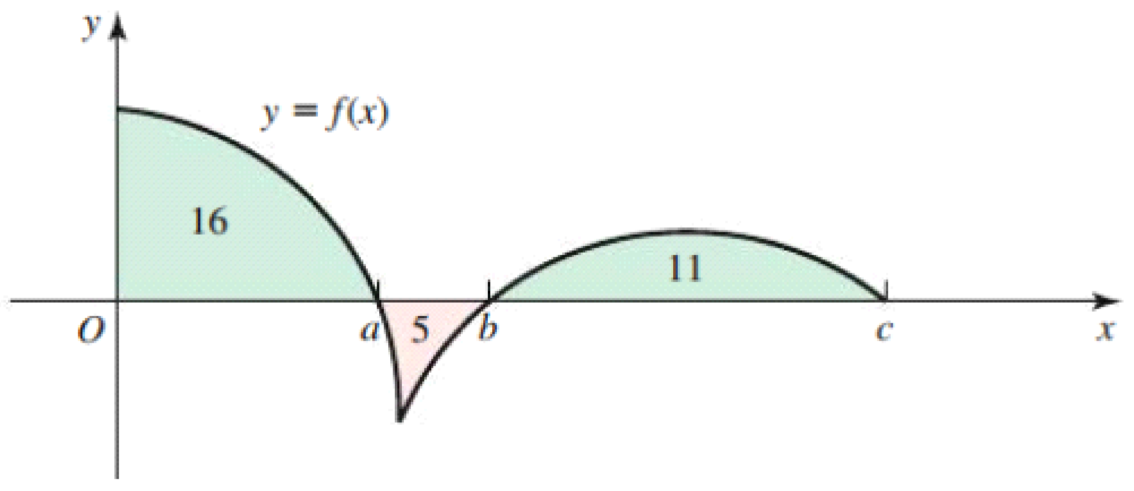
Chapter 5.3 Fundamental Theorem of Calculus

Wednesday, December 2, 2020 11:15 PM

Warm-Up / Poll Q.

Definite integrals from graphs *The figure shows the areas of regions bounded by the graph of f and the x-axis. Evaluate the following integrals.*



a) $\int_a^c f(x)\,dx = 11 - 5 = 6$

b) $\int_a^0 f(x)\,dx = -\int_0^a f(x)\,dx = -16$

use Prop. #2 below

c) $\int_c^0 |f(x)|\,dx = -\int_0^c |f(x)|\,dx = -(16+5+11) = -32$

use Prop. #2 below

use Prop. #6 below

Recall: From 11/30 Class

Table 5.4 Properties of definite integrals

Let f and g be integrable functions on an interval that contains a, b, and p.

1. $\int_a^a f(x)\,dx = 0$ Definition
2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ Definition
3. $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
4. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, for any constant c
5. $\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$
6. The function $|f|$ is integrable on $[a, b]$, and $\int_a^b |f(x)|\,dx$ is the sum of the areas of the regions bounded by the graph of f and the x-axis on $[a, b]$.

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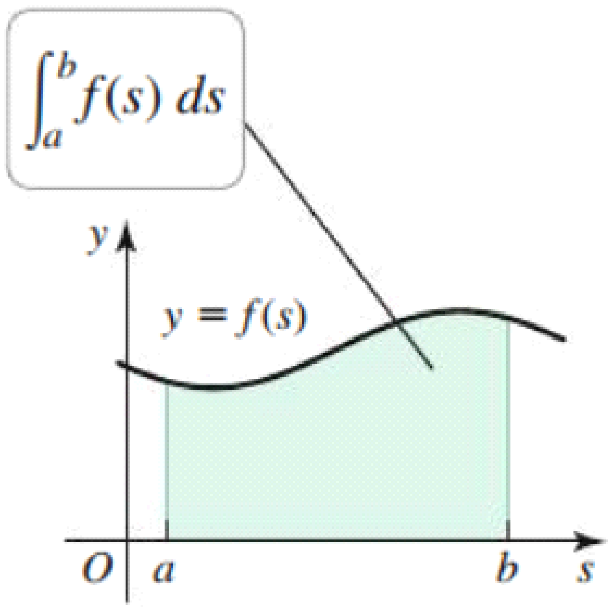
Introduction:

