Warm up - 9/21

Sunday, September 20, 2020 10:02 PM

Q: Did you review the midterm1-part2 notes sent on Friday?

Two Special Limits

Our principal goal is to determine derivative formulas for sin x and cos x. To do this, we use two special limits.

Template

THEOREM 3.10 Trigonometric Limits

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$

Evaluate the following limit. **Hint**: Use the Special trigonometric limits as seen above.

$$\lim_{x\to -3} \frac{\sin(x+3)}{x^2+8x+15} = \lim_{x\to -3}\left(\frac{\sin(x+3)}{(x+3)\cdot(x+5)}\right) = \lim_{x\to -3}\left(\frac{\sin(x+3)}{(x+3)}\cdot\frac{1}{(x+5)}\right)$$

5 3

$$= \underbrace{\lim_{x\to -3}\frac{\sin(x+3)}{(x+3)}}_{1}\cdot\underbrace{\lim_{x\to -3}\frac{1}{(x+5)}}_{DSP}$$

$$= 1\cdot\frac{1}{2} = \frac{1}{2}$$

9/21 2.4 Infinite Limits - continued

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0 → hole

| → V.A.

Finding vertical asymptotes *Find all vertical asymptotes* $x = a$ *of the following functions. For each value of a, determine* $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$, *and* $\lim_{x\to a} f(x)$.

$$f(x) = \frac{x^2 - 9x + 14}{x^2 - 5x + 6} = \frac{(x-7)\cancel{(x-2)}}{(x-3)\cancel{(x-2)}}, \quad x \neq 2$$

($f(2)$ is undefined, hole at $x=2$)

$x=2$ is NOT a V.A.

simplified

$$f(x) = \frac{x-7}{x-3}$$

To determine V.A: $x-3=0 \Rightarrow \boxed{x=3}$

DSP $3-7=-4$ (neg. non-zero #)

$$\lim_{x\to 3^-}\left(\frac{x-7}{x-3}\right) = \text{"}\frac{\text{neg. non-zero \#}}{\text{neg. \# app. 0}}\text{"} = +\infty$$

neg. # app. 0

DSP $3-7=-4$ (neg. non-zero #)

$$\lim_{x\to 3^+}\left(\frac{x-7}{x-3}\right) = \text{"}\frac{\text{neg. non-zero \#}}{\text{pos. \# app. 0}}\text{"} = -\infty$$

pos. # app. 0

$$\underbrace{\lim_{x\to 3}\left(\frac{x-7}{x-3}\right)}_{\text{DNE}} \quad \Big\} \quad \overbrace{\lim_{x\to 3^-}\left(\frac{x-7}{x-3}\right)}^{LL} \neq \overbrace{\lim_{x\to 3^+}\left(\frac{x-7}{x-3}\right)}^{RL}$$

$$\infty \neq -\infty$$

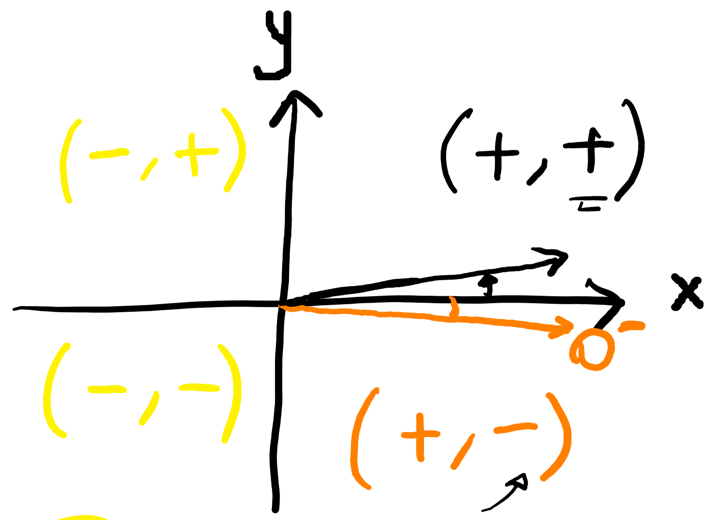
∴ Sign of the infinity matters!

