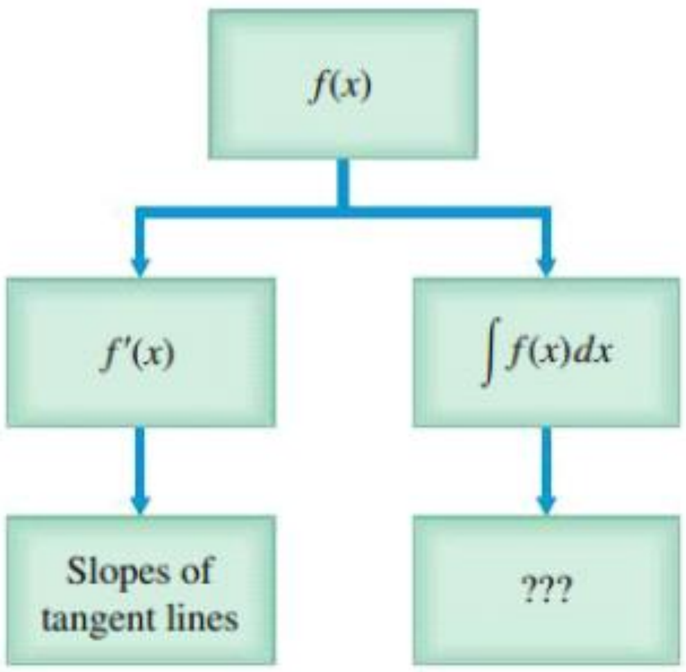
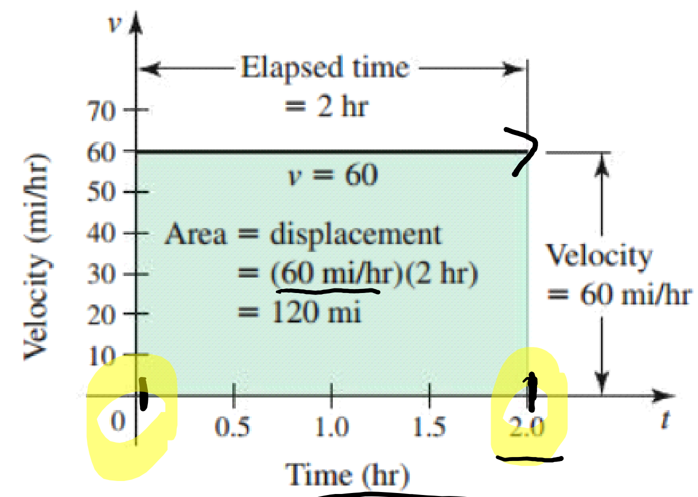
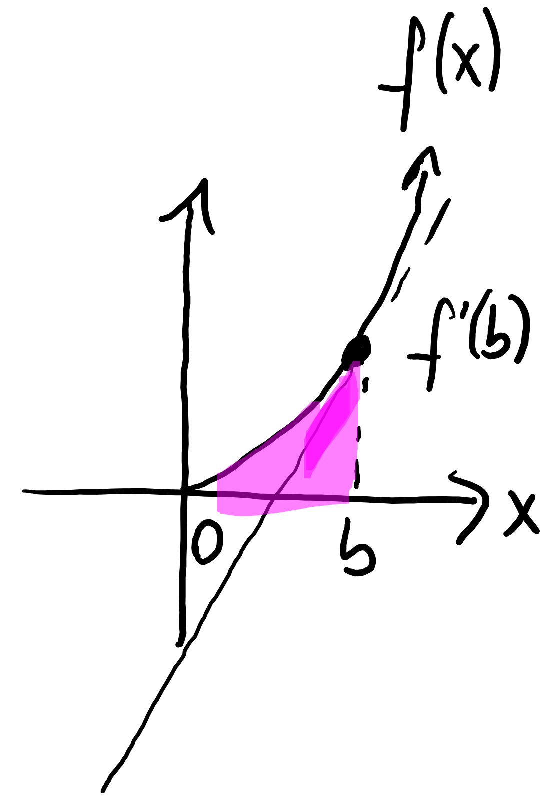
5.1 Approximating Areas under Curves

What is the geometric meaning of the integral?



Motivation: Area = displacement





$d = t \cdot v$

displacement

v = 60 mi/hr

v(t) constant

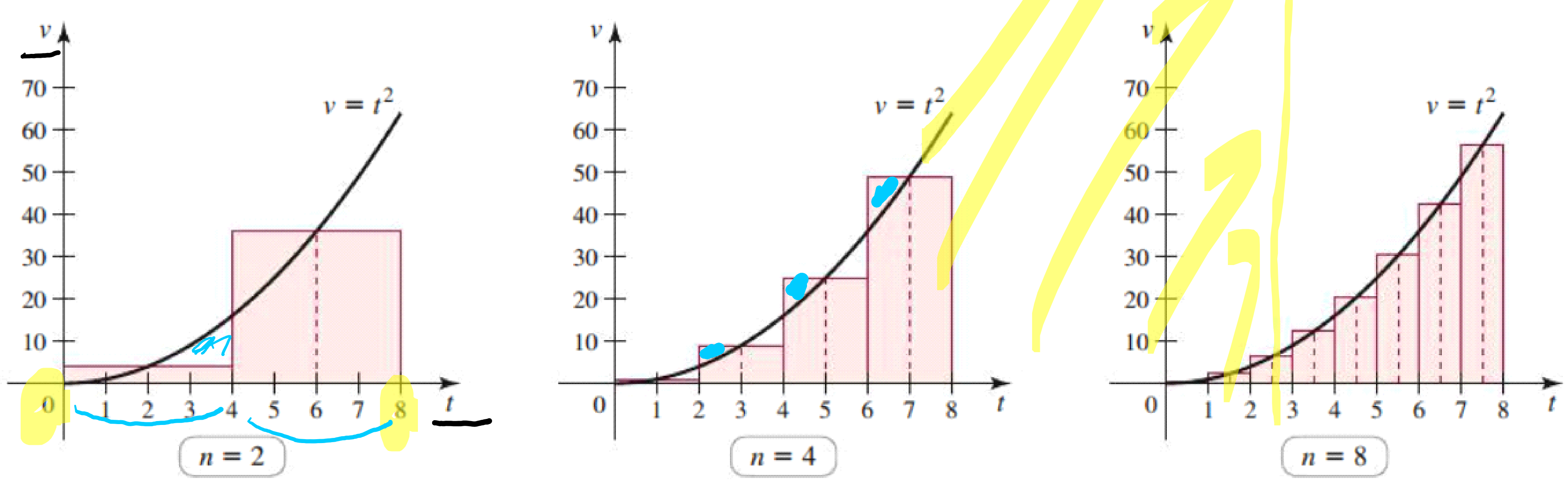
EXAMPLE 1 **Approximating the displacement** Suppose the velocity in m/s of an object moving along a line is given by the function $v = t^2$, where $0 \le t \le 8$. Approximate the displacement of the object by dividing the time interval $[0, 8]$ into n subintervals of equal length. On each subinterval, approximate the velocity with a constant equal to the value of v evaluated at the midpoint of the subinterval.

a. Begin by dividing $[0, 8]$ into $n = 2$ subintervals: $[0, 4]$ and $[4, 8]$.
b. Divide $[0, 8]$ into $n = 4$ subintervals: $[0, 2]$, $[2, 4]$, $[4, 6]$, and $[6, 8]$.
c. Divide $[0, 8]$ into $n = 8$ subintervals of equal length.

rectangles

inc. width of R

inc. # of rect.



