4.5 Group Activity Problems - Solutions

Guidelines for Optimization Problems

1. Read the problem carefully, identify the variables, and organize the given information with a picture.
2. Identify the objective function (the function to be optimized). Write it in terms of the variables of the problem.
3. Identify the constraint(s). Write them in terms of the variables of the problem.
4. Use the constraint(s) to eliminate all but one independent variable of the objective function.
5. With the objective function expressed in terms of a single variable, find the interval of interest for that variable.
6. Use methods of calculus to find the absolute maximum or minimum value of the objective function on the interval of interest. If necessary, check the endpoints.

8. What two nonnegative real numbers a and b whose sum is 23 maximize $a^2 + b^2$? Minimize $a^2 + b^2$?

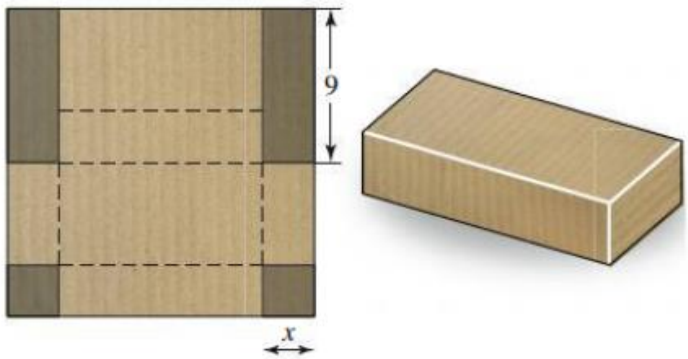
4.5.8 Let a and b be the two non-negative numbers. The constraint is $a + b = 23$, which gives $b = 23 - a$. The objective function to be maximized/minimized is the quantity $Q = a^2 + b^2$. Using $b = 23 - a$, we have $Q = a^2 + b^2 = a^2 + (23 - a)^2 = 2a^2 - 46a + 529$. Now a must be at least 0, and cannot exceed 23 (otherwise $b < 0$). Therefore we need to maximize $Q(a) = 2a^2 - 46a + 529$ for $0 \le a \le 23$. The critical points of

the objective function satisfy $Q'(a) = 4a - 46 = 0$, which has the solution $a = 23/2$. To find the absolute maximum/minimum of Q, we check the endpoints of $[0, 23]$ and the critical point $a = 23/2$. Observe that $Q(0) = Q(23) = 529$ and $Q(23/2) = 529/2$, so the absolute maximum occurs when $a, b = 0, 23$ or $23, 0$ and the absolute minimum occurs when $a = b = 23/2$.

17. Rectangles beneath a semicircle A rectangle is constructed with its base on the diameter of a semicircle with radius 5 and its two other vertices on the semicircle. What are the dimensions of the rectangle with maximum area?

4.5.17 Let the coordinates of the base of the rectangle be $(x, 0)$ and $(-x, 0)$ where $0 \le x \le 5$. Then the width of the rectangle is $2x$ and the height is $\sqrt{25 - x^2}$, so the area A is given by $A(x) = 2x\sqrt{25 - x^2}$. The critical points of this function satisfy $A'(x) = 2\sqrt{25 - x^2} + \dfrac{2x \cdot (-x)}{\sqrt{25 - x^2}} = \dfrac{2(25 - 2x^2)}{\sqrt{25 - x^2}} = 0$, which has unique solution $x = 5/\sqrt{2}$ in $(0, 5)$. We have $A(0) = A(5) = 0$, so the rectangle of maximum area has width $2x = 10/\sqrt{2}$ cm, height $y = \sqrt{25 - (25/2)} = 5/\sqrt{2}$ cm.

39. Designing a box Two squares of length x are cut out of adjacent corners of a 18″ × 18″ piece of cardboard and two rectangles of length 9 and width x are cut of out the other two corners of the cardboard (see figure). The resulting piece of cardboard is then folded along the dashed lines to form an enclosed box (see figure). Find the dimensions and volume of the largest box that can be formed in this way.



4.5.39 Because the length of the box (in inches) is $18 - 2x$, the width is $9 - x$, and the height is x, the volume of the box is $V(x) = x(9-x)(18-2x) = 2x^3 - 36x^2 + 162x$, where $0 \le x \le 9$. Taking the derivative of V, we have $V'(x) = 6x^2 - 72x + 162 = 6(x^2 - 12x + 27) = 6(x-3)(x-9)$. Setting the derivative equal to 0 and solving for x, we find that $x = 3$ and $x = 9$. Because $V(0) = V(9) = 0$ and $V(3) = 216$, the box of maximum volume is $12 \times 6 \times 3$ and has a volume of 216 in^3.