In 5.1, We approximated area under the curve by using Reimann sums (not that practical!).
In 5.2, We defined the definite integral in terms of limit of Reimann sums (as n --> infinity). We evaluated definite integrals by using properties and geometry. We related definite integral to "computing the area under the curve".
In 5.3, we will use ***The Fundamental Theorem of Calculus*** area to evaluate definite integrals in a practical and powerful way; discover the inverse relationship between differentiation and integration.

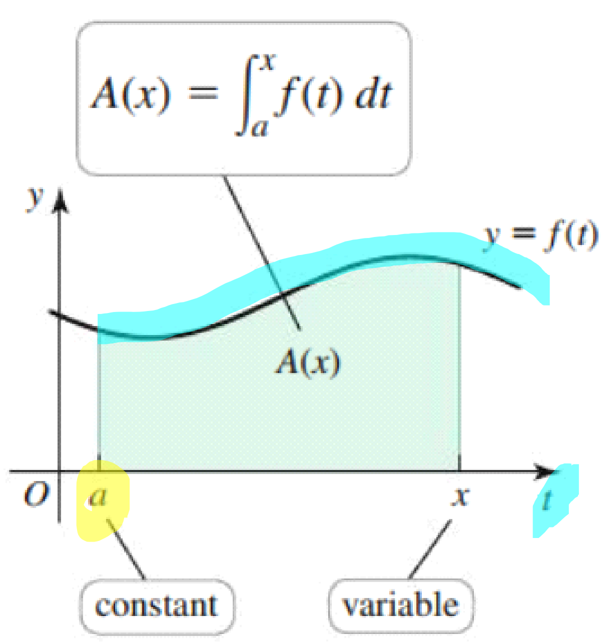
DEFINITION Definite Integral

A function f defined on $[a, b]$ is **integrable** on $[a, b]$ if $\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ exists and is unique over all partitions of $[a, b]$ and all choices of x_k^* on a partition. This limit is the **definite integral of *f* from *a* to *b***, which we write

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$



➤ The language "the area of the region bounded by the graph of a function" is often abbreviated as "the area under the curve."



DEFINITION Area Function

Let f be a continuous function, for $t \geq a$. The **area function for *f* with left endpoint *a*** is

$$A(x) = \int_a^x f(t)\,dt,$$

where $x \geq a$. The area function gives the net area of the region bounded by the graph of f and the t-axis on the interval $[a, x]$.

Chapter 5.3 Fundamental Theorem of Calculus

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THEOREM 5.3 (PART 1) Fundamental Theorem of Calculus
If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)\,dt, \quad \text{for} \quad a \le x \le b,$$

is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

which means that the area function of f is an antiderivative of f on $[a, b]$.

more practical

THEOREM 5.3 (PART 2) Fundamental Theorem of Calculus
If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a). = F(x)\Big|_a^b$$

Eval. def. integral of f:
Step1: Find any antiderivative of f, F
Step2: Compute $F(b)-F(a)$

It is customary and convenient to denote the difference $F(b) - F(a)$ by $F(x)\big|_a^b$. Using this shorthand, Part 2 of the Fundamental Theorem is summarized in Figure 5.43.

