End Behavior

The behavior of polynomials as $x \to \pm\infty$ is an example of what is often called *end behavior*. Having treated polynomials, we now turn to the end behavior of rational, algebraic, and transcendental functions.

EXAMPLE 3 **End behavior of rational functions** Use limits at infinity to determine the end behavior of the following rational functions.

a. $f(x) = \dfrac{3x+2}{x^2-1}$ **b.** $g(x) = \dfrac{40x^4+4x^2-1}{10x^4+8x^2+1}$ **c.** $h(x) = \dfrac{x^3-2x+1}{2x+4}$

c)

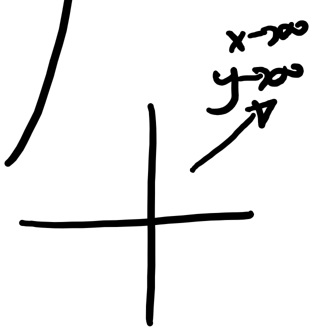
Step1) x → highest degree in the denominator

Step2) Div. ALL by x

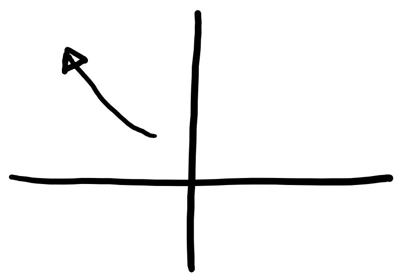
$$\lim_{x\to\infty}\left(\frac{\frac{x^3-2x+1}{x}}{\frac{2x+4}{x}}\right) = \lim_{x\to\infty}\left(\frac{x^2-2+\frac{1}{x}}{2+\frac{4}{x}}\right)$$

0 0

$$= \lim_{x\to\infty}\left(\frac{x^2-2}{2}\right) \overset{\text{"DSP"}}{=} \infty$$



$$\lim_{x\to-\infty}\left(\frac{\frac{x^3-2x+1}{x}}{\frac{2x+4}{x}}\right) = \lim_{x\to-\infty}\left(\frac{x^2-2}{2}\right) = \infty$$



No H.A.

- -- -- - - - .

Sunday, September 20, 2020 10:51 PM

THEOREM 2.7 End Behavior and Asymptotes of Rational Functions

Suppose $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function, where

$$p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_2x^2 + a_1x + a_0 \quad \text{and}$$
$$q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0,$$

with $a_m \neq 0$ and $b_n \neq 0$.

a. Degree of numerator less than degree of denominator If $m < n$, then $\lim_{x \to \pm\infty} f(x) = 0$, and $y = 0$ is a horizontal asymptote of f.

b. Degree of numerator equals degree of denominator If $m = n$, then $\lim_{x \to \pm\infty} f(x) = a_m/b_n$, and $y = a_m/b_n$ is a horizontal asymptote of f.

c. Degree of numerator greater than degree of denominator If $m > n$, then $\lim_{x \to \pm\infty} f(x) = \infty$ or $-\infty$, and f has no horizontal asymptote.

~~**d. Slant asymptote** If $m = n + 1$, then $\lim_{x \to \pm\infty} f(x) = \infty$ or $-\infty$, and f has no horizontal asymptote, but f has a slant asymptote.~~

e. Vertical asymptotes Assuming f is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeros of q.

H.A.

$x \to \pm\infty$

$f(x) \to ?$

$y = \_\_\_$

V.A.

$x \to a^+$ $x \to a^-$

$f(x) \to \pm\infty$

$x = \_\_\_$

$\cancel{\sqrt{-x^6}}$

$f(x) = \sqrt[2]{x^2} = |x|$ $\to x \geq 0,\ x$; $\to x < 0,\ -x$

EXAMPLE 5 **End behavior of an algebraic function** Use limits at infinity to determine the end behavior of $f(x) = \dfrac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$.

