Warm Up

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73. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

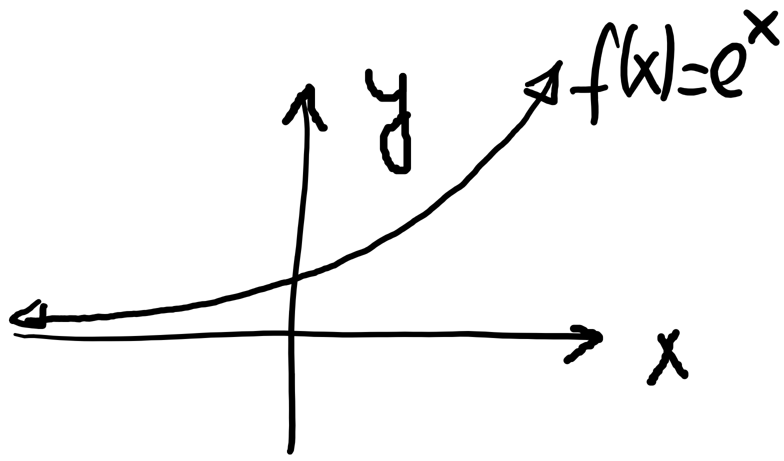
a. $\frac{d}{dx}(10^5) = 5 \cdot 10^4$.

b. The slope of a line tangent to $f(x) = e^x$ is never 0.

a. $\frac{d}{dx}(100,000) = 0$ False

b. $m_{tan} \stackrel{?}{=} f'(x) = e^x$

True



Q2
Given $h(x) = \sqrt{x}\left(\sqrt{x} - x^{3/2}\right)$, evaluate $h'(x)$ $\left(\frac{dh}{dx}\right)$

$$h(x) = x^{1/2}\left(x^{1/2} - x^{3/2}\right) = x^1 - x^2$$

$$h'(x) = \left(x^1 - x^2\right)' = 1 - 2 \cdot x^1$$

$x^0 = 1$

↳ derivative f.

$h'(2) \Rightarrow m_{tan}\Big|_{x=2}$

Higher Order Derivatives

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n^{th} $f^{(n)}$ der. of f vs f^n $[f(x)]^n$

DEFINITION Higher-Order Derivatives

Assuming $y = f(x)$ can be differentiated as often as necessary, the **second derivative** of f is

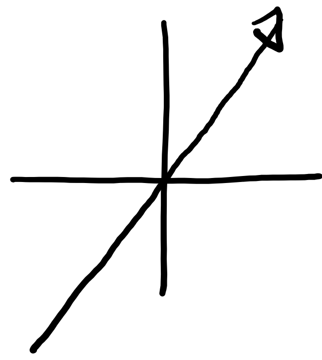
$$f''(x) = \frac{d}{dx}(f'(x)).$$

For integers $n \geq 1$, the ***n*th derivative** of f is

$$f^{(n)}(x) = \frac{d}{dx}(f^{(n-1)}(x)).$$

- Parentheses are placed around n to distinguish a derivative from a power. Therefore, $f^{(n)}$ is the nth derivative of f, and f^n is the function f raised to the nth power. By convention, $f^{(0)}$ is the function f itself.
- The notation $\frac{d^2f}{dx^2}$ comes from $\frac{d}{dx}\left(\frac{df}{dx}\right)$ and is read d 2 f dx squared.

Other common notations for the second derivative of $y = f(x)$ include $\frac{d^2y}{dx^2}$ and $\frac{d^2f}{dx^2}$; the notations $\frac{d^ny}{dx^n}$, $\frac{d^nf}{dx^n}$, and $y^{(n)}$ are used for the nth derivative of f.