EXAMPLE 6 **Limits of trigonometric functions** Analyze the following limits.

a. $\lim_{\theta \to 0^+} \cot \theta$ **b.** $\lim_{\theta \to 0^-} \cot \theta$

Recall: $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Unit Circle



$(x, y) \to (\cos\theta, \sin\theta)$

$\boxed{\sin 0 = 0}$

a.) $$\lim_{\theta \to 0^+} \left(\frac{\cos\theta}{\sin\theta}\right) \overset{\text{"DSP"}}{=} \text{"}\frac{\cos 0}{\sin 0^+}\text{"} = \text{"}\frac{1}{\text{pos. \# app. 0}}\text{"} = \text{"}\frac{\text{pos. non-zero \#}}{\text{pos. \# app. 0}}\text{"} = \infty$$

b.) $$\lim_{\theta \to 0^-} \left(\frac{\cos\theta}{\sin\theta}\right) \overset{\text{"DSP"}}{=} \text{"}\frac{\cos 0}{\sin 0^-}\text{"} = \text{"}\frac{1}{\text{neg. \# app. 0}}\text{"}$$

$$= \text{"}\frac{\text{pos. non-zero \#}}{\text{neg. \# app. 0}}\text{"} = -\infty$$

2.5 Limits at Infinity

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Limits at infinity occur when the independent variable becomes large in magnitude (such as $x \to \pm\infty$)

Limits at infinity determine the END BEHAVIOR of a function.

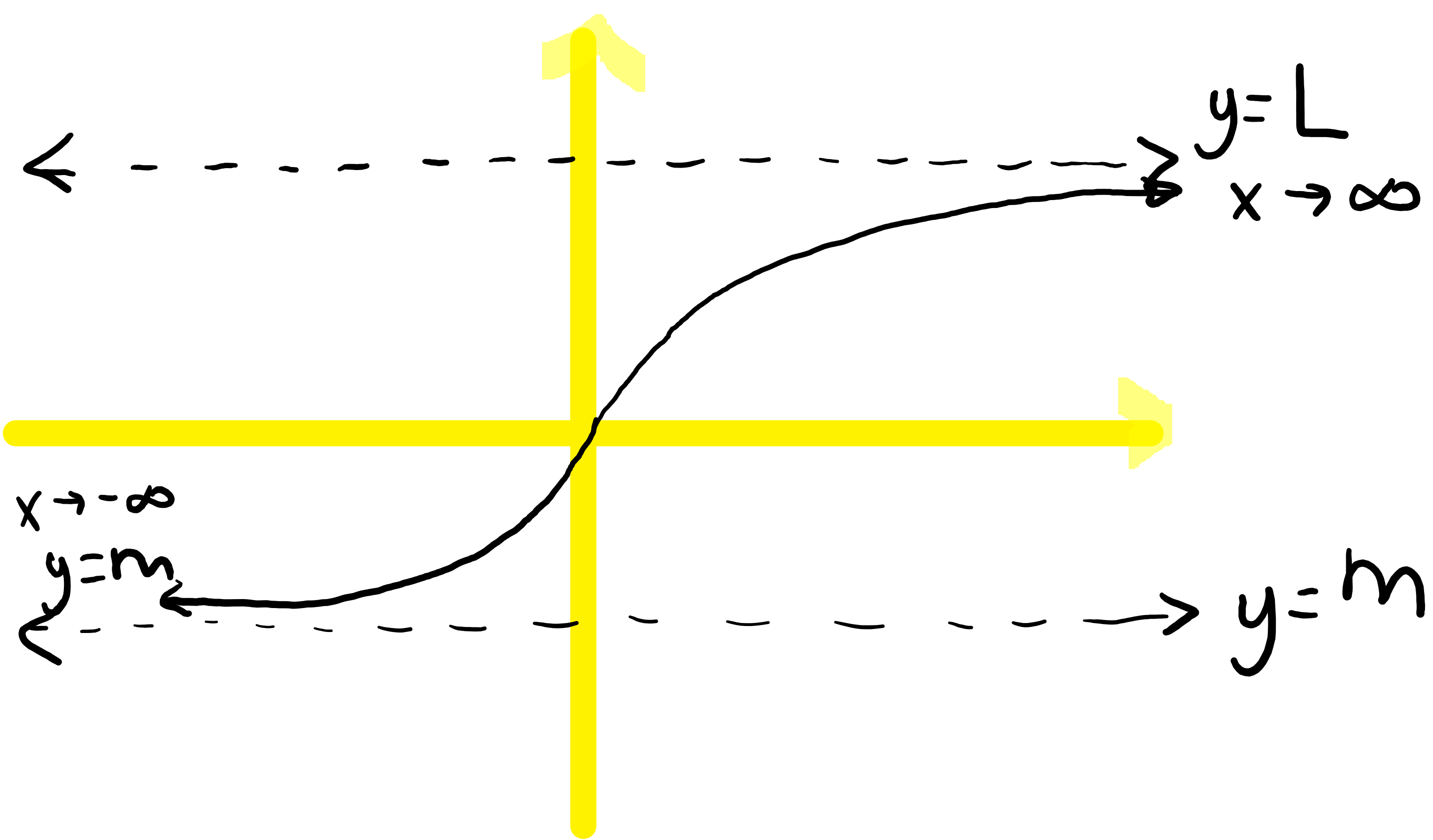
DEFINITION Limits at Infinity and Horizontal Asymptotes