(a) (b) (c)

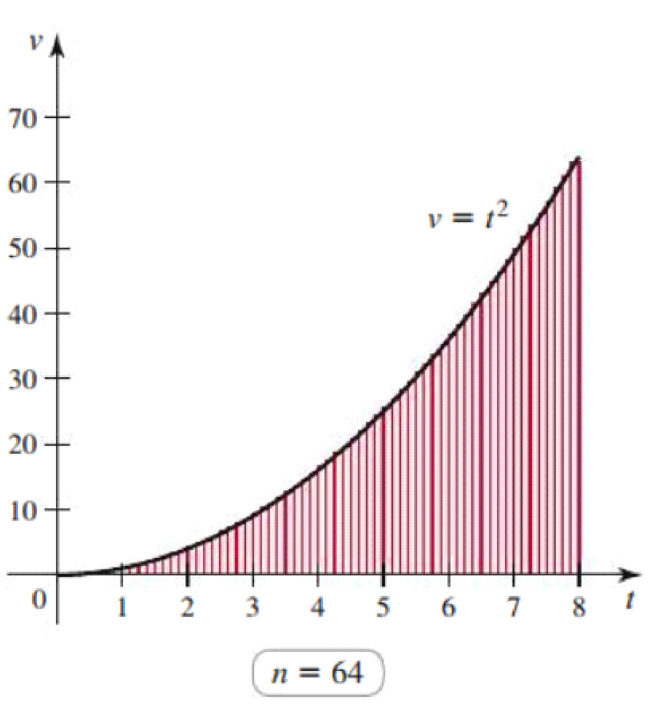


Table 5.1 **Approximations to the area under the velocity curve $v = t^2$ on $[0, 8]$**

Number of subintervals	Length of each subinterval	Approximate displacement (area under curve)
1	8 s	128.0 m
2	4 s	160.0 m
4	2 s	168.0 m
8	1 s	170.0 m
16	0.5 s	170.5 m
32	0.25 s	170.625 m
64	0.125 s	170.65625 m

Chapter 5

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Approximating Areas by using Reimann Sums

Summing the areas of the rectangles in Figure 5.8, we obtain an approximation to the area of R, which is called a **Riemann sum**:

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x.$$

Three notable Riemann sums are the *left*, *right*, and *midpoint Riemann sums*.

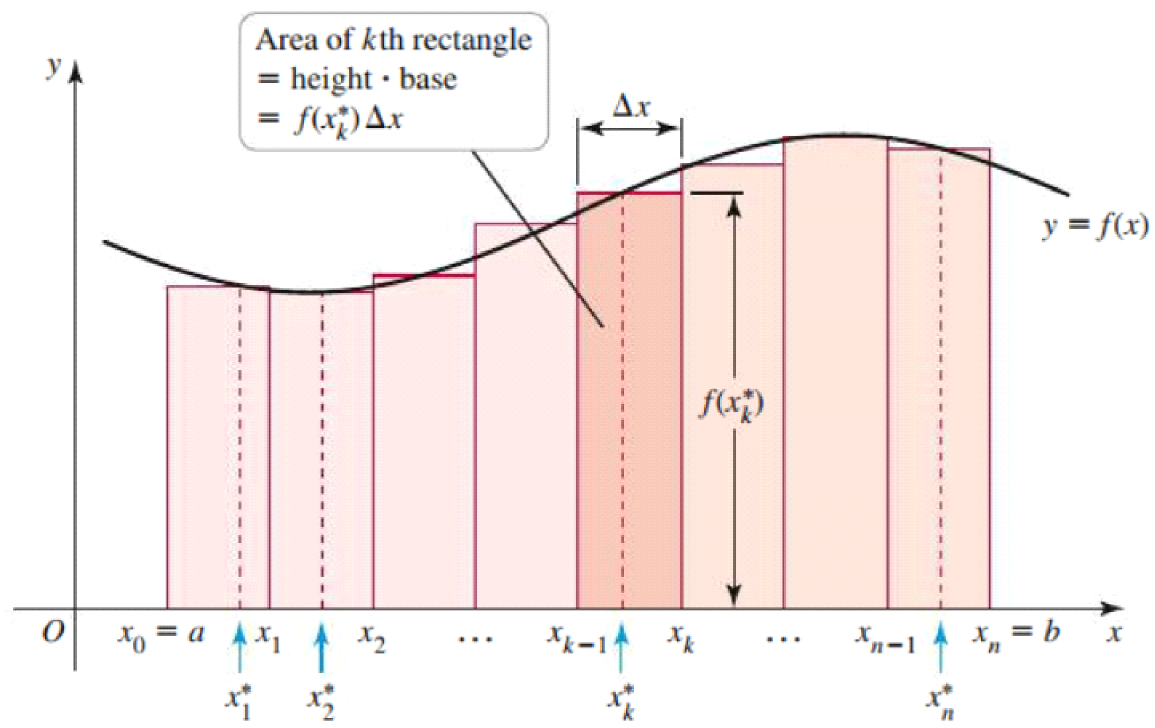
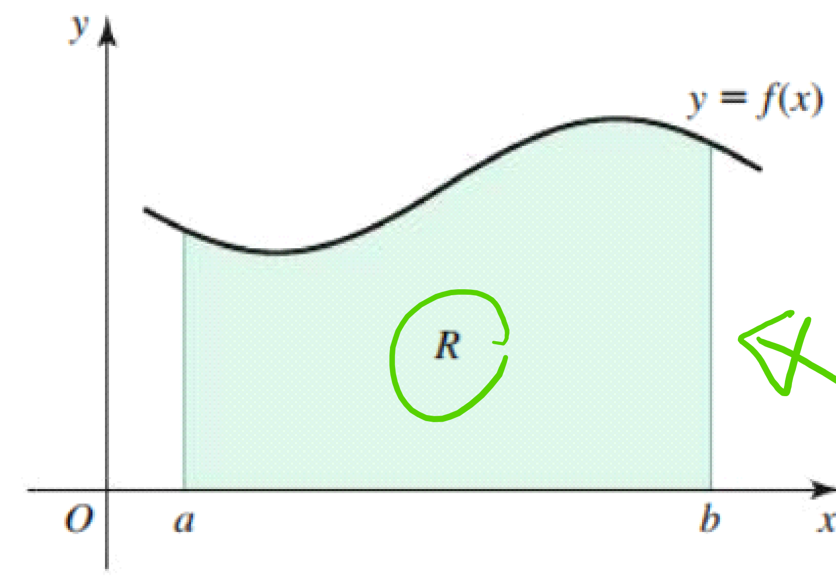