$y = 1x$

$y = mx + b$

EXAMPLE 6 Finding higher-order derivatives Find the third derivative of the following functions.

a. $f(x) = 3x^3 - 5x + 12$

b. $y = 3t + 2e^t$

a. $f'(x) = 3\cdot3\cdot x^2 - 5\cdot1 + 0 = 9x^2 - 5$

$f''(x) = \frac{d^2f}{dx^2} = \frac{d}{dx}\left(9x^2 - 5\right) = 9\cdot2\cdot x^1 = 18x$

$f'''(x) = \frac{d^3f}{dx^3} = \frac{d}{dx}\left(18x\right) = \boxed{18}$

b. $\frac{dy}{dt} = \frac{d}{dt}(3t + 2e^t) = 3 + 2\cdot e^t$

$\boxed{\frac{d(e^t)}{dt} = e^t}$

$\frac{d^2y}{dt^2} = (3 + 2e^t)' = 0 + 2e^t$; $\frac{d^3y}{dt^3} = \frac{d}{dt}(2e^t) = \boxed{2e^t}$

Find the derivative of $h(x) = \sqrt{x}\left(\sqrt{x} - x^{3/2}\right)$

$h(x) = x^{1/2}\left(x^{1/2} - x^{3/2}\right) = x^1 - x^2$

$h'(x) = (x - x^2)' = 1 - 2x$

3.4 The Product and Quotient Rules

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DIFFERENTIATION RULES

1. *Constant Rule:* If $f(x) = c$ (c constant), then $f'(x) = 0$.
2. *Power Rule:* If r is a real number, $\dfrac{d}{dx}x^r = rx^{r-1}$
3. *Constant Multiple Rule:* $\dfrac{d}{dx}(c \cdot f(x)) = c \cdot f'(x)$
4. *Sum Rule:* $\dfrac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
5. *Product Rule:* $\dfrac{d}{dx}[f(x)g(x)] \boxed{= f(x)g'(x) + f'(x)g(x)}$ ("take turns taking derivatives")
6. *Quotient Rule:* $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

EXAMPLE 1 **Using the Product Rule** Find and simplify the following derivatives.

a. $\dfrac{d}{dv}(v^2(2\sqrt{v} + 1))$ **b.** $\dfrac{d}{dx}(x^2e^x)$

$f'(v)$ $g(v)$ $f(v)$ $g'(v)$

$$\text{a)}\ \frac{d}{dv}\Big(\underbrace{v^2}_{f(v)}\cdot\underbrace{(2v^{1/2}+1)}_{g(v)}\Big) = 2v\cdot(2v^{1/2}+1) + v^2\left(\cancel{2}\cdot\frac{1}{\cancel{2}}\cdot v^{-1/2}\right)$$

$$= 2v(2v^{1/2}+1) + v^{2-1/2}$$

$$= 4v^{3/2} + 2v + 1v^{3/2} = 5v^{3/2} + 2v$$

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$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)\cdot g(x) - g'(x) f(x)}{(g(x))^2}$$

EXAMPLE 2 **Using the Quotient Rule** Find and simplify the following derivatives.

a. $\frac{d}{dx}\left(\frac{x^2 + 3x + 4}{x^2 - 1}\right)$ **b.** $\frac{d}{dx}(e^{-x})$ $= \frac{d}{dx}\left(\frac{1}{e^x}\right) = \frac{(1)'\cdot e^x - 1\cdot(e^x)'}{(e^x)^2} = \frac{-e^x}{e^x \cdot e^x} = \frac{-1}{e^x} = -e^{-x}$

$$\frac{1}{e^x} = e^{-x}$$

$$-\frac{1}{e^x} = -e^{-x}$$

a. $$\frac{d}{dx}\left(\frac{x^2+3x+4}{x^2-1}\right) = \frac{(x^2+3x+4)'\cdot(x^2-1) - (x^2+3x+4)\cdot(x^2-1)'}{(x^2-1)^2}$$

$$= \frac{(2x+3+0)(x^2-1) - (x^2+3x+4)\cdot(2x)}{(x^2-1)^2}$$

$$= \frac{\cancel{2x^3} - 2x + 3x^2 - 3 - (\cancel{2x^3} + 6x^2 + 8x)}{(x^2-1)^2}$$

$$= \frac{-2x + 3x^2 - 3 - 6x^2 - 8x}{(x^2-1)^2} = \frac{-3x^2 - 10x - 3}{(x^2-1)^2} = -\left(\frac{3x^2+10x+3}{(x^2-1)^2}\right)$$

EXAMPLE 3 **Finding tangent lines** Find an equation of the line tangent to the graph of $f(x) = \dfrac{x^2+1}{x^2-4}$ at the point $(3, 2)$. Plot the curve and tangent line.