If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write

$$\lim_{x\to\infty} f(x) = L.$$

We say the limit of $f(x)$ as x approaches infinity is L. In this case, the line $y = L$ is a **horizontal asymptote** of f (Figure 2.31). The limit at negative infinity, $\lim_{x\to-\infty} f(x) = M$, is defined analogously. When this limit exists, $y = M$ is a horizontal asymptote.

Graph of $f(x)$



EXAMPLE 1 **Limits at infinity** Evaluate the following limits.

a. $\lim_{x\to-\infty}\left(2+\frac{10}{x^2}\right)$ **b.** $\lim_{x\to\infty}\left(5+\frac{\sin x}{\sqrt{x}}\right)$

a. $\lim_{x\to-\infty}\left(2+\underbrace{\frac{10}{x^2}}\right) = \lim_{x\to-\infty}(2) + \lim_{x\to-\infty}\left(\frac{10}{x^2}\right)$

2

0

pos. # 10

pos. $(-\infty)^2$

pos. very large #

$= 2+0=2$

b. $\lim_{x\to\infty}\left(5+\frac{\sin x}{\sqrt{x}}\right) = \underbrace{\lim_{x\to\infty}(5)} + \underbrace{\lim_{x\to\infty}\left(\frac{\sin(x)}{\sqrt{x}}\right)}$

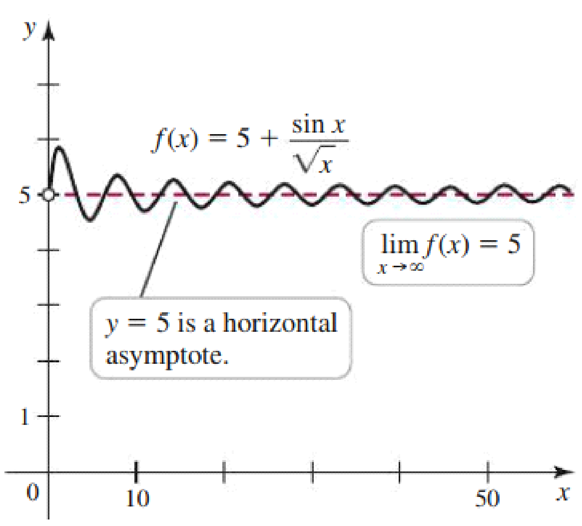
$x\to\infty$ $\quad \frac{-1}{\sqrt{x}} \le \frac{\sin(x)}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$

" $\frac{-1}{\text{very big \#}}$ " $\quad$ " $\frac{1}{\text{very big \#}}$ "

$0 \le \frac{\sin(x)}{\sqrt{x}} \le 0$

$$\lim_{x\to\infty}\frac{\sin(x)}{\sqrt{x}} = 0$$

$= 5+0=5$



DEFINITION Infinite Limits at Infinity

If $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, then we write

$$\lim_{x\to\infty} f(x) = \infty.$$

The limits $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$, and $\lim_{x\to-\infty} f(x) = -\infty$ are defined similarly.

THEOREM 2.6 Limits at Infinity of Powers and Polynomials

Let n be a positive integer and let p be the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, where $a_n \neq 0$.

1. $\lim_{x\to\pm\infty} x^n = \infty$ when n is even.
2. $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to-\infty} x^n = -\infty$ when n is odd.
3. $\lim_{x\to\pm\infty} \frac{1}{x^n} = \lim_{x\to\pm\infty} x^{-n} = 0.$ " $\frac{1}{\text{very big \#}} \to 0$ "
4. $\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_n x^n = \pm\infty$, depending on the degree of the polynomial and the sign of the leading coefficient a_n.

Limits of a polynomial is determined by the highest power of x.

Recall: Identify leading term correctly!