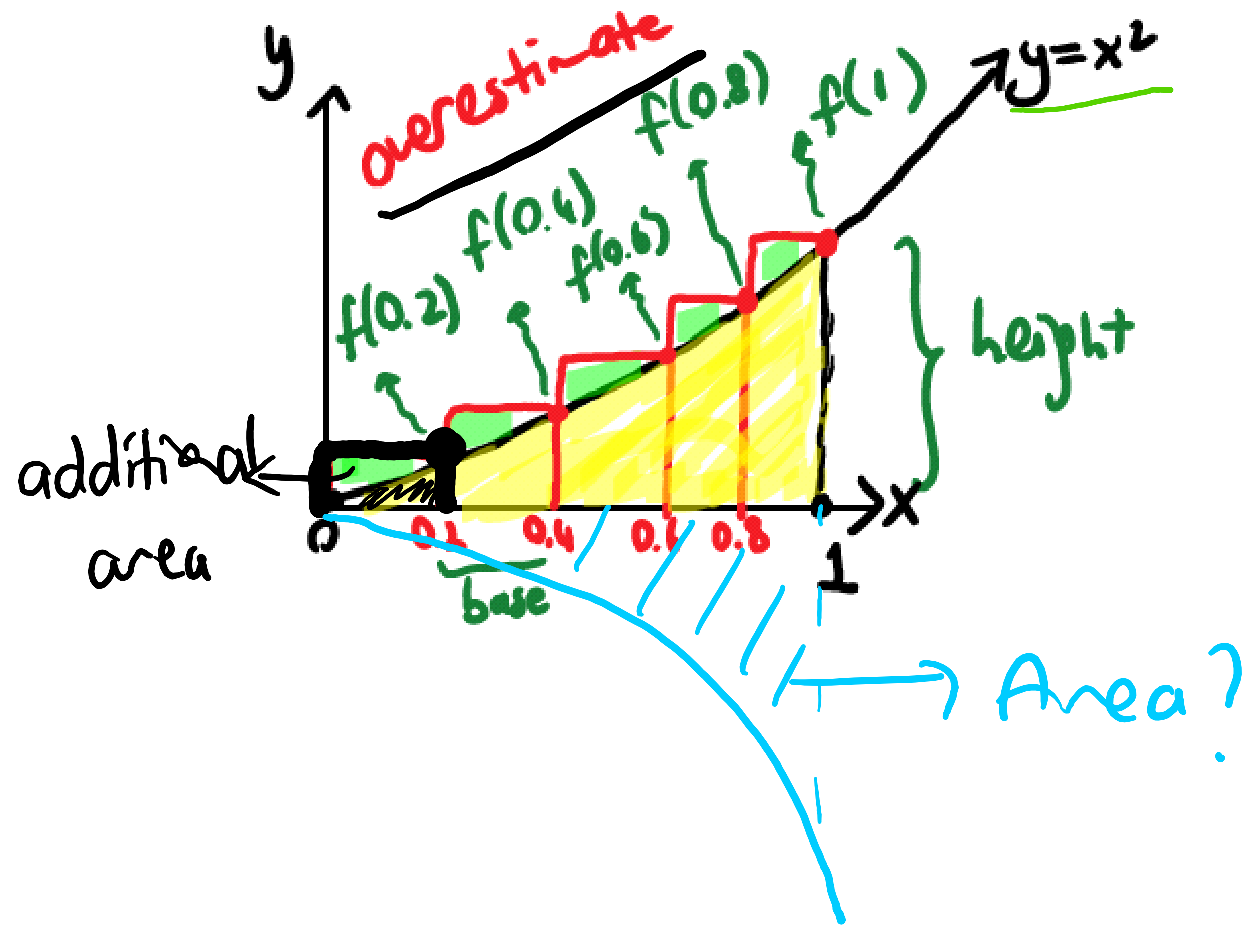
Figure 5.8



➤ The language "the area of the region bounded by the graph of a function" is often abbreviated as "the area under the curve."

bounded → definite $\int_a^b f(x)\cdot dx$ vs. $\int f(x)dx$ indefinite

Exp1) Estimate the area under the graph of $y=f(x)=x^2$ and above the x-axis on $[0,1]$ with $n=5$ rectangles.



5.2 Definite Integrals

We introduced Riemann sums in Section 5.1 as a way to approximate the area of a region bounded by a curve $y = f(x)$ and the x-axis on an interval $[a, b]$. In that discussion, we assumed f to be nonnegative on the interval. Our next task is to discover the geometric meaning of Riemann sums when f is negative on some or all of $[a, b]$. Once this matter is settled, we proceed to the main event of this section, which is to define the *definite integral*. With definite integrals, the approximations given by Riemann sums become exact.

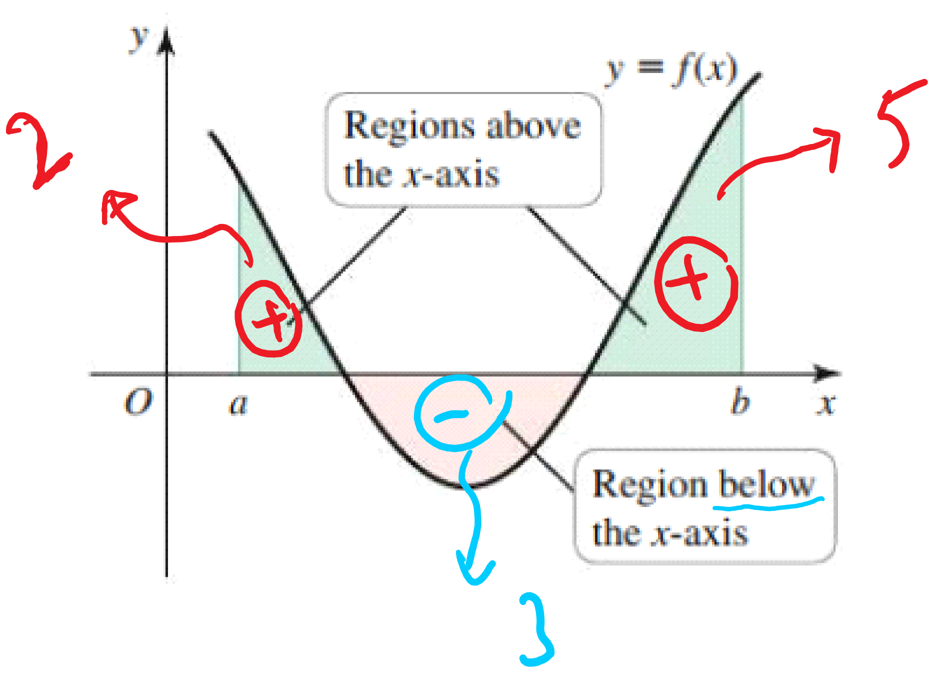
Net Area

How do we interpret Riemann sums when f is negative at some or all points of $[a, b]$? The answer follows directly from the Riemann sum definition.