$$f'(x)\Big|_{(3,2)}$$

$$m_{tan}\Big|_{x=3} = \lim_{x\to 3} \frac{f(x)-f(3)}{x-3}$$

$$f'(x) = \frac{(x^2+1)'(x^2-4)-(x^2+1)(x^2-4)'}{(x^2-4)^2}$$

$$= \frac{2x(x^2-4)-(x^2+1)\cdot 2x}{(x^2-4)^2} = \frac{\cancel{2x^3}-8x-\cancel{2x^3}-2x}{(x^2-4)^2}$$

$$\boxed{f'(x) = \frac{-10x}{(x^2-4)^2}}$$

Slope of the tan. line at $\underbrace{(3,2)}$ is:

$$m_{tan}\Big|_{x=3} = f'(3) = \frac{-10\cdot 3}{(9-4)^2} = \frac{-30}{25} \overset{\div 5}{=} \frac{-6}{5}$$

$$\boxed{y-2 = \frac{-6}{5}(x-3)}$$

EXAMPLE 4 **Using the Power Rule** Find the following derivatives.

c. $\frac{d}{dz}\left(\sqrt[3]{z}\, e^{z}\right)$ **d.** $\frac{d}{dx}\left(\frac{3x^{5/2}}{2x^2+4}\right)$

$$d)\quad \frac{d}{dx}\left(\frac{3x^{5/2}}{2x^2+4}\right) = \frac{(3x^{5/2})'\cdot(2x^2+4) - (3x^{5/2})\cdot(2x^2+4)'}{(2x^2+4)^2}$$

$$= \frac{3\cdot\frac{5}{2}\cdot x^{3/2}(2x^2+4) - (3x^{5/2})\cdot 4x}{(2x^2+4)^2}$$

$$= \frac{\frac{15}{\cancel{2}}\cdot x^{3/2}\cdot \cancel{2}x^2 + \frac{15}{\cancel{2}}x^{3/2}\cdot \cancel{4}^{\,2} - 12x^{7/2}}{(2x^2+4)^2}$$

$$3x^3+3x^2 = 3x^2(x+1)$$

$$= \frac{15\cdot x^{7/2} + 30x^{3/2} - 12x^{7/2}}{(2x^2+4)^2}$$

$$= \frac{3x^{7/2}+30x^{3/2}}{(2x^2+4)^2}$$

$$= \frac{3x^{3/2}(x^2+10)}{(2x^2+4)^2}$$

3.5 Derivatives of Trigonometric Functions

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USEFUL DERIVATIVES

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\cot x) = -\csc^2 x$
5. $\frac{d}{dx}(\sec x) = \sec x \tan x$
6. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

$$\frac{d}{dx}(e^x) = e^x$$

EXAMPLE 2 **Derivatives involving trigonometric functions** Calculate dy/dx for the following functions.

a. $y = e^x \cdot \cos x$

c. $y = \dfrac{1 + \sin x}{1 - \sin x}$

a. $y = e^x \cdot \cos x \Rightarrow \dfrac{dy}{dx} = (e^x)' \cdot \cos x + e^x \cdot (\cos x)'$

$$= e^x \cdot \cos x + e^x(-\sin x)$$

$$= e^x(\cos x - \sin x)$$

c. $y = \dfrac{1+\sin x}{1-\sin x} \Rightarrow \dfrac{dy}{dx} = \dfrac{(1+\sin x)' \cdot (1-\sin x) - (1+\sin x)(1-\sin x)'}{(1-\sin x)^2}$

$$\frac{dy}{dx} = \frac{\cos x \cdot (1-\sin x) + (1+\sin x) \cdot (+\cos x)}{(1-\sin x)^2}$$

$$= \frac{\cos x - \cancel{\sin x \cdot \cos x} + \cos x + \cancel{\sin x \cdot \cos x}}{(1-\sin x)^2} = \frac{2\cos x}{(1-\sin x)^2}$$

$f'(x)$ m_{tan} $P(x,y)$ } $m_{tan}|_{x=a}$ $\Rightarrow$ eq.

Find an equation of the line tangent to the following curves at the given value of x.