Step1) $\sqrt[2]{x^6} = |x^3|$

$x \geq 0,\ x^3$ $\quad x \to \infty \quad$ Case 1

$x < 0,\ -x^3$ $\quad x \to -\infty \quad$ case 2

Step2) Div. ALL by x^3 or $-x^3$

Case 1)

$$\lim_{x\to\infty}\left(\frac{\dfrac{10x^3-3x^2+8}{x^3}}{\sqrt{\dfrac{25x^6+x^4+2}{x^6}}}\right) = \lim_{x\to\infty}\left(\frac{10-\dfrac{3}{x}+\dfrac{8}{x^3}}{\sqrt{25+\dfrac{1}{x^2}+\dfrac{2}{x^6}}}\right)$$

$\dfrac{3}{x} \to 0$, $\dfrac{8}{x^3} \to 0$, $\dfrac{1}{x^2} \to 0$, $\dfrac{2}{x^6} \to 0$

$$= \lim_{x\to\infty}\left(\frac{10}{\sqrt{25}}\right) = \frac{10}{5} = 2$$

case 2)

$$\lim_{x\to-\infty}\left(\frac{\dfrac{10x^3-3x^2+8}{-x^3}}{\sqrt{\dfrac{25x^6+x^4+2}{x^6}}}\right) = \lim_{x\to-\infty}\left(\frac{-10+\dfrac{3}{x}-\dfrac{8}{x^3}}{\sqrt{25+\dfrac{1}{x^2}+\dfrac{2}{x^6}}}\right)$$

$\dfrac{3}{x} \to 0$, $\dfrac{8}{x^3} \to 0$, $\dfrac{1}{x^2} \to 0$, $\dfrac{2}{x^6} \to 0$