DEFINITION Net Area

Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the areas of the parts of R that lie above the x-axis *minus* the sum of the areas of the parts of R that lie below the x-axis on $[a, b]$.

$8 + (-3) = 5$



Net Area $= 5 + 2 + (-3) = 4$

7

In the more general case that f is positive on part of $[a, b]$, we get positive contributions to the sum where f is positive and negative contributions to the sum where f is negative. In this case, Riemann sums approximate the area of the regions that lie above the x-axis *minus* the area of the regions that lie *below* the x-axis (Figure 5.19). This difference between the positive and negative contributions is called the *net area*; it can be positive, negative, or zero.

The Definite Integral

Riemann sums for f on $[a, b]$ give *approximations* to the net area of the region bounded by the graph of f and the x-axis between $x = a$ and $x = b$, where $a < b$. How can we make these approximations exact? If f is continuous on $[a, b]$, it is reasonable to expect the Riemann sum approximations to approach the exact value of the net area as the number of subintervals $n \to \infty$ and as the length of the subintervals $\Delta x \to 0$ (Figure 5.20). In terms of limits, we write

height width

$$\text{net area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

$n \to \infty$
$\Delta x \to 0$
width of rect.

DEFINITION Definite Integral

A function f defined on $[a, b]$ is **integrable** on $[a, b]$ if $\lim_{\Delta\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ exists and is unique over all partitions of $[a, b]$ and all choices of x_k^* on a partition. This limit is the **definite integral of f from a to b**, which we write

$$\int_a^b f(x)\,dx = \lim_{\Delta\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

As $n \to \infty$ $(\Delta x \to 0)$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(a+k\cdot\Delta x)\cdot\Delta x = A$$

$S_n \to$ approximate sum by using Right Riemann sum

$A \to$ exact area under the graph of $f(x)$ on $[a,b]$

$A \to$ is called the integral of f on $[a, b]$

$$\boxed{A = \int_a^b f(x)\cdot dx}$$

$a \to$ lower limit of integration
$b \to$ upper limit of integration
$x \to$ variable of integration
$f(x) \to$ integrand

The variable of integration is a dummy variable that is completely internal to the integral. It does not matter what the variable of integration is called, as long as it does not conflict with other variables that are in use. Therefore, the integrals in Figure 5.22 all have the same meaning.

$$\int_a^b f(x)\,dx = \int_a^b f(\omega)\,d\omega = \int_a^b f(s)\,ds$$

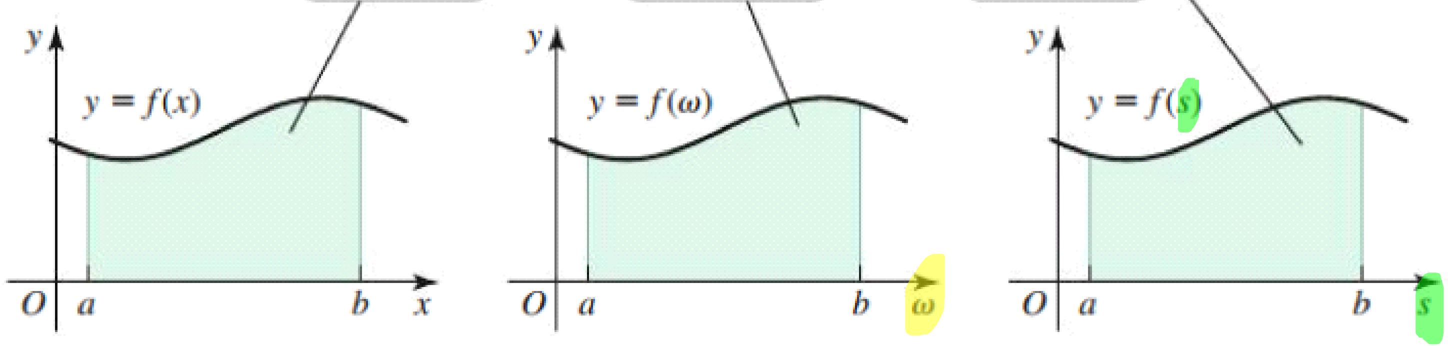


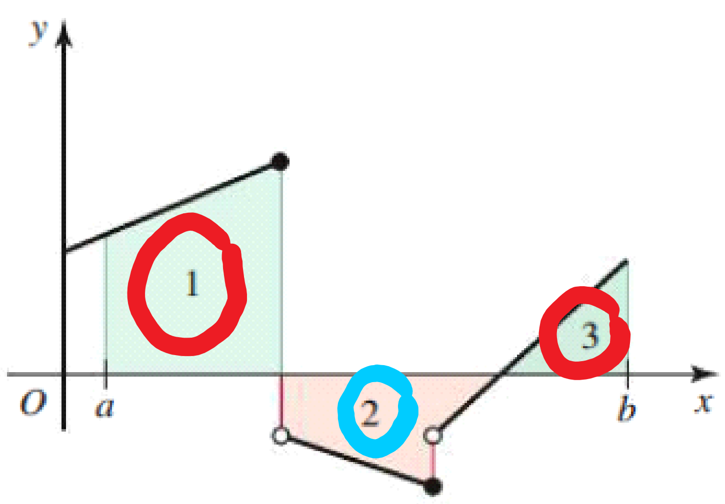
Figure 5.22

Evaluating Definite Integrals

Net area $= \int_a^b f(x)\,dx$

$=$ area above x-axis (Regions 1 and 3)
$-$ area below x-axis (Region 2)

$= 1 + 3 - 2$



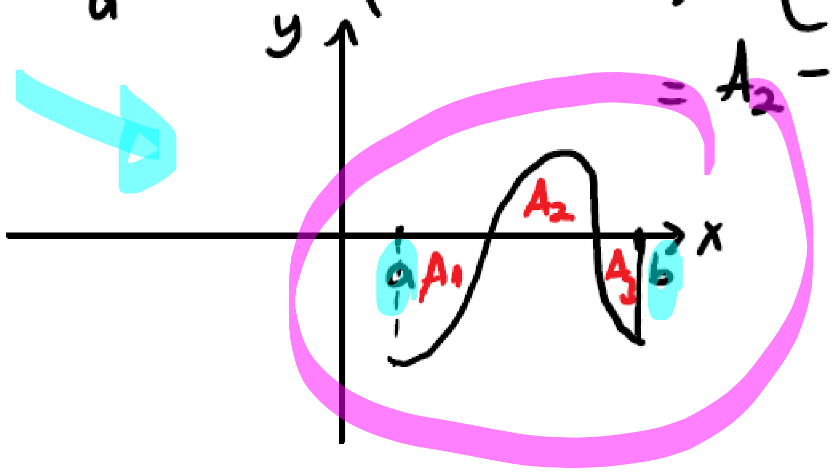
A bounded piecewise continuous function is integrable.