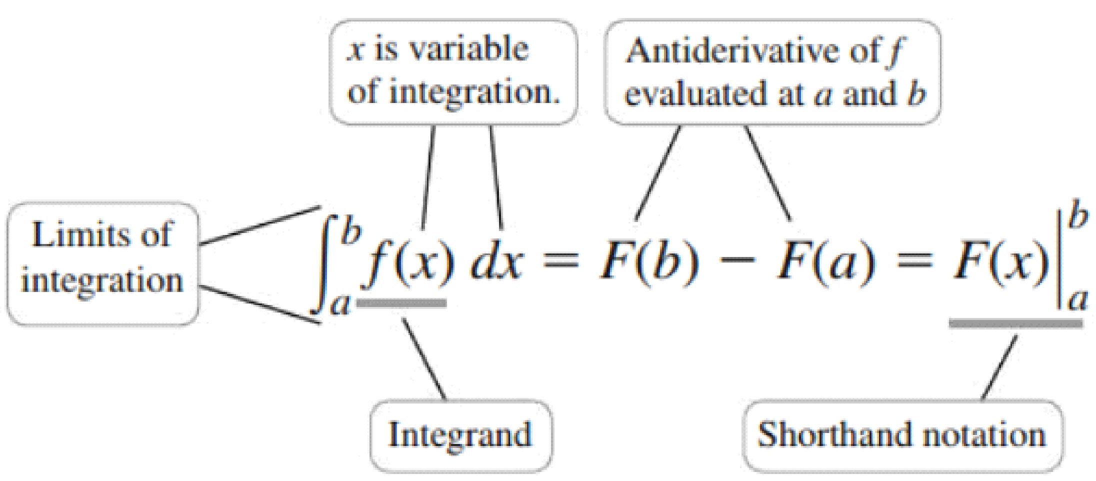

Figure 5.43

Steps
1) $F(t)$
2) $F(2)-F(1)$

Definite integrals *Evaluate the following integrals using the Fundamental Theorem of Calculus.*

$$\int_1^2 \underbrace{\frac{3}{t}}_{f(t)}\,dt$$

Hint: Think backwards to determine the antiderivative of 3/t

$$[\ln(t)]' = \frac{1}{t}, \quad t > 0$$

$$f(t) = \frac{3}{t}$$

$F(t)$ is an antider. of $f(t)$

1) $F(t) = 3\cdot\ln|t| + C$
$= 3\cdot\ln(t)$, $\underbrace{1 < t < 2}_{\text{pos.}}$

2) $3\cdot\ln 2 - 3\cdot\cancel{\ln 1}^{\,0}$
$3\cdot\ln 2 = \ln 2^3 = \ln 8$

both are OK.

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The Inverse Relationship Between Differentiation and Integration It is worth pausing to observe that the two parts of the Fundamental Theorem express the inverse relationship between differentiation and integration. Part 1 of the Fundamental Theorem says

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

or the derivative of the integral of f is f itself.

Noting that f is an antiderivative of f', Part 2 of the Fundamental Theorem says

$$\int_a^b f'(x)\,dx = f(b) - f(a),$$

or the definite integral of the derivative of f is given in terms of f evaluated at two points. In other words, the integral "undoes" the derivative.

EXAMPLE 3 **Evaluating definite integrals** Evaluate the following definite integrals using the Fundamental Theorem of Calculus, Part 2. Interpret each result geometrically.

a. $\int_0^{10} (60x - 6x^2)\,dx$ **b.** $\int_0^{2\pi} 3\sin x\,dx$ **c.** $\int_{1/16}^{1/4} \frac{\sqrt{t} - 1}{t}\,dt$

