3.3-3.5, 3.9 Group Activity Problems - Solutions

DIFFERENTIATION RULES

1. *Constant Rule:* If $f(x) = c$ (c constant), then $f'(x) = 0$.
2. *Power Rule:* If r is a real number, $\frac{d}{dx}x^r = rx^{r-1}$
3. *Constant Multiple Rule:* $\frac{d}{dx}(c \cdot f(x)) = c \cdot f'(x)$
4. *Sum Rule:* $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
5. *Product Rule:* $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$
6. *Quotient Rule:* $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

DEFINITION **Rate of Change and the Slope of the Tangent Line**

The **average rate of change** in f on the interval $[a, x]$ is the slope of the corresponding secant line:

$$m_{sec} = \frac{f(x) - f(a)}{x - a}.$$

The **instantaneous rate of change** in f at a is

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \qquad (1)$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists. The **tangent line** is the unique line through $(a, f(a))$ with slope m_{tan}. Its equation is

$$y - f(a) = m_{tan}(x - a).$$

ALTERNATIVE DEFINITION **Rate of Change and the Slope of the Tangent Line**

The **average rate of change** in f on the interval $[a, a + h]$ is the slope of the corresponding secant line:

$$m_{sec} = \frac{f(a + h) - f(a)}{h}.$$

The **instantaneous rate of change** in f at a is

$$m_{tan} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}, \qquad (2)$$

which is also the **slope of the tangent line** at $(a, f(a))$, provided this limit exists.

Dr. Tabanli's Spring 2020 Exam#1 Question

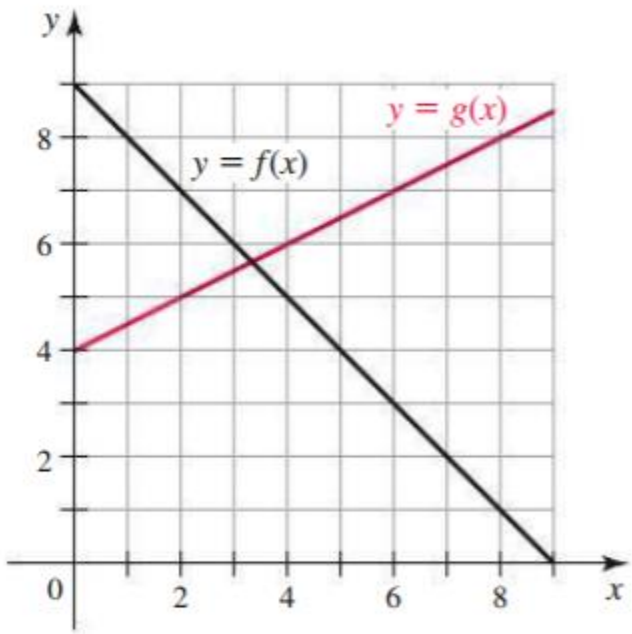
14. For both parts of this problem let $f(x) = 3x^2 - 4x + 2$.

(a) (2 points) Calculate $f'(4)$ by using derivative rules to receive full credit.

(b) (8 points) Calculate $f'(4)$ by using the limit definition of derivative and proper notation to receive full credit. If you simply quote a derivative rule without using the limit definition, you will receive no credit.

8. If $f'(0) = 6$ and $g(x) = f(x) + e^x + 1$, find $g'(0)$.

9–11. *Let* $F(x) = f(x) + g(x)$, $G(x) = f(x) - g(x)$, *and* $H(x) = 3f(x) + 2g(x)$, *where the graphs of f and g are shown in the figure. Find each of the following.*



9. $F'(2)$ **10.** $G'(6)$ **11.** $H'(2)$

3.3.8 $g'(x) = f'(x) + e^x$, so $g'(0) = f'(0) + e^0 = 6 + 1 = 7$.

3.3.9 $F'(2) = f'(2) + g'(2) = -1 + \frac{1}{2} = -\frac{1}{2}$.

3.3.10 $G'(6) = f'(6) - g'(6) = -1 - \frac{1}{2} = -\frac{3}{2}$.