Properties of Definite Integrals

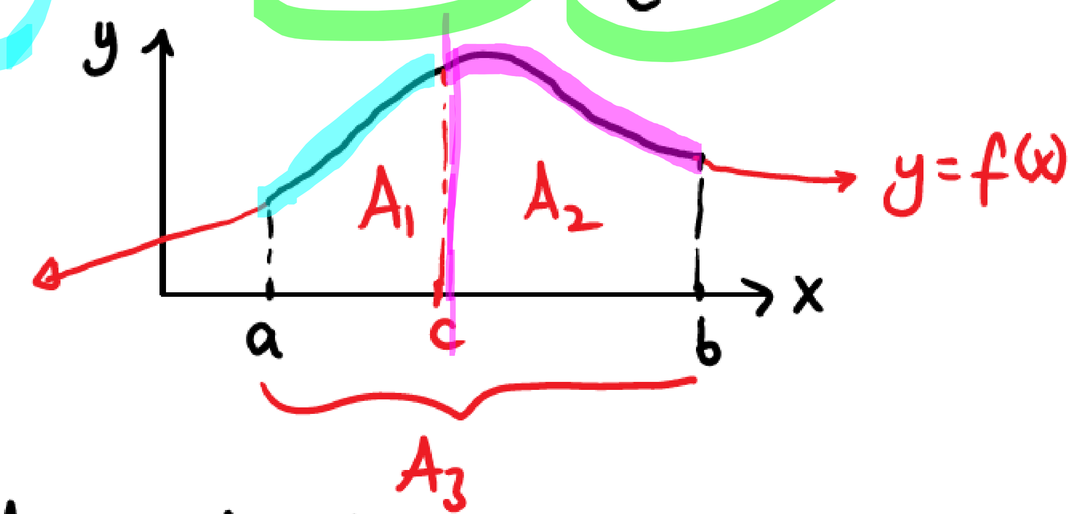
1) $\int_a^a f(x)\,dx = 0$

2) $\int_a^b f(x)\,dx = \left(\text{area above the x-axis}\right) - \left(\text{area below the x-axis}\right)$

$= A_2 - (A_1 + A_3)$



3) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

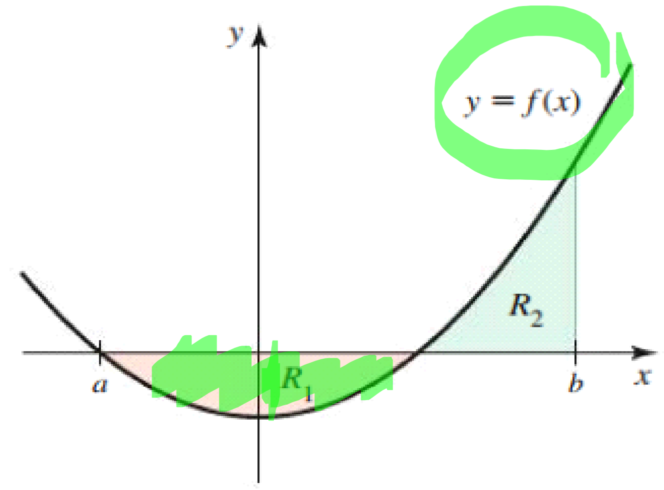


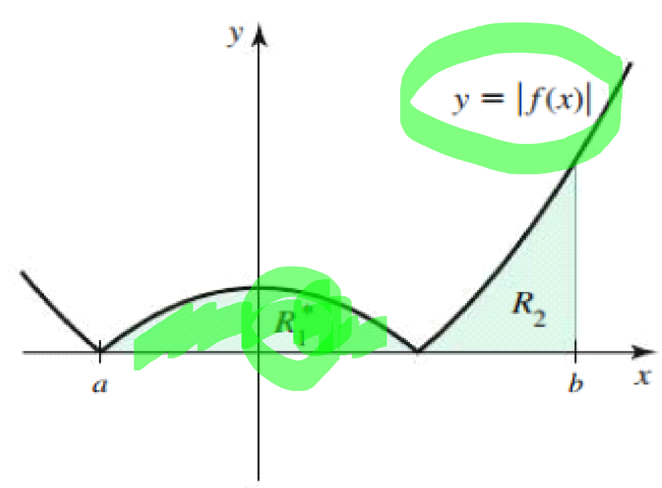
$A_3 = A_1 + A_2$

Table 5.4 Properties of definite integrals

Let f and g be integrable functions on an interval that contains a, b, and p.

1. $\int_a^a f(x)\,dx = 0$ Definition
2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ Definition
3. $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
4. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, for any constant c
5. $\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$
6. The function $|f|$ is integrable on $[a, b]$, and $\int_a^b |f(x)|\,dx$ is the sum of the areas of the regions bounded by the graph of f and the x-axis on $[a, b]$.





$\int_a^b |f(x)|\,dx = \text{area of } R_1^* + \text{area of } R_2$
$= \text{area of } R_1 + \text{area of } R_2$

EXAMPLE 3 **Evaluating definite integrals using geometry** Use familiar area formulas to evaluate the following definite integrals.

a. $\int_2^4 (2x + 3)\,dx$ **b.** $\int_1^6 (2x - 6)\,dx$ **c.** $\int_3^4 \sqrt{1 - (x - 3)^2}\,dx$

y-int: 3 ; m = 2

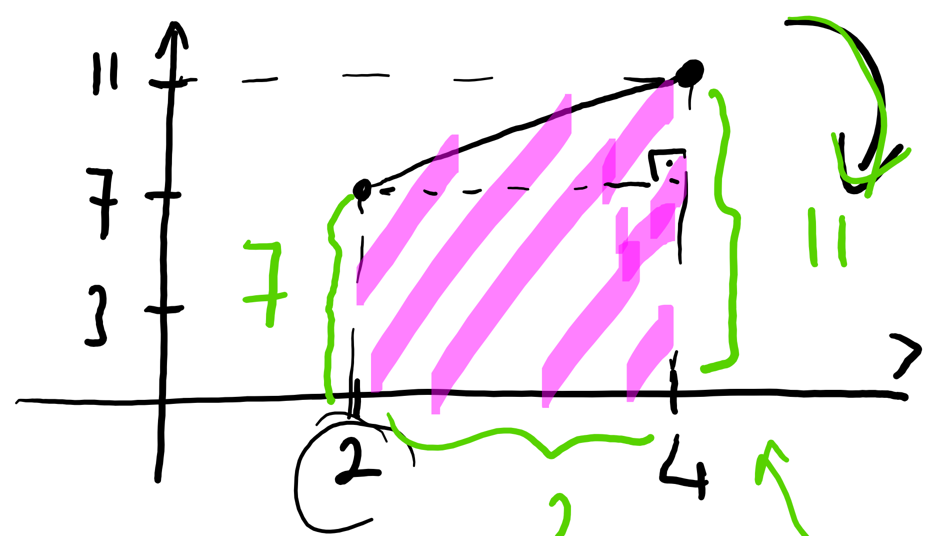
a) $\underbrace{\int_a^b f(x)\,dx}_{\text{template}}$