① Find antider. $F(x)$

② $F(b) - F(a)$

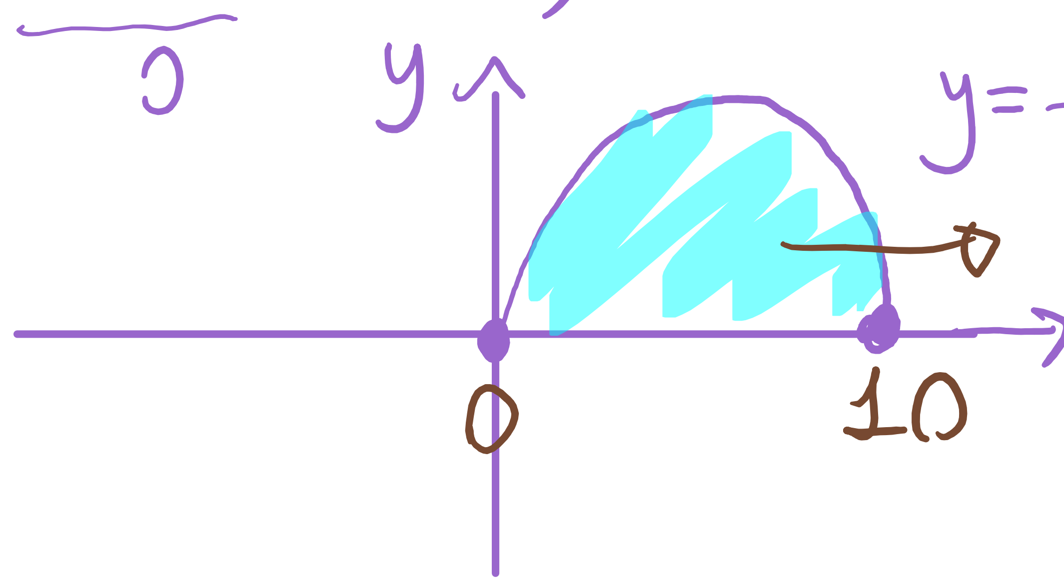
a) $$\int_0^{10} \underbrace{(60x - 6x^2)}_{f(x)}\,dx = \underbrace{\left(60\cdot\frac{x^2}{2} - 6\cdot\frac{x^3}{3}\right)}_{F(x)}\Bigg|_0^{10} = \left(30x^2 - 2x^3\right)\Bigg|_0^{10}$$

$$= \underbrace{\left(30\cdot 10^2 - 2\cdot 10^3\right)}_{F(10)} - \underbrace{\left(30\cdot 0^2 - 2\cdot 0^3\right)}_{F(0)}$$

$$= \left(30\cdot 100 - 2\cdot 1000\right) - 0 = (3000 - 2000) = 1000$$

$f(x) = \underbrace{60x - 6x^2}_{} = \underbrace{6x}_{0}\underbrace{(10 - x)}_{0} = 0$; zeros of $f(x)$: $x = 0, 10$