Steps
1) find $f'(x)$
2) find $f'\left(\frac{\pi}{6}\right) \to m_{tan}|_{x=\frac{\pi}{6}}$
3) set up. eq.
use $\left(\frac{\pi}{6}, f\left(\frac{\pi}{6}\right)\right)$

73. $y = 1 + 2\sin x$; $\boxed{x = \frac{\pi}{6}}$

$f(x) = 1 + 2\sin x$

Steps

1) $f'(x) = 2 \cdot \cos x$

2) $f'\left(\frac{\pi}{6}\right) = 2 \cdot \cos\frac{\pi}{6} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3} = m_{tan}|_{x=\frac{\pi}{6}}$

$x = \frac{\pi}{6}$, $f\left(\frac{\pi}{6}\right) = 1 + 2 \cdot \sin\frac{\pi}{6} = 1 + 2 \cdot \frac{1}{2} = 2$ $\left(\frac{\pi}{6}, 2\right)$

$$y - 2 = \sqrt{3} \cdot \left(x - \frac{\pi}{6}\right)$$

$$y - y_1 = m_{tan}(x - x_1)$$

3.9 Derivatives of Logarithmic and Exponential Functions

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$$(\ln(x))' = \frac{x'}{x} = \frac{1}{x}$$

THEOREM 3.15 Derivative of ln x

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \text{ for } x > 0 \qquad \frac{d}{dx}(\ln |x|) = \frac{1}{x}, \text{ for } x \neq 0$$

If u is differentiable at x and $u(x) \neq 0$, then

$$\frac{d}{dx}(\ln |u(x)|) = \frac{u'(x)}{u(x)}.$$

EXAMPLE 1 Derivatives involving ln x Find $\frac{dy}{dx}$ for the following functions.

a. $y = \ln 4x$ **b.** $y = x \ln x$ **c.** $y = \ln |\sec x|$ **d.** $y = \frac{\ln x^2}{x^2}$

a. $y = \ln(4x)$

$$\frac{dy}{dx} = \frac{(4x)'}{4x} = \frac{4}{4x} = \frac{1}{x}$$

b. $y = x \cdot \ln x$

Use product rule!

$$\frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

c. $y = \ln|\sec x|$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\sec x)}{\sec x} = \frac{\cancel{\sec x} \cdot \tan x}{\cancel{\sec x}}$$

$$= \tan x$$

$$[\ln(3x+5)]' = \frac{(3x+5)'}{3x+5}$$

$$= \left(\frac{3}{3x+5}\right)$$

THEOREM 3.16 **Derivative of b^x**

If $b > 0$ and $b \neq 1$, then for all x,

$$\frac{d}{dx}(b^x) = b^x \ln b.$$

$\cancel{(x^{1/12})'}$

↳ base is var.

EXAMPLE 2 **Derivatives with b^x** Find the derivative of the following functions.

a. $f(x) = 3^x$ **b.** $g(t) = 108 \cdot 2^{t/12}$

a. $f(x) = \underbrace{3}_{b}{}^{x}$

$$\frac{d}{dx}\left(3^x\right) = 3^x \cdot \ln 3$$

b. $g(t) = 108 \cdot 2^{t/12}$

$$\frac{d}{dt}\left(108 \cdot \underbrace{2^{t/12}}_{b}\right)_{(\#)} = 108 \cdot 2^{t/12} \cdot \ln 2$$

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$$(\ln(u))' = \frac{u'}{u}$$

EXAMPLE 8 **Logarithmic differentiation** Let $f(x) = \dfrac{(x^2+1)^4 \cdot e^x}{x^2+4}$ and compute $f'(x)$.

$$\ln e^x = x$$

$$\ln(f(x)) = \ln\left(\frac{(x^2+1)^4 \cdot e^x}{(x^2+4)}\right)$$

$$= \ln(x^2+1)^4 + \ln e^x - \ln(x^2+4)$$

$$\frac{d}{dx}\left(\ln(f(x))\right) = \frac{d}{dx}\left(4 \cdot \ln(x^2+1) + x - \ln(x^2+4)\right)$$

$$\cancel{f(x)} \cdot \left(\frac{f'(x)}{f(x)}\right) = \left(4 \cdot \frac{2x}{x^2+1} + 1 - \frac{2x}{x^2+4}\right) \cdot f(x)$$

$$f'(x) = \left(4 \cdot \frac{2x}{x^2+1} + 1 - \frac{2x}{x^2+4}\right) \cdot \frac{(x^2+1)^4 \cdot e^x}{(x^2+4)}$$