4.7 Group Activity Problems Solutions

THEOREM 4.12 L'Hôpital's Rule
Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies if $x \to a$ is replaced with $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

THEOREM 4.13 L'Hôpital's Rule (∞/∞)
Suppose f and g are differentiable on an open interval I containing a, with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies for $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

(Other/Secondary) Indeterminate Forms

$1^\infty, 0^0, \infty^0$ $\quad \infty - \infty \quad 0 \cdot \infty$

use ln properties

Procedure:
1) "DSP" to obtain an indeterminate form
2) Re-write "other indeterminate forms" into "primary indeterminate form"
3) Use L.R.

6. Which of the following limits can be evaluated without l'Hôpital's Rule? Evaluate each limit.

a. $\lim_{x \to 0} \frac{\sin x}{x^3 + 2x + 1}$ **b.** $\lim_{x \to 0} \frac{\sin x}{x^3 + 2x}$

4.7.6

a. L'Hôpital's Rule is not needed. This is not an indeterminate form. $\lim_{x \to 0} \frac{\sin x}{x^2 + 2x + 1} = \frac{0}{1} = 0.$

b. L'Hôpital's Rule is needed. $\lim_{x \to 0} \frac{\sin x}{x^3 + 2x} = \lim_{x \to 0} \frac{\cos x}{3x^2 + 2} = \frac{1}{2}.$

17–83. Limits *Evaluate the following limits. Use l'Hôpital's Rule when it is convenient and applicable.*

22. $\lim_{x \to 0} \frac{e^x - 1}{x^2 + 3x}$

24. $\lim_{x \to \infty} \frac{4x^3 - 2x^2 + 6}{\pi x^3 + 4}$

4.7.22 L'Hôpital's rule gives $\lim_{x\to 0} \frac{e^x - 1}{x^2 + 3x} = \lim_{x\to 0} \frac{e^x}{2x+3} = \frac{1}{3}$.

4.7.24 Apply L'Hôpital's rule three times:

$$\lim_{x\to\infty} \frac{4x^3 - 2x^2 + 6}{\pi x^3 + 4} = \lim_{x\to\infty} \frac{12x^2 - 4x}{3\pi x^2} = \lim_{x\to\infty} \frac{24x - 4}{6\pi x} = \lim_{x\to\infty} \frac{24}{6\pi} = \frac{4}{\pi}.$$

44. $\lim_{x\to 1} \frac{x^n - 1}{x - 1}$, n is a positive integer

4.7.44 L'Hôpital's rule gives $\lim_{x\to 1} \frac{x^n - 1}{x - 1} = \lim_{x\to 1} \frac{nx^{n-1}}{1} = n$.

51. $\lim_{x\to\infty} \frac{x^2 - \ln(2/x)}{3x^2 + 2x}$

52. $\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\sqrt{x-1}} \right)$

4.7.51 Applying L'Hôpital's rule twice gives:

$$\lim_{x\to\infty} \frac{x^2 - \ln(2/x)}{3x^2 + 2x} = \lim_{x\to\infty} \frac{2x + (1/x)}{6x + 2} = \lim_{x\to\infty} \frac{2 - 1/x^2}{6} = \frac{2}{6} = \frac{1}{3}.$$

4.7.52 Observe that $\lim_{x\to 1+} \left(\frac{1}{x-1} - \frac{1}{\sqrt{x-1}} \right) = \lim_{x\to 1+} \frac{1 - \sqrt{x-1}}{x-1} = \infty$.

80. $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$, for a constant a

81. $\lim_{x\to 0}\left(e^{ax}+x\right)^{1/x}$, for a constant a

4.7.80 Note that $\ln(1+a/x)^x = x\ln(1+a/x)$, so we evaluate

$$L=\lim_{x\to\infty} x\ln(1+a/x)=\lim_{x\to\infty}\frac{\ln(1+a/x)}{1/x}=\lim_{x\to\infty}\frac{\frac{1}{1+a/x}\cdot\frac{-a}{x^2}}{\frac{-1}{x^2}}=\lim_{x\to\infty}\frac{a}{1+a/x}=a$$

by L'Hôpital's rule. Therefore $\lim_{x\to\infty}(1+a/x)^x=e^L=e^a$.

4.7.81 Note that $\ln(e^{ax}+x)^{1/x}=\frac{1}{x}\ln(e^{ax}+x)$, so we evaluate

$$L=\lim_{x\to 0}\frac{\ln(e^{ax}+x)}{x}=\lim_{x\to 0}\frac{(ae^{ax}+1)/(e^{ax}+x)}{1}=\frac{(a+1)/1}{1}=a+1.$$

Therefore, $\lim_{x\to 0}(e^{ax}+x)^{1/x}=e^{a+1}$.

93. $\lim_{x\to 0}\frac{a^x-b^x}{x}$, for positive constants a and b

94. $\lim_{x\to 0}(1+ax)^{b/x}$, for positive constants a and b

4.7.93 Apply L'Hôpital's rule: $\lim_{x\to 0}\frac{a^x-b^x}{x}=\lim_{x\to 0}\frac{(\ln a)a^x-(\ln b)b^x}{1}=\ln a-\ln b.$

4.7.94 Note that $\ln(1+ax)^{b/x}=b\ln(1+ax)/x$, so we evaluate

$$L=\lim_{x\to 0}\frac{b\ln(1+ax)}{x}=b\lim_{x\to 0}\frac{\frac{a}{1+ax}}{1}=ab$$

by L'Hôpital's rule. Therefore $\lim_{x\to 0}(1+ax)^{b/x}=e^L=e^{ab}$.

105. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. By l'Hôpital's Rule, $\lim_{x \to 2} \frac{x - 2}{x^2 - 1} = \lim_{x \to 2} \frac{1}{2x} = \frac{1}{4}$.

b. $\lim_{x \to 0} x \sin x = \lim_{x \to 0} f(x)g(x) = \lim_{x \to 0} f'(x) \lim_{x \to 0} g'(x) = (\lim_{x \to 0} 1)(\lim_{x \to 0} \cos x) = 1$.

c. $\lim_{x \to 0^+} x^{1/x}$ is an indeterminate form.

4.7.105

a. False. $\lim_{x\to 2} x^2 - 1 = 3$, so L'Hôpital's rule does not apply. In fact, $\lim_{x\to 2} \frac{x-2}{x^2-1} = \frac{0}{3} = 0$.

b. False. L'Hôpital's rule does not say $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f'(x) \lim_{x\to a} g'(x)$. In fact, $\lim_{x\to 0} x \sin x = 0 \cdot 0 = 0$.

c. False. This limit has the form $0^\infty = 0$.