$= -6x^2 + 60$



$y = f(x) = 60x - 6x^2$

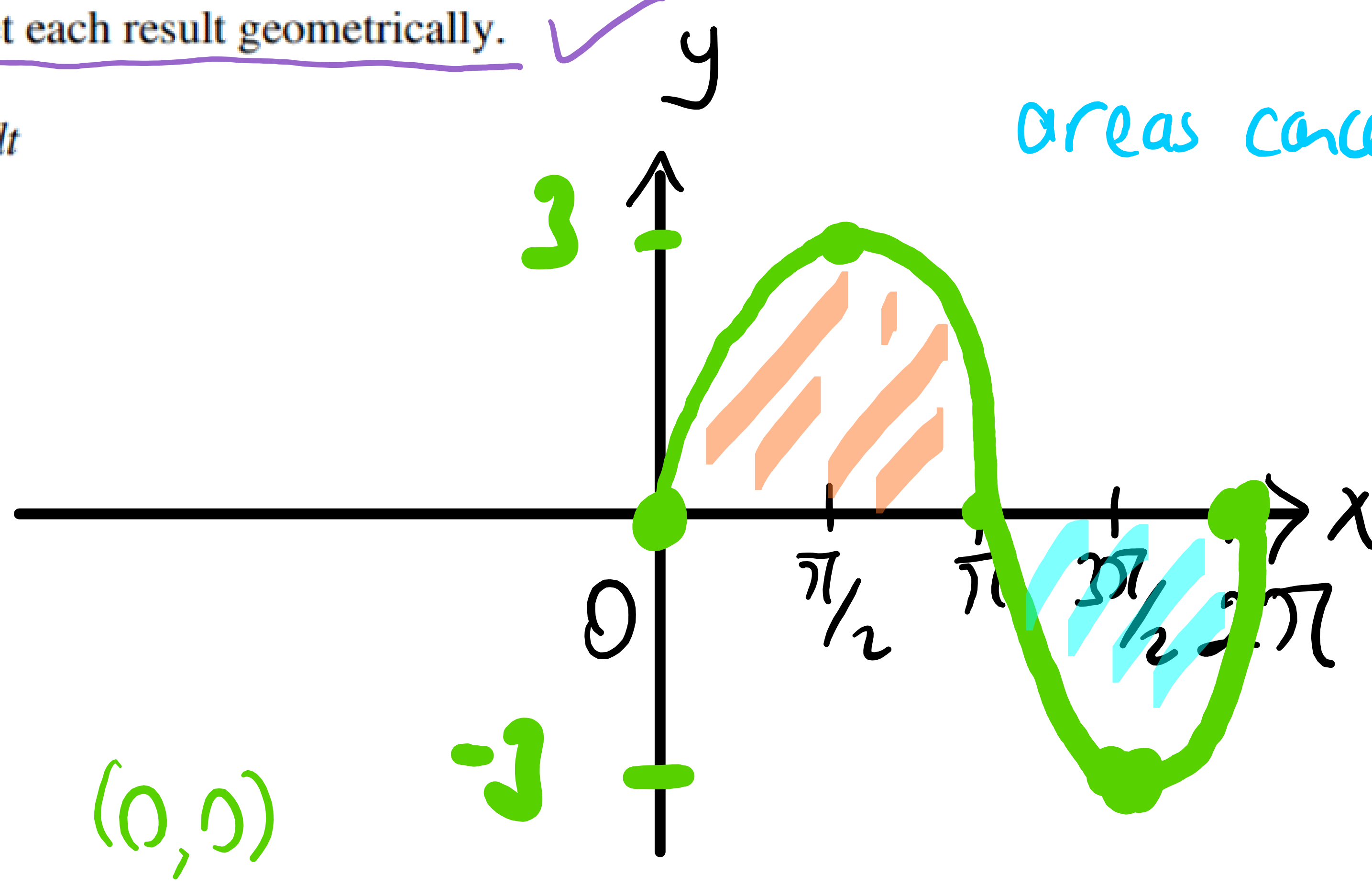
$\text{Area} = \int_0^{10} (60x - 6x^2)\,dx$

$[0, 10]$

EXAMPLE 3 **Evaluating definite integrals** Evaluate the following definite integrals using the Fundamental Theorem of Calculus, Part 2. Interpret each result geometrically.

a. $\int_0^{10} (60x - 6x^2)\,dx$ **b.** $\int_0^{2\pi} 3\sin x\,dx$ **c.** $\int_{1/16}^{1/4} \frac{\sqrt{t} - 1}{t}\,dt$

areas cancel out



b) $\int_0^{2\pi} \underbrace{3 \cdot \sin x}_{f(x)} \cdot dx = 0$

$x=0 \quad f(0) = 3 \cdot \sin 0 = 0 \quad (0,0)$

$x=\pi/2 \quad f(\pi/2) = 3 \cdot \sin\frac{\pi}{2} = 3 \quad (\pi/2, 3)$

$x=\pi \quad f(\pi) = 0 \quad (\pi, 0)$

$x=\frac{3\pi}{2} \quad f\left(\frac{3\pi}{2}\right) = 3 \cdot \sin\left(\frac{3\pi}{2}\right) = -3 \quad (3\pi/2, -3)$

$x=2\pi \quad f(2\pi) = 0 \quad (2\pi, 0)$

$(\cos x)' = -\sin x$

$(-\cos x)' = \sin x$

$\int_0^{2\pi} 3 \cdot \sin x \cdot dx = -3 \cdot \cos x \Big|_0^{2\pi}$

$= -3(\cos 2\pi - \cos 0) = -3(1 - 1) = 0$

Recall: $(\ln(t))' = \frac{1}{t}, t>0$

EXAMPLE 3 **Evaluating definite integrals** Evaluate the following definite integrals using the Fundamental Theorem of Calculus, Part 2. Interpret each result geometrically.