3.3.11 $H'(2) = 3f'(2) + 2g'(2) = 3(-1) + 2(\frac{1}{2}) = -2$.

18. The line tangent to the graph of f at $x = 3$ is $y = 4x - 2$ and the line tangent to the graph of g at $x = 3$ is $y = -5x + 1$. Find the values of $(f + g)(3)$ and $(f + g)'(3)$.

3.3.18 $(f+g)(3) = f(3) + g(3) = 10 - 14 = -4$ and $(f+g)'(3) = f'(3) + g'(3) = 4 - 5 = -1$.

66. Finding slope locations Let $f(x) = 2e^x - 6x$.

a. Find all points on the graph of f at which the tangent line is horizontal.

b. Find all points on the graph of f at which the tangent line has slope 12.

3.3.66

a. The slope of the tangent line is given by $f'(x) = 2e^x - 6$. This is equal to zero when $2e^x = 6$, which occurs for $x = \ln 3$. The point on the graph is therefore $(\ln 3, 6 - 6\ln 3)$.

b. The slope of the tangent line is 12 when $2e^x - 6 = 12$, or $e^x = 9$. This occurs for $x = \ln 9$. The point on the graph is therefore $(\ln 9, 18 - 6\ln 9)$.

77. Tangent line given Determine the constants b and c such that the line tangent to $f(x) = x^2 + bx + c$ at $x = 1$ is $y = 4x + 2$.

3.3.77 For $f(x) = x^2 + bx + c$ we have $f'(x) = 2x + b$, so $f'(1) = 2 + b$. Because the slope of $4x + 2$ is 4, we require $2 + b = 4$, so $b = 2$. Also, because the value of $4x + 2$ at $x = 1$ is 6, we must have $f(1) = 1 + 2 + c = 6$, so $c = 3$. Thus the curve $f(x) = x^2 + 2x + 3$ has $y = 4x + 2$ as its tangent line at $x = 1$.

74. Tangent lines Suppose $f(2) = 2$ and $f'(2) = 3$. Let $g(x) = x^2 f(x)$ and $h(x) = \dfrac{f(x)}{x - 3}$.

a. Find an equation of the line tangent to $y = g(x)$ at $x = 2$.

b. Find an equation of the line tangent to $y = h(x)$ at $x = 2$.

3.3.74

a. The slope of the tangent line to g at x is given by $g'(x) = 2x + f'(x)$, so $g'(3) = 6 + f'(3) = 10$. The point on the curve $y = g(x)$ at $x = 3$ is $(3, 9 + f(3)) = (3, 9 + 1) = (3, 10)$. Thus the equation of the tangent line at this point is $y - 10 = 10(x - 3)$, or $y = 10x - 20$.

b. The slope of the tangent line to h at x is given by $h'(x) = 3f'(x)$, so $h'(3) = 3f'(3) = 3 \cdot 4 = 12$. The point on the curve $y = h(x)$ at $x = 3$ is $(3, 3 \cdot f(3)) = (3, 3)$. Thus the equation of the tangent line at this point is $y - 3 = 12(x - 3)$, or $y = 12x - 33$.

3.4 Group Activity Problems

Find and simplify the derivatives.

38. $y = \left(2\sqrt{x} - 1\right)(4x + 1)^{-1}$

3.4.38

$$y' = \frac{d}{dx}\left(\frac{2\sqrt{x}-1}{4x+1}\right) = \frac{(4x+1)\left(\frac{1}{\sqrt{x}}\right) - (2\sqrt{x}-1)4}{(4x+1)^2}$$
$$= \frac{4\sqrt{x} + \frac{1}{\sqrt{x}} - 8\sqrt{x} + 4}{(4x+1)^2} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{-4x + 1 + 4\sqrt{x}}{\sqrt{x}(4x+1)^2}.$$

54. $f(z) = \left(\frac{z^2 + 1}{z}\right)e^z$

3.4.54

$$f'(z) = \left(\frac{z(2z) - (z^2+1)}{z^2}\right)e^z + \left(\frac{z^2+1}{z}\right)e^z = e^z\left(\left(\frac{z^2-1}{z^2}\right) + \left(\frac{z^2+1}{z}\right)\right)$$
$$= e^z\left(\frac{z^3 + z^2 + z - 1}{z^2}\right).$$