$\underbrace{f(x) = \overbrace{2x+3}^{mx+b}}$

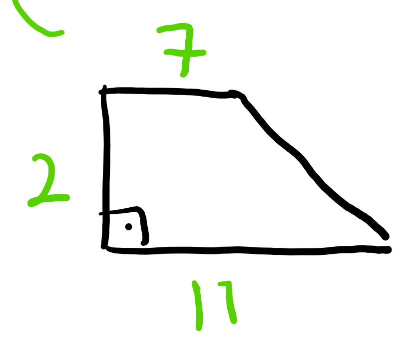
$[2, 4]$

$f(2) = 7$

$f(4) = 11$



$A_{trapezoid} = \frac{(b_1+b_2)h}{2}$



$$\int_2^4 (2x+3)\,dx = A = \frac{(7+11)\cdot \cancel{2}}{\cancel{2}} = 18$$

b) $\int_1^6 (2x-6)\,dx = 9 - 4 = 5$

Net Area

$f(x) = 2x - 6 \rightarrow y = mx + b$: $m = 2$, $b = -6$

$[1, 6]$

$f(1) = -4$

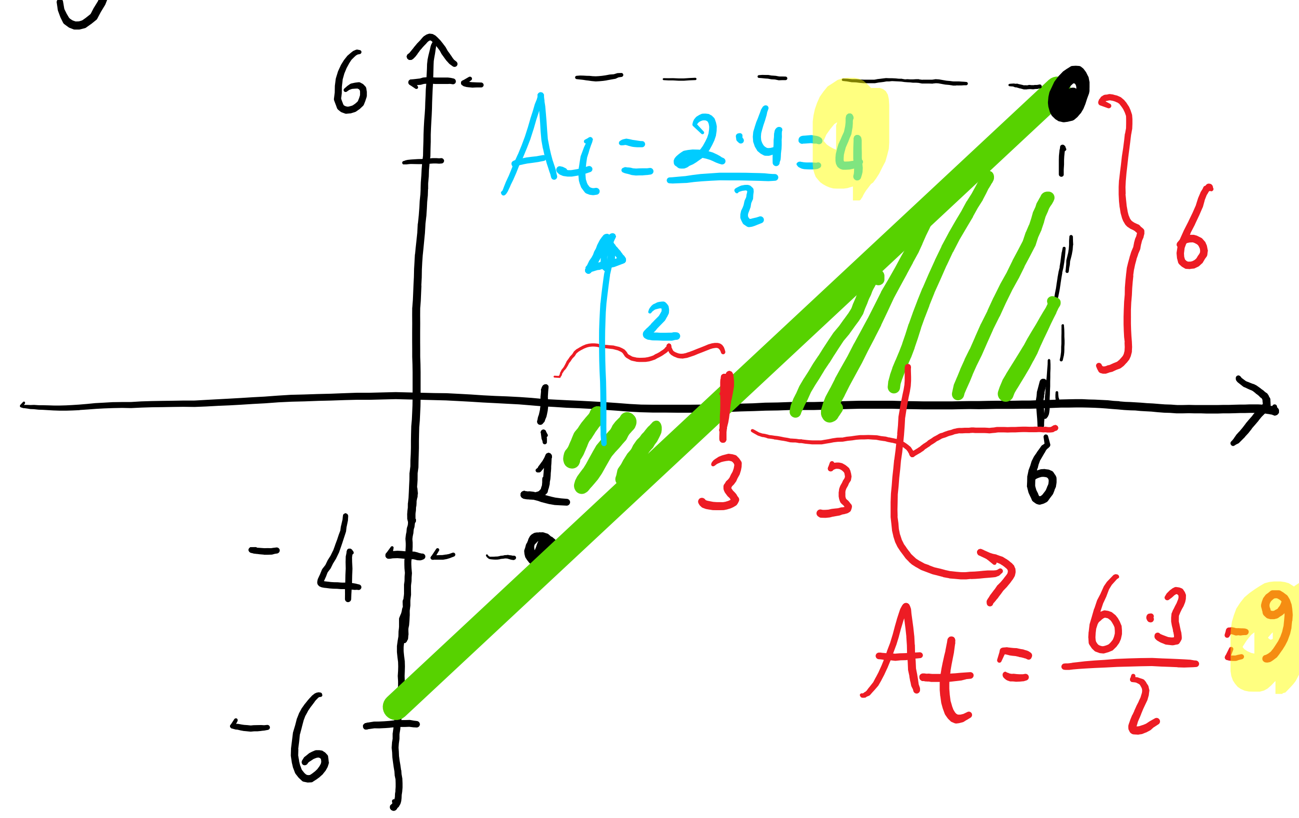
$f(6) = 6$

x-int: $y = 0$

$f(x) = 0 = y = 2x - 6$

$0 = 2x - 6$

$x = 3$



Definite area is the NET AREA of the entire region.

EXAMPLE 3 **Evaluating definite integrals using geometry** Use familiar area formulas to evaluate the following definite integrals.

a. $\int_2^4 (2x+3)\,dx$ b. $\int_1^6 (2x-6)\,dx$ c. $\int_3^4 \sqrt{1-(x-3)^2}\,dx$