a. $\int_0^{10} (60x - 6x^2)\,dx$ **b.** $\int_0^{2\pi} 3\sin x\,dx$ **c.** $\int_{1/16}^{1/4} \frac{\sqrt{t}-1}{t}\,dt$

c) $f(t) = \frac{\sqrt{t}-1}{t} = \frac{t^{1/2}-1}{t} = \frac{t^{1/2}}{t^1} - \frac{1}{t} = t^{-1/2} - \frac{1}{t}$ *

step 1) $F(t) = \frac{t^{1/2}}{1/2} - \ln|t| + C$

step 2) $F\left(\frac{1}{4}\right) - F\left(\frac{1}{16}\right)$

↳ C's will be cancelled out

$$\int_{1/16}^{1/4} \frac{\sqrt{t}-1}{t}\,dt = \left(2t^{1/2} - \ln t\right)\Big|_{1/16}^{1/4}$$

$$= \left(2\left(\frac{1}{4}\right)^{1/2} - \ln\left(\frac{1}{4}\right)\right) - \left(2\cdot\left(\frac{1}{16}\right)^{1/2} - \ln\left(\frac{1}{16}\right)\right)$$

$$= 2\cdot\frac{1}{2} - \ln(4)^{-1} - \left(2\cdot\frac{1}{4} - \ln(16)^{-1}\right)$$

$$= 1 + \ln 4 - \frac{1}{2} + (-\ln 16) = \frac{1}{2} + \ln\left(\frac{4}{16}\right)$$

$$= \frac{1}{2} + \ln\left(\frac{1}{4}\right)$$

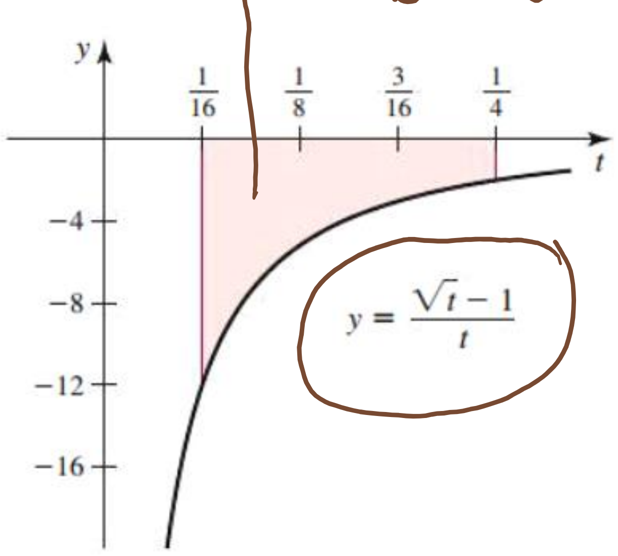
$$= \frac{1}{2} + \ln(4)^{-1}$$

$$= \frac{1}{2} - \ln 4$$

$$\approx \boxed{-0.8863}$$

Recall: $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

0.8863



You try it! / Poll Q

Definite integrals *Evaluate the following integrals using the Fundamental Theorem of Calculus. Sketch the graph of the integrand and shade the region whose net area you have found.*

$\int_{-2}^{3}(x^2 - x - 6)\,dx$

$(x-3)(x+2)$

$x=3,-2$

$\cup$

$$= \left(\frac{x^3}{3} - \frac{x^2}{2} - 6x\right)\Bigg|_{-2}^{3}$$

$$= \left(\frac{3^3}{3} - \frac{3^2}{2} - 6\cdot 3\right) - \left(\frac{(-2)^3}{3} - \frac{(-2)^2}{2} - 6(-2)\right)$$