$r(x) = -10 + 6x^3 + \underbrace{5x^4}_{\text{L.T.}}$

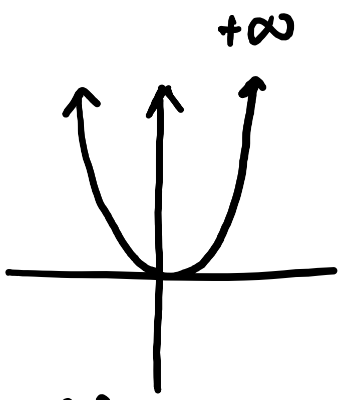
$5x^4 \to$ leading term

$r(x)$ in standard form:

or $r(x) = 5x^4 + 6x^3 - 10$

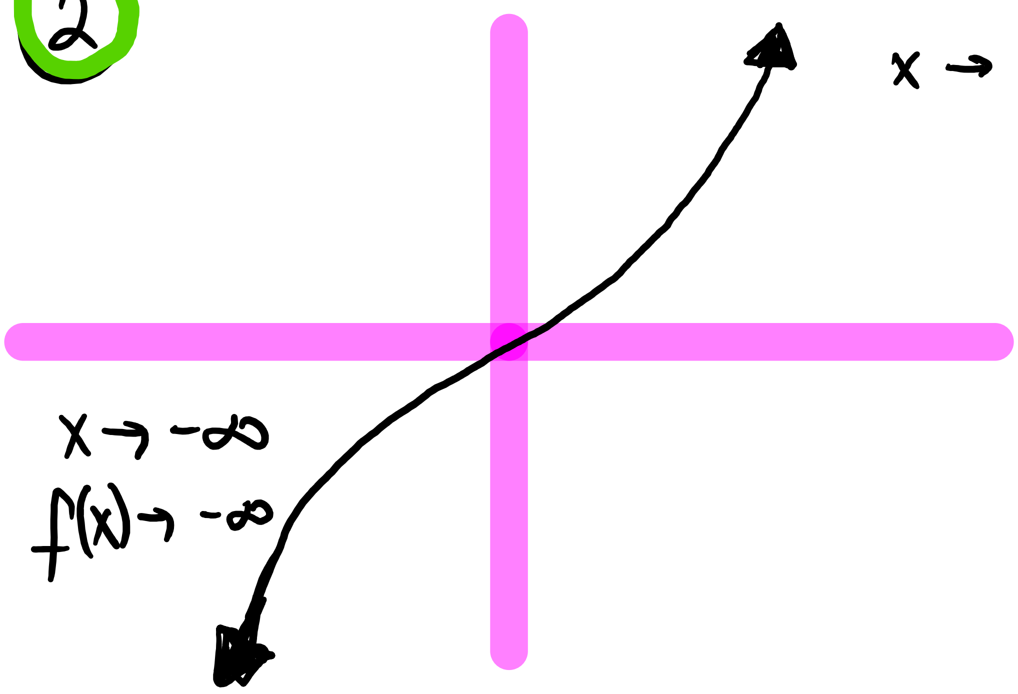
1)

$f(x) = x^2$



$\lim_{x\to\mp\infty} x^{2\to\text{even}} = \infty$

2



$x\to\infty$; $f(x)\to\infty$

$f(x) = x^{3\to\text{odd}}$

$x\to-\infty$

$f(x)\to-\infty$

EXAMPLE 2 **Limits at infinity** Determine the limits as $x \to \pm\infty$ of the following functions.

a. $p(x) = 3x^4 - 6x^2 + x - 10$

b. $q(x) = -2x^3 + 3x^2 - 12$

a. $$\lim_{x \to \infty} \left(\underline{\underline{3x^4}} - 6x^2 + x - 10 \right) = \infty$$

$$\lim_{x \to -\infty} \left(3x^4 \cancel{- 6x^2 + x - 10} \right) = \lim_{x \to -\infty} \left(\underbrace{3x^4}_{\to \infty} \right) = +\infty$$

EXAMPLE 2 **Limits at infinity** Determine the limits as $x \to \pm\infty$ of the following functions.

a. $p(x) = 3x^4 - 6x^2 + x - 10$

b. $q(x) = -2x^3 + 3x^2 - 12$

$$\text{b. } \lim_{x\to\infty}\left(-2x^3+3x^2-12\right) = \lim_{x\to\infty}\left(\underbrace{-2}_{\text{neg.}}\underbrace{x^3}_{\to\infty}\right) = -\infty$$