$$= \lim_{x\to-\infty}\left(\frac{-10}{\sqrt{25}}\right) = \frac{-10}{5} = -2$$

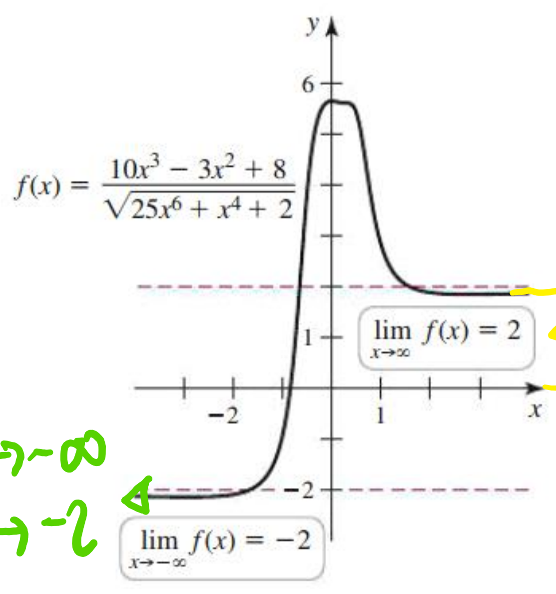


Figure 2.42

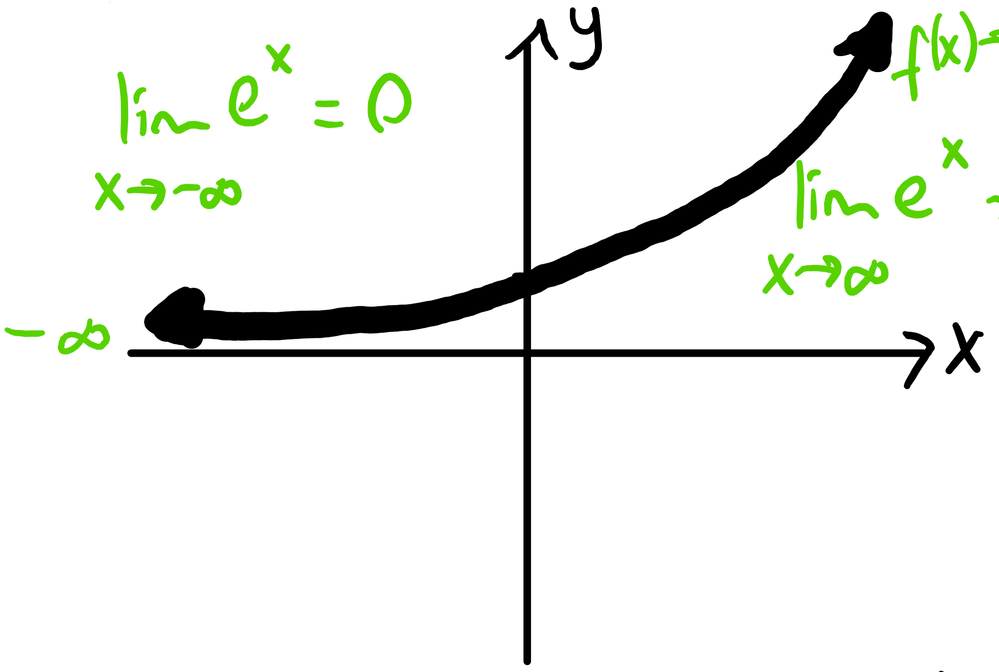
$x \to \infty$
$f(x) \to 2$

$x \to -\infty$
$y \to -2$

H.A: $y = 2,\ y = -2$

Sunday, September 20, 2020 10:58 PM

End Behavior for e^x, e^{-x}, and $\ln x$

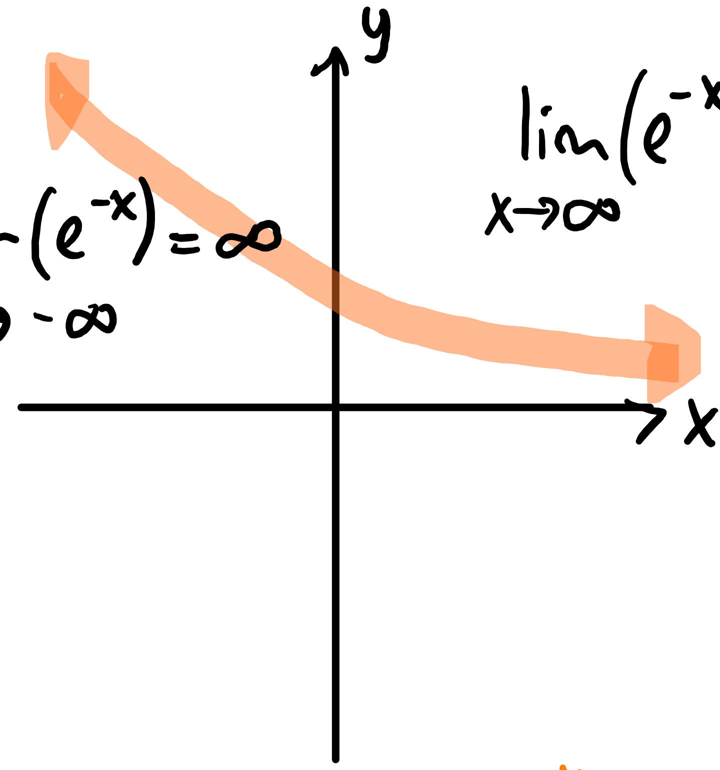


$\lim_{x \to -\infty} e^x = 0$

$\lim_{x \to \infty} e^x = \infty$

$f(x) = e^x$ $(0, e^0)$

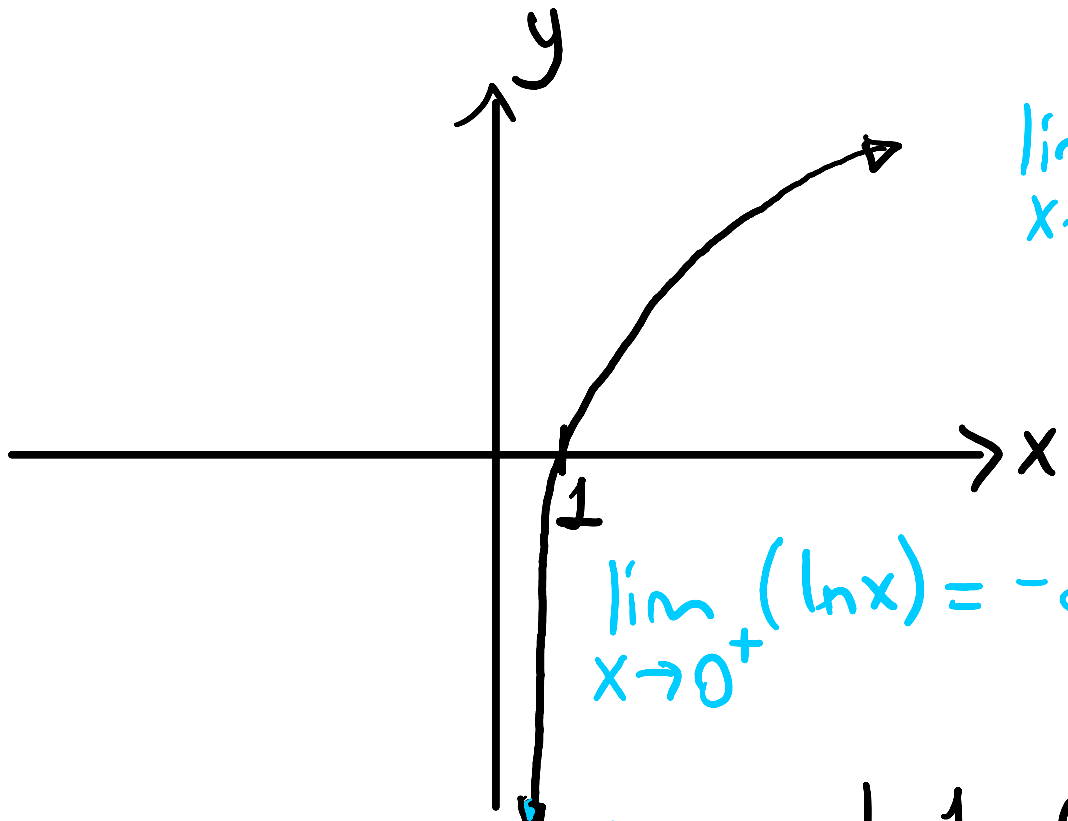
$(0, 1)$



$\lim_{x \to -\infty} (e^{-x}) = \infty$

$\lim_{x \to \infty} (e^{-x}) = 0$

$g(x) = e^{-x}$



$\lim_{x \to \infty} (\ln x) = \infty$

$\lim_{x \to 0^+} (\ln x) = -\infty$

$h(x) = \ln(x)$ $\ln 1 = 0$ $(1, 0)$ x-int.

D: $(0, \infty)$

THEOREM 2.8 End Behavior of e^x, e^{-x}, and $\ln x$

The end behavior for e^x and e^{-x} on $(-\infty, \infty)$ and $\ln x$ on $(0, \infty)$ is given by the following limits:

$$\lim_{x \to \infty} e^x = \infty \quad \text{and} \quad \lim_{x \to -\infty} e^x = 0,$$