$$= \left(9 - \frac{9}{2} - 18\right) - \left(\frac{-8}{3} - 2 + 12\right)$$

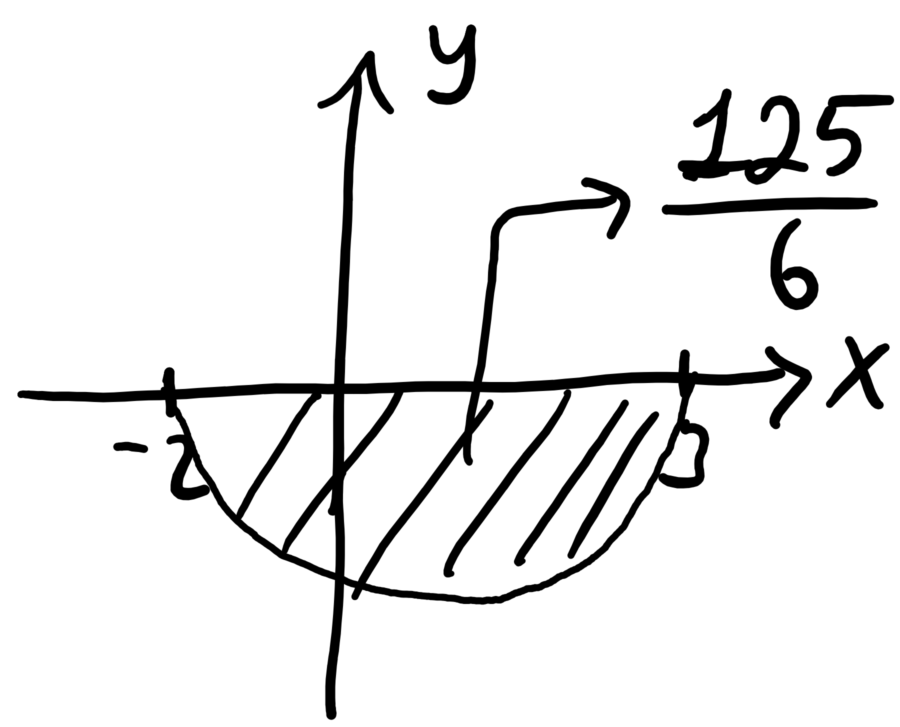
$$= \left(-9 - \frac{9}{2}\right) - \left(\frac{-8}{3} + 10\right) = -9 - \frac{9}{\underset{(3)}{2}} + \frac{8}{\underset{(2)}{3}} - 10$$

$$= -19 + \left(\frac{-27+16}{6}\right) = -\frac{19}{\underset{(6)}{1}} + \left(\frac{-11}{6}\right) = \frac{-114-11}{6} = \frac{-125}{6}$$

Important

$-2^2 \neq (-2)^2$

$-4 \neq 4$



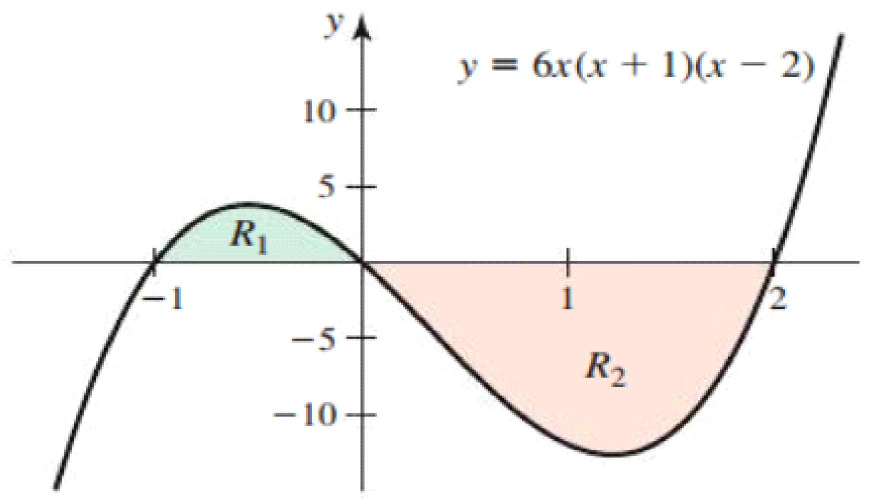


Figure 5.48

EXAMPLE 4 **Net areas and definite integrals** The graph of $f(x) = 6x(x+1)(x-2)$ is shown in Figure 5.48. The region R_1 is bounded by the curve and the x-axis on the interval $[-1, 0]$, and R_2 is bounded by the curve and the x-axis on the interval $[0, 2]$.

a. Find the *net area* of the region between the curve and the x-axis on $[-1, 2]$. → $R_1 - R_2$

b. Find the *area* of the region between the curve and the x-axis on $[-1, 2]$.

