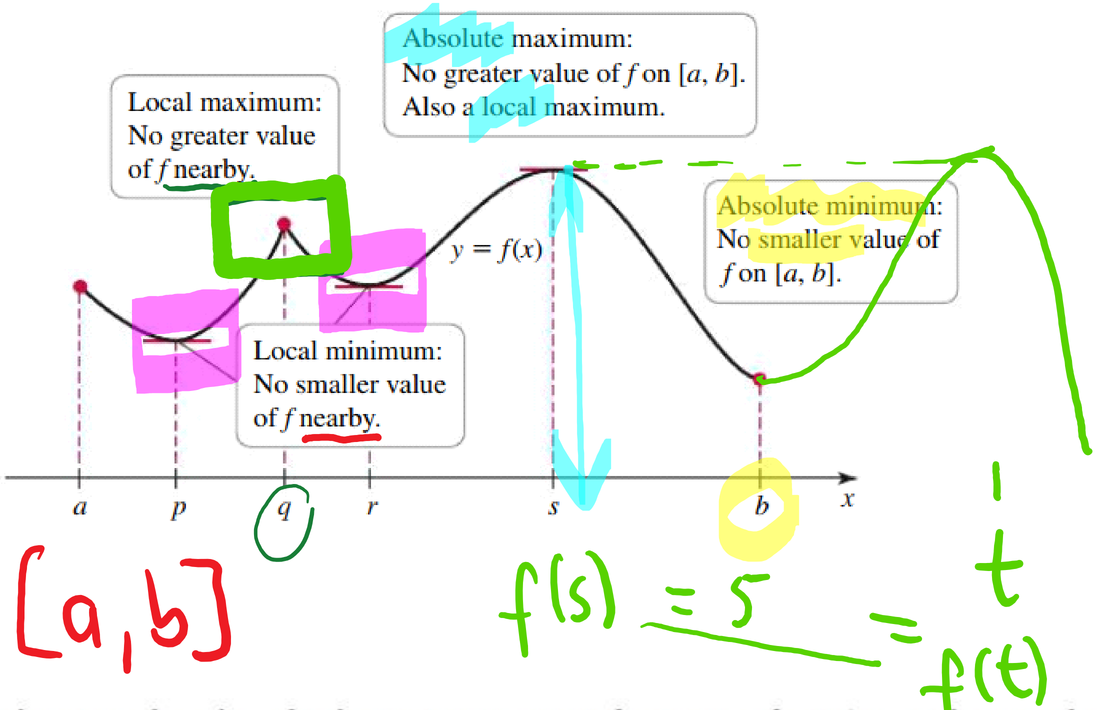


Figure 4.5

Local MIN/MAX CAN NOT occur at the endpoints.

✓ q → local max?

special behavior at q, where its value is greatest *among values at nearby points*, and at p and r, where its value is least *among values at nearby points*. A point at which a function takes on the maximum or minimum value among values at nearby points is important.

> **DEFINITION Local Maximum and Minimum Values**
> Suppose c is an interior point of some interval I on which f is defined. If $f(c) \geq f(x)$ for all x in I, then $f(c)$ is a **local maximum** value of f. If $f(c) \leq f(x)$ for all x in I, then $f(c)$ is a **local minimum** value of f.

In this text, we adopt the convention that local maximum values and local minimum values occur only at interior points of the interval(s) of interest. For example, in Figure 4.5, the minimum value that occurs at the endpoint b is not a local minimum. However, it is the absolute minimum of the function on $[a, b]$.

In this text, we adopt the convention that local maximum values and local minimum values occur only at interior points of the interval(s) of interest. For example, in Figure 4.5, the minimum value that occurs at the endpoint b is not a local minimum. However, it is the absolute minimum of the function on $[a, b]$.

EXAMPLE 2 Locating various maxima and minima Figure 4.6 shows the graph of a function defined on $[a, b]$. Identify the location of the various maxima and minima using the terms *absolute* and *local*.