$x^2+y^2=r^2$ Eq. of circle

c) $\int_3^4 +\sqrt{1-(x-3)^2}\cdot dx$

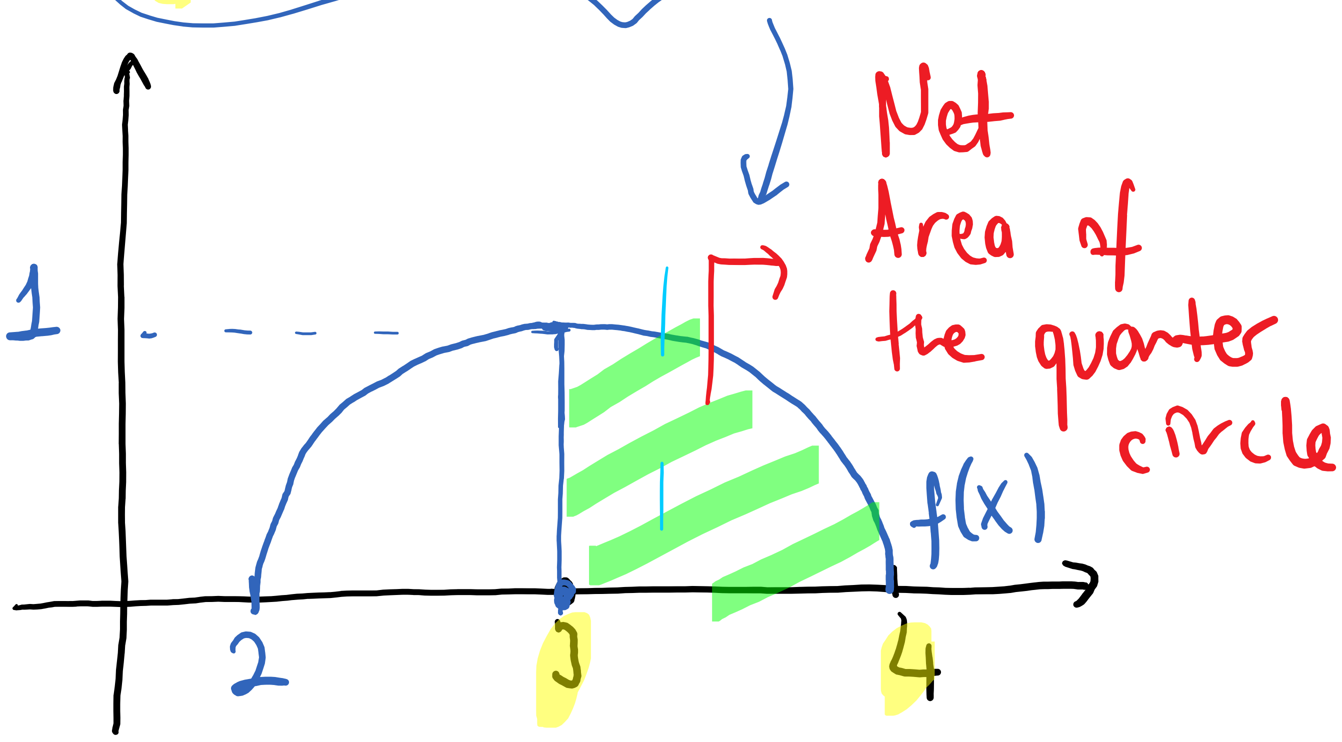
$f(x) = \left(\sqrt{1-(x-3)^2}\right)^2 = (y)^2$

$1-(x-3)^2 = y^2$

$1^2 = (x-3)^2 + (y-0)^2$

$\hookrightarrow r=1$ Center $(3,0)$

Circle centered at $(3,0)$ w/ $r=1$



Not drawn to scale

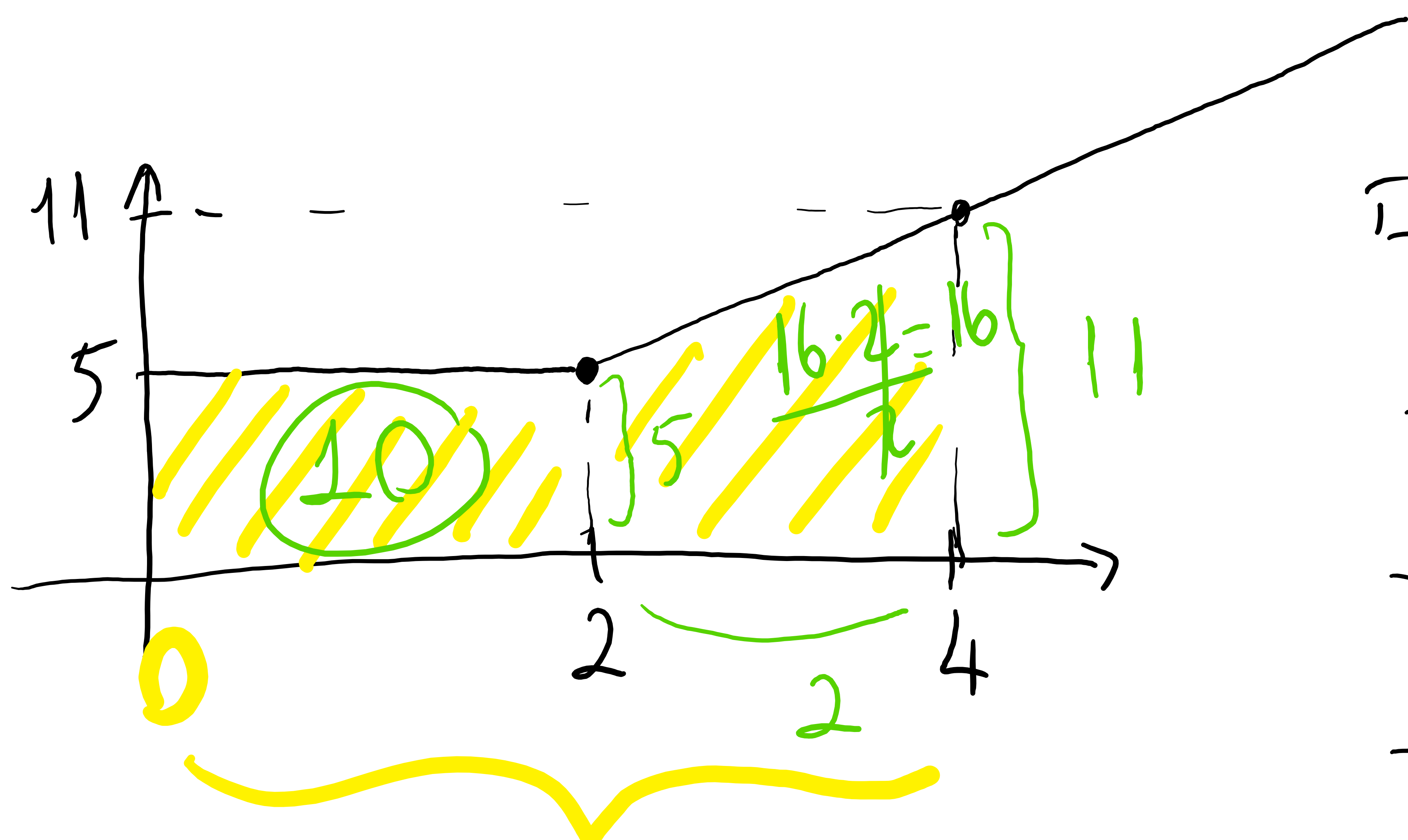
$A_{circle} = \pi r^2$, $A_{q\text{-}circle} = \dfrac{\pi r^2}{4}$

$$\int_3^4 \sqrt{1-(x-3)^2}\cdot dx = A = \frac{\pi \cdot 1^2}{4} = \frac{\pi}{4}$$