↳ Total Area $= R_1 + R_2$

a) $$R_1 - R_2 = \int_{-1}^{2} \underbrace{6x(x+1)(x-2)}_{f(x)}\,dx = \int_{-1}^{2} 6x(x^2-2x+x-2)\,dx$$

$$= \int_{-1}^{2} 6x(x^2-x-2)\,dx = \int_{-1}^{2} \underbrace{(6x^3-6x^2-12x)}_{f(x)}\,dx = \left(\frac{6x^4}{4} - \frac{6x^3}{3} - \frac{12x^2}{2}\right)\Bigg|_{-1}^{2}$$

1) $F(x)$

2) $F(2) - F(-1)$

$$= \left(\frac{3x^4}{2} - 2x^3 - 6x^2\right)\Bigg|_{-1}^{2} = \underbrace{\left(\frac{3}{2}(2)^4 - 2(2)^3 - 6\cdot 2^2\right)}_{F(2)} - \underbrace{\left(\frac{3}{2}\cdot(-1)^4 - 2(-1)^3 - 6(-1)^2\right)}_{F(-1)}$$

$$= -27/2$$

b) $R_1 + R_2 = \int_{-1}^{2} |f(x)|\,dx$

$$R_1 = \int_{-1}^{0} (6x^3 - 6x^2 - 12x)\,dx = \left(\frac{3}{2}x^4 - 2x^3 - 6x^2\right)\Big|_{-1}^{0} = 0 - \left(\frac{3}{2}(-1)^4 - 2(-1)^3 - 6(-1)^2\right)$$

$$= 0 - \left(\frac{3}{2} + 2 - 6\right) = \frac{5}{2}$$

$$R_2 = \int_{0}^{2} (6x^3 - 6x^2 - 12x)\,dx = \left(\frac{3}{2}x^4 - 2x^3 - 6x^2\right)\Big|_{0}^{2} = \left(\frac{3}{2}\cdot 2^4 - 2\cdot 2^3 - 6\cdot 2^2\right) - 0$$

$$= \left(\frac{3}{\cancel{2}}\cancel{16}^{\,8} - 16 - 24\right) = -16$$

The area of R_2 is $-(-16) = 16$

The total (combined) area of R_1 and R_2 is:

$$\frac{5}{2} + 16 = \frac{5+32}{2} = \boxed{\frac{37}{2}}$$

EXAMPLE 5 **Derivatives of integrals** Use Part 1 of the Fundamental Theorem to simplify the following expressions.

a. $\frac{d}{dx}\int_1^x \sin^2 t\,dt$ **b.** $\frac{d}{dx}\int_x^5 \sqrt{t^2+1}\,dt$ **c.** $\frac{d}{dx}\int_0^{x^2} \cos t^2\,dt$

recitation

a) $\frac{d}{dx}\int_1^x \sin^2 t \cdot dt = \sin^2 x$ $t \to$ dummy var.

x

b) $\frac{d}{dx}\int_x^5 \sqrt{t^2+1} \cdot dt = -\frac{d}{dx}\int_5^x \sqrt{t^2+1} \cdot dt$

$= \boxed{-\sqrt{x^2+1}}$

Chapter 5.3 Fundamental Theorem of Calculus

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a - C, b - B, c - D, d - A

You try it! / Poll Q (Please annotate as a-A etc.)

Matching functions with area functions Match the functions f, whose graphs are given in a–d, with the area functions $A(x) = \int_0^x f(t)\,dt$, whose graphs are given in A–D.