$$\lim_{x \to \infty} e^{-x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} e^{-x} = \infty,$$

$$\lim_{x \to 0^+} \ln x = -\infty \quad \text{and} \quad \lim_{x \to \infty} \ln x = \infty.$$

Dr. Tabanli's Spring 2020 Exam2 Question:

Thursday, September 17, 2020 8:11 AM

Find all horizontal asymptotes of $f(x) = \dfrac{8-3x}{2x+\sqrt{25x^2+x+13}}$. Write "NONE" if f has no horizontal asymptotes.

method#1

Step1) Identify the highest (degree) exponent in the denominator

Step2) Divide ALL by

USE LIMITS!

x^1 is the highest exponent in the denominator

Case#1: $x \to \infty$ $\quad \sqrt{x^2} = x \quad (x \geq 0)$

$$\lim_{x\to\infty}\left(\frac{\frac{8-3x}{x}}{\frac{2x+}{x}\sqrt{\frac{25x^2+x+13}{x^2}}}\right) = \lim_{x\to\infty}\left(\frac{\frac{8}{x}^{\to 0}-3}{2+\sqrt{25+\frac{1}{x}^{\to 0}+\frac{13}{x^2}^{\to 0}}}\right)$$

$$\lim_{x\to\infty}\left(\frac{-3}{2+\sqrt{25}}\right) = \lim_{x\to\infty}\left(\frac{-3}{7}\right) = \frac{-3}{7}$$