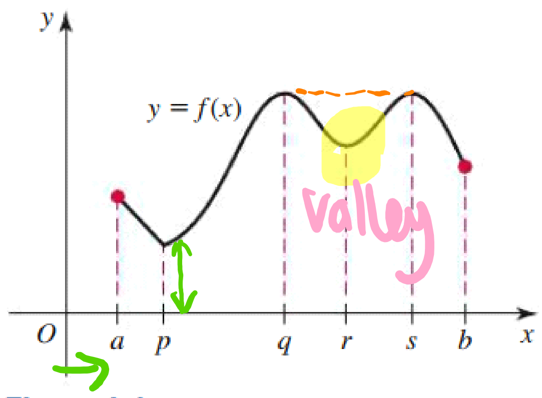


Figure 4.6

Local min/max CAN NOT occur at the endpoints.

$f(q) = f(s)$

valley at $x=r$

(Abs) Global MIN at $x=p$ $(f(p))$

Local MIN at $x=p$

(Abs) Global MAX at $x=q, s$

Local MAX at $x=q, s$

No global MIN/MAX at $x=a, b$

Local MIN at $x=r$

Local MAX at $x=q, s$

(valley → local min)

(peak → local max)

THEOREM 4.2 Local Extreme Value Theorem
If f has a local maximum or minimum value at c and $f'(c)$ exists, then $f'(c) = 0$.

Local extrema can also occur at points c where $f'(c)$ does not exist. Figure 4.8 shows two such cases, one in which c is a point of discontinuity and one in which f has a corner point at c. Because local extrema may occur at points c where $f'(c) = 0$ or where $f'(c)$ does not exist, we make the following definition.

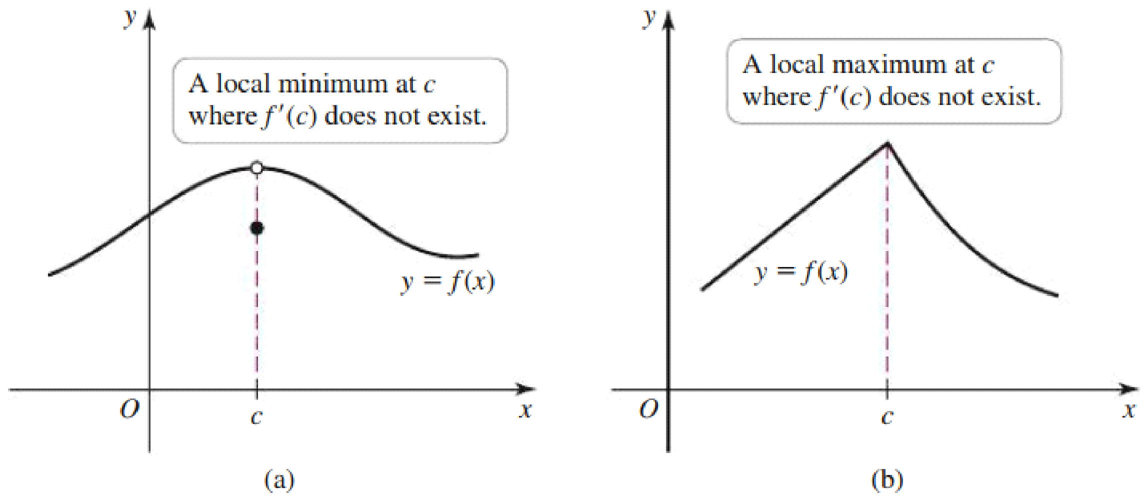


Figure 4.8

DEFINITION Critical Point
An interior point c of the domain of f at which $f'(c) = 0$ or $f'(c)$ fails to exist is called a **critical point** of f.

Note that the converse of Theorem 4.2 is not necessarily true. It is possible that $f'(c) = 0$ at a point without a local maximum or local minimum value occurring there (Figure 4.9a). It is also possible that $f'(c)$ fails to exist, with no local extreme value occurring at c (Figure 4.9b). Therefore, critical points are *candidates* for the location of local extreme values, but you must determine whether they actually correspond to local maxima or minima. This procedure is discussed in Section 4.3.