Use geometry to evaluate the definite integral:

$$\int_0^4 f(x)\,dx, \quad \text{where } f(x) = \begin{cases} 5, & \text{if } x \leq 2 \\ 3x-1, & \text{if } x > 2 \end{cases}$$

Hint: Break down the area by using the piece-wise f. intervals



If $x > 2$

$f(x) = 3x - 1$

$f(2) = 5$

$f(4) = 3 \cdot 4 - 1 = 11$

Net Area $= 10 + 16 = 26$

Chapter 5

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EXAMPLE 4 **Definite integrals from graphs** Figure 5.28 shows the graph of a function f with the areas of the regions bounded by its graph and the x-axis given. Find the values of the following definite integrals.

a. $\int_a^b f(x)\,dx$ **b.** $\int_b^c f(x)\,dx$ **c.** $\int_a^c f(x)\,dx$ **d.** $\int_b^d f(x)\,dx$

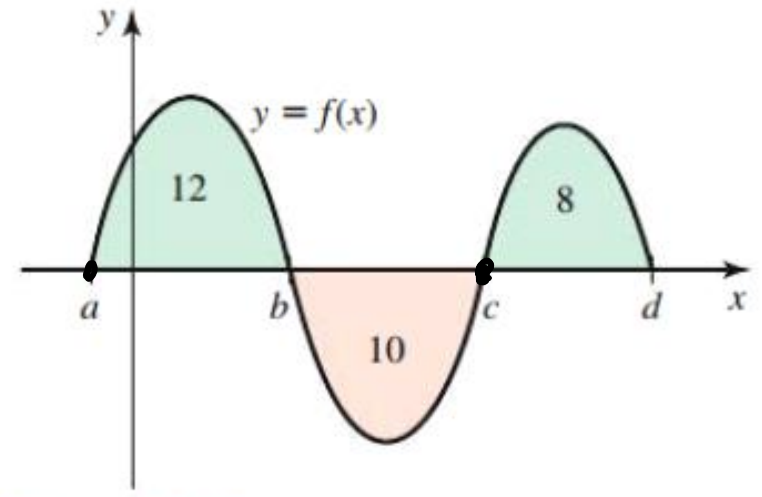


Figure 5.28

a) 12

b) -10

c) $\int_b^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx = 12 + (-10) = 2$

d) $\int_b^d f(x)dx = \int_b^c f(x)dx + \int_c^d f(x)dx = -10 + 8 = -2$

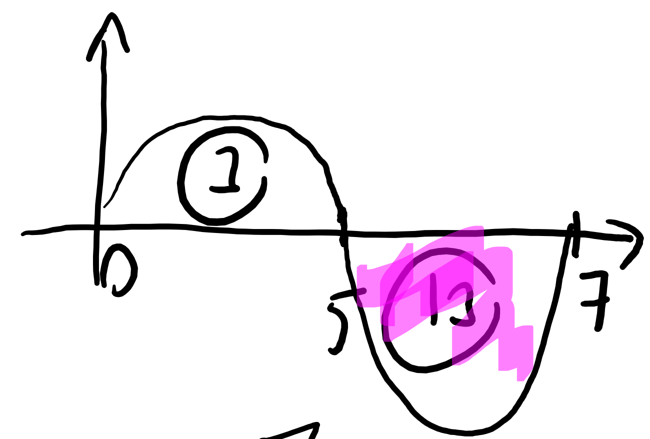
EXAMPLE 5 **Properties of integrals** Assume $\int_0^5 f(x)\,dx = 3$ and $\int_0^7 f(x)\,dx = -10$. given
Evaluate the following integrals, if possible.

a. $\int_0^7 2f(x)\,dx$ **b.** $\int_5^7 f(x)\,dx$ **c.** $\int_5^0 f(x)\,dx$ **d.** $\int_7^0 6f(x)\,dx$ **e.** $\int_0^7 |f(x)|\,dx$

a) $\int_0^7 2\cdot f(x)\,dx = 2\cdot \int_0^7 f(x)\,dx = -20$

b) $\int_5^7 f(x)\,dx = \int_0^7 f(x)\,dx - \int_0^5 f(x)\,dx$

$= -10 - 3 = -13$



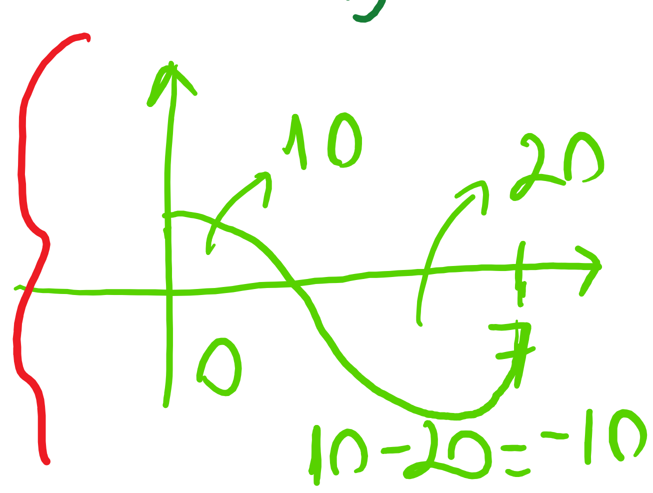
c) $\int_5^0 f(x)\,dx = -\int_0^5 f(x)\,dx = -3$

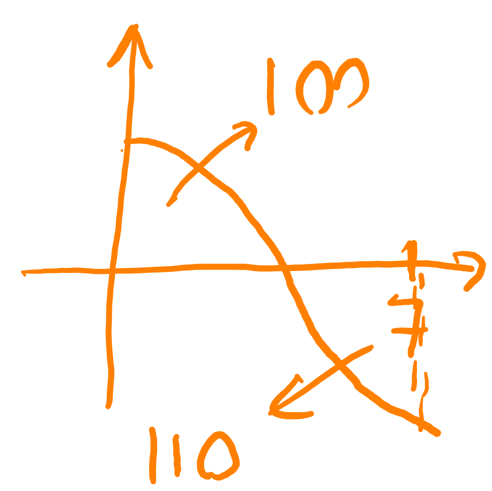
d) $\int_7^0 6\cdot f(x)\,dx = -6\int_0^7 f(x)\,dx = -6\cdot(-10) = 60$

e) $\int_0^7 |f(x)|\,dx$ given: $\int_0^7 f(x)\,dx = \underbrace{-10}_{\text{net area}}$

The integral CAN NOT be evaluated. →

10+20
30
?





100 − 110
= −10

100+110
= 210 ?