Case #2: $x \to -\infty$ $\sqrt{x^2} = -x$ $(x < 0)$

$$\lim_{x \to -\infty} \left(\frac{\frac{8-3x}{-x}}{\frac{2x}{-x} + \sqrt{\frac{25x^2+x+13}{x^2}}} \right) = \lim_{x \to -\infty} \left(\frac{\frac{-8}{x} + 3}{-2 + \sqrt{25 + \frac{1}{x} + \frac{13}{x^2}}} \right)$$

0 5

$$= \lim_{x \to -\infty} \left(\frac{3}{-2+5} \right) = 1$$

Alternative (more Practical) method:

Find all horizontal asymptotes of $f(x) = \dfrac{8-3x}{2x+\sqrt{25x^2+x+13}}$. Write "NONE" if f has no horizontal asymptotes.

Focused on factoring more:

$$f(x) = \frac{8-3x}{2x+\sqrt{x^2\left(25+\frac{1}{x}+\frac{13}{x^2}\right)}} = \frac{8-3x}{2x+\underbrace{|x|}\cdot\sqrt{25+\frac{1}{x}+\frac{13}{x^2}}}$$

recall $\sqrt{x^2} = |x|$

Case#1 As $x \to \infty$ $|x| = x$;

$$\lim_{x\to\infty} \frac{8-3x}{2x+x\sqrt{25+\frac{1}{x}+\frac{13}{x^2}}} = \lim_{x\to\infty} \frac{\cancel{x}\left(\frac{8}{x}-3\right)}{\cancel{x}\left(2+1\cdot\sqrt{25+\frac{1}{x}+\frac{13}{x^2}}\right)}$$

$$= \lim_{x\to\infty}\left(\frac{-3}{2+5}\right) = \frac{-3}{7}$$

Case#2 As $x \to -\infty$ $|x| = -x$;

$$\lim_{x\to-\infty} \frac{8-3x}{2x-x\sqrt{25+\frac{1}{x}+\frac{13}{x^2}}} = \lim_{x\to-\infty} \frac{\cancel{x}\left(\frac{8}{x}-3\right)}{\cancel{x}\left(2-1\cdot\sqrt{25+\frac{1}{x}+\frac{13}{x^2}}\right)}$$

$$= \lim_{x\to-\infty}\left(\frac{-3}{2-5}\right) = \frac{-3}{-3} = 1$$

2.6 Continuity

Wednesday, September 23, 2020 6:22 PM

Informally, a function f is continuous at "a" if the graph of f doesn't have a hole or a break at a.

Definition: Continuity at a Point

A function f is continuous at a

if $$\lim_{x \to a} f(x) = f(a)$$

Continuity Checklist

In order for f to be continuous at a, the following three conditions must hold.

1. $f(a)$ is defined (a is in the domain of f).
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$ (the value of f equals the limit of f at a).

If f is continuous at a, then $\lim_{x \to a} f(x) = f(a)$, and direct substitution may be used to evaluate $\lim_{x \to a} f(x)$.

Note that when f is defined on an open interval containing a (except possibly at a), we say that f has a **discontinuity** at a (or that a is a **point of discontinuity**) if f is not continuous at a.

Checklist: #1) $f(a)$ #2) LL=RL # 3) $f(a) = \lim_{x \to a} f(x)$

EXAMPLE 1 **Points of discontinuity** Use the graph of f in Figure 2.48 to identify values of x on the interval $(0, 7)$ at which f has a discontinuity.

open interval

Hint: Check the open interval and state the type of discontinuities.