28. $f(x) = e^x \sqrt[3]{x}$

3.4.28 $f'(x) = e^x \cdot \sqrt[3]{x} + e^x \cdot \frac{1}{3}x^{-2/3} = e^x\left(\frac{3x+1}{3x^{2/3}}\right).$

72–73. First and second derivatives *Find $f'(x)$ and $f''(x)$.*

72. $f(x) = \dfrac{x}{x+2}$

3.4.72

$$f'(x) = \frac{d}{dx}\left(\frac{x}{x+2}\right) = \frac{(x+2)-x}{(x+2)^2} = \frac{2}{(x+2)^2} = \frac{2}{x^2+4x+4}.$$

$$\begin{aligned} f''(x) = \frac{d}{dx}\left(\frac{2}{x^2+4x+4}\right) &= \frac{(x^2+4x+4)\cdot 0 - 2(2x+4)}{(x+2)^4} \\ &= \frac{-4(x+2)}{(x+2)^4} = -\frac{4}{(x+2)^3} = -\frac{4}{x^3+6x^2+12x+8}. \end{aligned}$$

3.5 Group Activity Problems

Dr. Tabanli's Spring 2020 Exam#1 Question

12. Calculate $f'(x)$. After calculating the derivative, do not simplify your answer.

(b) $f(x) = \dfrac{x^2 - 9}{\cos(x-3)}$

Find and simplify the derivatives.

37. $y = x \cos x \sin x$

3.5.37 $\dfrac{dy}{dx} = \cos x \sin x + x(-\sin x)\sin x + x \cos x \cos x = \sin x \cos x - x \sin^2 x + x \cos^2 x = \dfrac{1}{2}\sin 2x + x \cos 2x.$

66–71. Trigonometric limits *Evaluate the following limits or state that they do not exist. (Hint: Identify each limit as the derivative of a function at a point.)*

68. $\lim_{h \to 0} \dfrac{\sin\left(\dfrac{\pi}{6} + h\right) - \dfrac{1}{2}}{h}$

3.5.68 $\lim_{h\to 0} \dfrac{\sin(\pi/6+h) - (1/2)}{h} = \lim_{h\to 0} \dfrac{\sin(\pi/6+h) - \sin \pi/6}{h} = \dfrac{d}{dx} \sin x \Big|_{x=\pi/6} = \cos(\pi/6) = \sqrt{3}/2.$

84. Continuity of a piecewise function Let

$$f(x) = \begin{cases} \dfrac{3 \sin x}{x} & \text{if } x \neq 0 \\ a & \text{if } x = 0. \end{cases}$$

For what values of a is f continuous?

3.5.84 f is continuous at 0 if and only if $\lim_{x\to 0} f(x) = f(0)$. Because $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{3\sin x}{x} = 3$, we require $a = 3$ in order for f to be continuous.

3.9 Group Activity Problems

Evaluate:

32. $y = \ln\left(e^x + e^{-x}\right)$

34. $y = e^x x^e$

70. $f(x) = \ln \frac{2x}{(x^2+1)^3}$

3.9.32 $\frac{d}{dx}(\ln(e^x + e^{-x})) = \frac{1}{e^x + e^{-x}}(e^x - e^{-x}) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

3.9.34 $y' = e^x x^e + e^{x+1} x^{e-1} = e^x x^{e-1}(x + e)$.

3.9.70 $f'(x) = \frac{d}{dx}(\ln 2x - 3\ln(x^2+1)) = \frac{1}{x} - \frac{6x}{x^2+1}$.

87. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The derivative of $\log_2 9$ is $1/(9 \ln 2)$.

b. $\ln(x+1) + \ln(x-1) = \ln(x^2-1)$, for all x.

c. The exponential function 2^{x+1} can be written in base e as $e^{2\ln(x+1)}$.

3.9.87

a. False. $\log_2 9$ is a constant, so its derivative is 0.

b. False. If $x < -1$, then the right-hand side is defined while the left-hand side isn't.

c. False. The correct way to write that function would be $e^{(x+1)\ln 2}$.