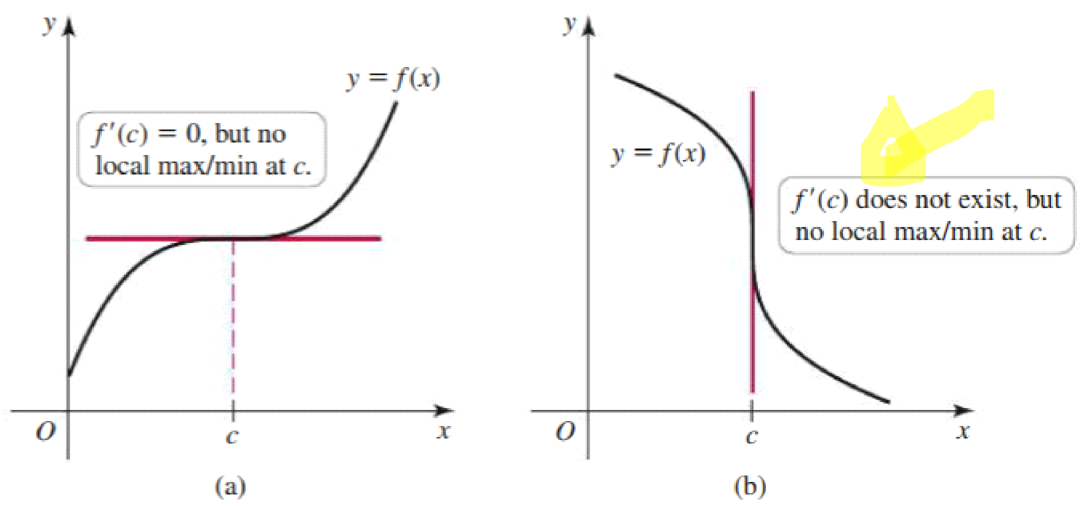
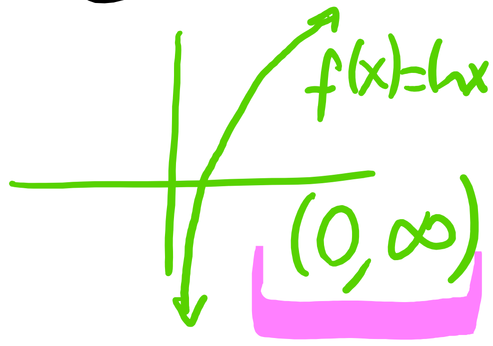


Figure 4.9

EXAMPLE 3 **Locating critical points** Find the critical points of $f(x) = x^2 \ln x$.

f, I

$f'(x)=0$ or DNE



$f(x) = x^2 \cdot \ln x$

Use product rule to diff:

$$f'(x) = 2x \cdot \ln x + x^2 \cdot \frac{1}{x} = 2x \cdot \ln x + x = x(2\ln x + 1)$$

$$f'(x) = x \cdot (2\ln x + 1) = 0$$

$\cancel{x=0}$

$2\ln x + 1 = 0 \Rightarrow \ln x = -1/2$

$x = e^{-1/2}$

x=0 is NOT in the domain of f(x)

$\boxed{x = e^{-1/2} \text{ is a critical point.}}$

$(x,y) \to (x, x^2 \cdot \ln x)$

$\boxed{\left(e^{-1/2}, \frac{-1}{2e}\right)}$

$f(e^{-1/2})$

$$f(e^{-1/2}) = (e^{-1/2})^2 \cdot \ln(e^{-1/2})$$

$$= e^{-1} \cdot \frac{-1}{2} \cdot \cancel{\ln e^1}$$

$$= \frac{-1}{2e}$$

Locating Absolute Maxima and Minima

Theorem 4.1 guarantees the existence of absolute extreme values of a continuous function on a closed interval $[a, b]$, but it doesn't say where these values are located. Two observations lead to a procedure for locating absolute extreme values.

- An absolute extreme value in the interior of an interval is also a local extreme value, and we know that local extreme values occur at the critical points of f.
- Absolute extreme values may also occur at the endpoints of the interval of interest.

These two facts suggest the following procedure for locating the absolute extreme values of a continuous function on a closed interval.

PROCEDURE **Locating Absolute Extreme Values on a Closed Interval**

Assume the function f is continuous on the closed interval $[a, b]$.

1. Locate the critical points c in (a, b), where $f'(c) = 0$ or $f'(c)$ does not exist. These points are candidates for absolute maxima and minima.
2. Evaluate f at the critical points and at the endpoints of $[a, b]$.
3. Choose the largest and smallest values of f from Step 2 for the absolute maximum and minimum values, respectively.

Note that the preceding procedure box does not address the case in which f is continuous on an open interval. If the interval of interest is an open interval, then absolute extreme values—if they exist—occur at interior points.