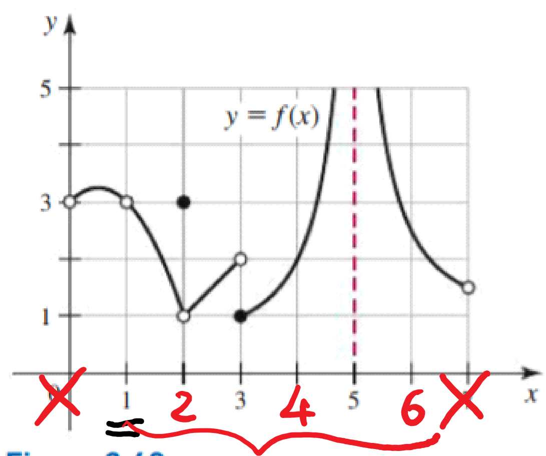


Figure 2.48

@ $x = 1$

$f(1)$ is undefined, failed #1

$\therefore$ point of discontinuity at $x = 1$

@ $x = 2$

#1) $f(2) = 3$

#2) $\lim_{x \to 2^-} f(x) = 1 = \lim_{x \to 2^+} f(x)$

#3) $3 \neq 1$, failed #3

$\therefore$ point of discontinuity at $x = 2$

3 checklist

EXAMPLE 2 **Continuity at a point** Determine whether the following functions are continuous at a. Justify each answer using the continuity checklist.

a. $f(x) = \dfrac{3x^2 + 2x + 1}{x - 1}$; $a = 1$ **b.** $g(x) = \dfrac{3x^2 + 2x + 1}{x - 1}$; $a = 2$

c. $h(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$; $a = 0$

a) $f(1)$ is undefined. Point of discont. at $x=1$

b) #1) $g(2) = \dfrac{3\cdot 2^2+2\cdot 2+1}{2-1} = \dfrac{3\cdot 4+4+1}{1} = 17$

#2 $\lim\limits_{x\to 2} g(x) \overset{\text{"DSP"}}{=} 17$

#3 $g(2) = \lim\limits_{x\to 2} g(x) = 17$ ✓

$\boxed{\sin\left(\frac{1}{x}\right)\cdot x \neq \sin 1}$

c) $h(x) = \left\{ \begin{array}{ll} x\cdot\sin\left(\frac{1}{x}\right), & \text{if } x\neq 0 \\ 0, & \text{if } x = 0 \end{array} \right\}$ $a = 0$

#1 $h(0) = 0$ ✓

#2 $\lim\limits_{x\to 0} h(x)$ ✓

$x\cdot -1 \leq \sin\left(\frac{1}{x}\right)\cdot x \leq 1\cdot x$

$\lim\limits_{x\to 0}\left(-x \leq \underbrace{x\cdot\sin\left(\frac{1}{x}\right)}_{\substack{h(x)\\ x\neq 0}} \leq x \right)$

$\boxed{\lim\limits_{x\to 0} h(x) = 0}$

#3 $h(0) = \lim\limits_{x\to 0} h(x) = 0$ ✓

@ $x = 3$ (Jump Discontinuity)

#1 $f(3) = 1$

#2 $\lim_{x \to 3^-} f(x) = 2$, $\lim_{x \to 3^+} f(x) = 1$

$2 \neq 1$

failed #2 . STOP!

$\lim_{x \to 3} f(x)$ DNE

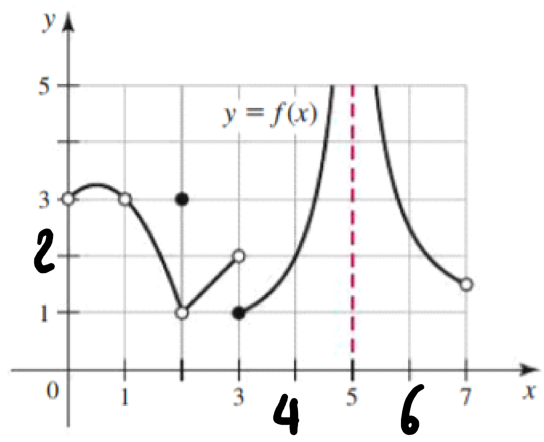


Figure 2.48

@ $x = 4$, $x = 6$ (Continuous)

$x = 4 \Rightarrow$ $f(4) = \lim_{x \to 4} f(x) = 2$ ✓

$x = 6 \Rightarrow$ $f(6) = \lim_{x \to 6} f(x) = 2.5$ ✓

@ $x = 5$ (Infinite Discont.)