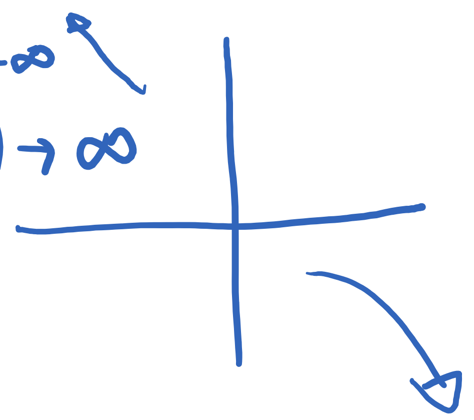
$$\lim_{x\to-\infty}\left(-2x^3+3x^2-12\right) = \lim_{x\to-\infty}\left(\underbrace{-2x^3}_{\to+\infty}\right) = +\infty$$

Recall: leading coefficient test from precalc!

rises to the left

$x \to -\infty$

$f(x) \to \infty$



$x \to \infty$

$f(x) \to -\infty$

falls to the right

An effective strategy for determining the limits of rational functions at infinity is to divide both the numerator and denominator by x^n, where *n* is the highest degree of the polynomial in the denominator.

End Behavior

The behavior of polynomials as $x \to \pm\infty$ is an example of what is often called *end behavior*. Having treated polynomials, we now turn to the end behavior of rational, algebraic, and transcendental functions.

$(x \to \pm\infty)$

EXAMPLE 3 **End behavior of rational functions** Use limits at infinity to determine the end behavior of the following rational functions.

a. $f(x) = \dfrac{3x + 2}{x^2 - 1}$ **b.** $g(x) = \dfrac{40x^4 + 4x^2 - 1}{10x^4 + 8x^2 + 1}$ **c.** $h(x) = \dfrac{x^3 - 2x + 1}{2x + 4}$

a. $\lim_{x\to\infty}\left(\dfrac{3x+2}{x^2-1}\right)$

1) Det. the highest degree in the denominator: x^2

2) Div. ALL by x^2

$$\lim_{x\to\infty}\left(\dfrac{\frac{3x}{x^2}+\frac{2}{x^2}}{\frac{x^2}{x^2}-\frac{1}{x^2}}\right) = \lim_{x\to\infty}\left(\dfrac{\frac{3}{x}+\frac{2}{x^2}}{1-\frac{1}{x^2}}\right)$$

DSP

$\frac{3}{x} \to 0$, $\frac{2}{\text{very large pos \#}} \to 0$

DSP

$1 - \frac{1}{\text{very large pos. \#}} \to 0$

$$= \frac{0}{1} = 0$$

$$\lim_{x\to-\infty}\left(\dfrac{\frac{3}{x}+\frac{2}{x^2}}{1-\frac{1}{x^2}}\right) = 0$$

$\boxed{\text{H.A: } y=0}$

End Behavior

The behavior of polynomials as $x \to \pm\infty$ is an example of what is often called *end behavior*. Having treated polynomials, we now turn to the end behavior of rational, algebraic, and transcendental functions.

EXAMPLE 3 **End behavior of rational functions** Use limits at infinity to determine the end behavior of the following rational functions.

a. $f(x) = \dfrac{3x + 2}{x^2 - 1}$ **b.** $g(x) = \dfrac{40x^4 + 4x^2 - 1}{10x^4 + 8x^2 + 1}$ **c.** $h(x) = \dfrac{x^3 - 2x + 1}{2x + 4}$

b. $$\lim_{x\to\infty}\left(\frac{\frac{40x^4}{x^4}+\frac{4x^2}{x^4}-\frac{1}{x^4}}{\frac{10x^4}{x^4}+\frac{8x^2}{x^4}+\frac{1}{x^4}}\right)$$

$x^4 \to$ highest degree in the denom.

div. ALL by x^4

$$\lim_{x\to\infty}\left(\frac{4+\frac{4}{x^2}\to 0-\frac{1}{x^4}\to 0}{1+\frac{8}{x^2}\to 0+\frac{1}{x^4}\to 0}\right) = 4$$

$$\lim_{x\to-\infty}\left(\frac{\frac{40x^4+4x^2-1}{x^4}}{\frac{10x^4+8x^2+1}{x^4}}\right)$$

div. ALL by x^4

$$\lim_{x\to-\infty}\left(\frac{40+\frac{4}{x^2}\to 0-\frac{1}{x^4}\to 0}{10+\frac{8}{x^2}\to 0+\frac{1}{x^4}\to 0}\right) = 4$$

H.A: $y=4$