Steps

1) c_1, c_2, c_3 in I

2) $f(c_1), f(c_2), f(c_3),$
$f(a), f(b)$

3) compare
greatest → ABS. MAX

smallest → ABS. MIN.

PROCEDURE **Locating Absolute Extreme Values on a Closed Interval**

Assume the function f is continuous on the closed interval $[a, b]$.

1. Locate the critical points c in (a, b), where $f'(c) = 0$ or $f'(c)$ does not exist. These points are candidates for absolute maxima and minima.
2. Evaluate f at the critical points and at the endpoints of $[a, b]$.
3. Choose the largest and smallest values of f from Step 2 for the absolute maximum and minimum values, respectively.

EXAMPLE 4 **Absolute extreme values** Find the absolute maximum and minimum values of the following functions.

a. $f(x) = x^4 - 2x^3$ on the interval $[-2, 2]$

b. $g(x) = x^{2/3}(2 - x)$ on the interval $[-1, 2]$

a) Step1 Find critical points c in $(-2, 2)$

$f'(x) = 4x^3 - 6x^2 = 0$ or DNE

$f'(x) = 2x^2(2x-3) = 0 \Rightarrow \underbrace{x=0, \quad x=3/2}_{\text{Cadidates for ABS. min/max}}$

critical points

Step2 Eval. $f(x)$ at $x = 0, 3/2, -2, 2$

$f(x) = x^4 - 2x^3$

$f(0) = 0$

$f(3/2) = \frac{-27}{16}$

$f(-2) = 32$

$f(2) = 0$

Smallest value (ABS. min) at $(3/2, -27/16)$

greatest value (ABS. max) at $(-2, 32)$

$x=0$ is not ABS. min nor ABS. max.

PROCEDURE **Locating Absolute Extreme Values on a Closed Interval**

Assume the function f is continuous on the closed interval $[a, b]$.

1. Locate the critical points c in (a, b), where $f'(c) = 0$ or $f'(c)$ does not exist. These points are candidates for absolute maxima and minima.
2. Evaluate f at the critical points and at the endpoints of $[a, b]$.
3. Choose the largest and smallest values of f from Step 2 for the absolute maximum and minimum values, respectively.

Recall:

$x^{2/3} \cdot x^{1} = x^{5/3}$

EXAMPLE 4 **Absolute extreme values** Find the absolute maximum and minimum values of the following functions.

a. $f(x) = x^4 - 2x^3$ on the interval $[-2, 2]$

b. $g(x) = x^{2/3}(2 - x)$ on the interval $[-1, 2]$

b. Step 1 $g'(x) = 0$ or DNE $[-1,2]$

$g(x) = 2 \cdot x^{2/3} - x^{5/3}$

Use power rule: $g'(x) = 2 \cdot \frac{2}{3} \cdot x^{-1/3} - \frac{5}{3} \cdot x^{2/3}$

$= \frac{4}{3} \cdot x^{-1/3} - \frac{5}{3} \cdot x^{2/3}$

$= \frac{4}{3x^{1/3}}(1) - \frac{5}{3}x^{2/3}(x^{1/3})$ $5x^1$

$= \frac{4 - 5x}{3x^{1/3}}$

$g'(x) = 0$ or DNE

$g'(x) = \frac{4-5x}{3x^{1/3}} = 0$ $\overset{0}{\Rightarrow}$ $4 - 5x = 0 \Rightarrow 4 = 5x \Rightarrow x = 4/5$

$\overset{\text{DNE}}{\Rightarrow}$ $3 \cdot x^{1/3} = 0 \Rightarrow x = 0$

The critical points are $x = 0, 4/5$ on $[-1,2]$

Step2) critical points at $x=0, 4/5$

endpoints are $x=-1,2$ $\underbrace{[-1,2]}_{\text{interval}}$

$g(x)= x^{2/3}(2-x)$

$g(0) = 0$

$g(4/5) = \left(\frac{4}{5}\right)^{2/3}\left(2-4/5\right) \approx 1.03$

$g(-1) = (-1)^{2/3}(2+1) = \sqrt[3]{(-1)^2} \cdot 3 = 3$

$g(2) = 2^{2/3}(2-2) = 0$

Step3) compare ALL $g(0), g(4/5), g(-1), g(2)$

ABS. MAX. at $x=-1$ $P(-1, 3)$

ABS. MIN at $x=0,2$ $P(0,\underline{0}), P(2,\underline{0})$