#1 $f(5)$ undefined

#2 $\lim_{x \to 5} f(x) = +\infty$

Point of discontinuity at $x = 5$

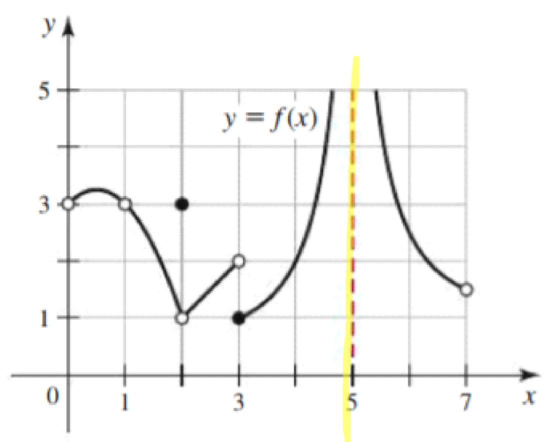


Figure 2.48

The following theorems make it easier to test various combinations of functions for continuity at a point.

THEOREM 2.9 Continuity Rules

If f and g are continuous at a, then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.

a. $f + g$

b. $f - g$

c. cf

d. fg

e. f/g, provided $g(a) \neq 0$

f. $(f(x))^n$

THEOREM 2.10 Polynomial and Rational Functions

a. A polynomial function is continuous for all x.

b. A rational function (a function of the form $\frac{p}{q}$, where p and q are polynomials) is continuous for all x for which $q(x) \neq 0$.

EXAMPLE 3 **Applying the continuity theorems** For what values of x is the function $f(x) = \frac{x}{x^2 - 7x + 12}$ continuous?

$-3 \quad -4$

$$f(x) = \frac{x}{(x-3)(x-4)} \neq 0$$

$(x = 3, 4)$

f is cont. for all x except x=3 and x=4

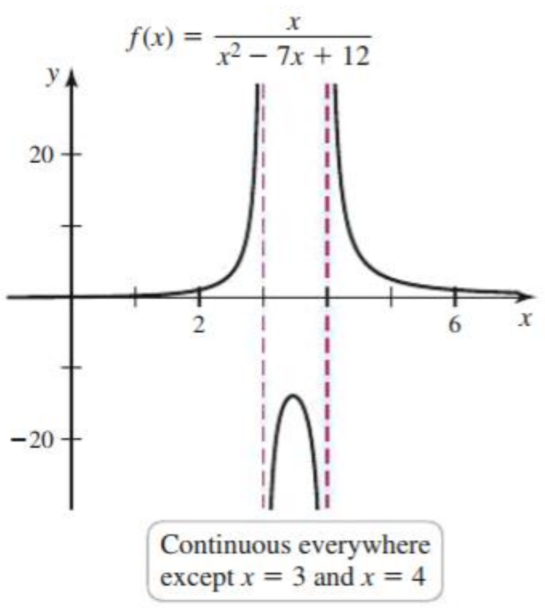


Figure 2.49

THEOREM 2.11 Continuity of Composite Functions at a Point
If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

Recall
$f(g(x)) = (f \circ g)(x)$

EXAMPLE 4 Limit of a composition Evaluate $\lim_{x \to 0} \left(\frac{x^4 - 2x + 2}{x^6 + 2x^4 + 1} \right)^{10}$.

Observe that:

$x \to 0^-$, $x \to 0^+$
$x^6, x^4 > 0$

denominator will not be equal to zero!

$$\lim_{x \to 0} \left(\frac{x^4 - 2x + 2}{x^6 + 2x^4 + 1} \right)^{10} \overset{DSP}{=} \left(\frac{0 - 0 + 2}{0 + 0 + 1} \right)^{10} = \underline{2^{10}}$$

THEOREM 2.12 Limits of Composite Functions

1. If g is continuous at a and f is continuous at $g(a)$, then
$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x) \right).$$
2. If $\lim_{x \to a} g(x) = L$ and f is continuous at L, then
$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x) \right).$$