Experimental Results on Upper Bounds for Vertex Pi-Lights*

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Abstract

The problem of illuminating a simple n-gon with $cn, c < 1$ π-lights is open, whereas a lower bound of $\lfloor \frac{3}{5}n \rfloor$ is known. We provide an algorithm for placing π-lights, and experimental results that support the conjecture that $\lfloor \frac{3}{5}n \rfloor$ lights are also sufficient. We also prove that $\lfloor \frac{5}{6}n \rfloor$ π-lights suffice if some may be *outward facing*[4].

1 The Open Problem

Consider the problem of illuminating a simple n-gon P with vertex π-lights [7]. A π-light placed at a reflex vertex v cannot illuminate the entire interior of the angle. An *inward-facing* light illuminates a subangle of 180^o (Fig. 1a&b). If one edge incident on v bounds the subangle, the light is *fully turned*. An *outward-facing* light illuminates two convex subangles which collectively measure less than 180^o (Fig. 1c).

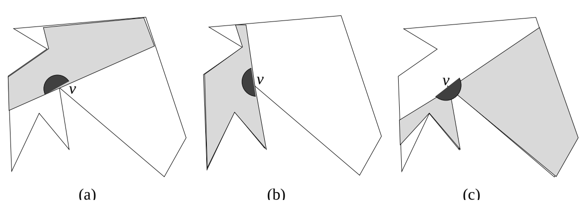

Figure 1: (a) Inward facing (b) Fully turned (c) Outward facing.

The existence of a sub-linear (i.e., cn for $c < 1$) upper bound on the number of inward facing vertex π-lights needed to light a simple polygon is uncertain. The only known upper bound for an n-gon is $n - 2$, obtained by removing an ear from the polygon, placing a light on the removed vertex, and proceeding inductively. Monotone mountains of n vertices can be illuminated with $\lceil n/2 \rceil - 1$ π-lights [5]. The best known lower bound of $\lfloor \frac{3}{5}n \rfloor$ was shown by F. Santos for a family of polygons with $5k+1$ vertices, which require $3k$ π-lights [7] (see Fig. 2). We conjecture that this bound is tight. In Section 2, we present an algorithm for placing inward facing vertex π-lights and conjecture that it always places at most $\lfloor \frac{3}{5}n \rfloor$ of them and provide experimental corroboration for this conjecture. Section 3 verifies that, if outward facing π-lights are allowed, $\lfloor \frac{5}{6}n \rfloor$ lights suffice.

2 An Algorithm and Its Results

Let $v_1, \ldots, v_n$ be the vertices of P in counterclockwise order. Place inward facing π-lights thus:

Initialization:
1. Arbitrarily triangulate the polygon. All triangles are considered "unlit."
2. For each convex vertex $v_i \in P$, let the weight t_i of the vertex be the number of triangles incident at v_i.
3. For each reflex vertex $v_j \in P$, let l_j be the number of complete unlit triangles incident on v_j that are illuminated by a π-light on v_j that lights the left of the left ray $\overrightarrow{v_{j-1}v_j}$. Let r_j be the number of complete unlit triangles incident on v_j that are illuminated by a π-light on v_j that lights the left of the right ray $\overrightarrow{v_j v_{j+1}}$. Then the weight of the vertex $t_j = \max(l_j, r_j)$.

Place lights until $t_j = 0$ for all $1 \le j \le n$:
1. Place a light at a vertex v_m of maximum weight. If a tie, choose a convex vertex over a reflex one. Set $t_m = 0$.
2. For every vertex v_j of non-zero weight of a triangle that has just been lit, update the value of t_j as follows: In the clockwise (resp. counterclockwise) ordering of vertices incident on v_j, let (v_j, v_i) (resp. (v_j, v_k)) be the first edge incident on an unlit triangle. Now treat $\overrightarrow{v_i v_j}$ as the left ray and $\overrightarrow{v_j v_k}$ as the right ray of the initialization step, and update t_j accordingly.
3. For every unlit triangle incident on v_m, if one of the other two vertices of the triangle has weight zero, recursively place a light on the remaining vertex v_i of non-zero weight so that it covers this triangle, even if that placement does not cover t_i triangles.

The lights placed by the algorithm are inward-facing but may or may not be fully turned. We implemented our algorithm in `C++` using LEDA 4.1 for Unix. Experimental results from the implementation support our conjecture:

Conjecture: $\lfloor \frac{3}{5}n \rfloor$ π-lights suffice to light an n-gon.
If g is the number of guards placed by our algorithm on an n-gon, the ratio g/n was smaller than 0.6 for each

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[4] Since submission, we have learned of a recent result by Csaba Toth [6] showing that $\lfloor \frac{3}{4}n \rfloor$ outward-facing π-lights suffice to illuminate an n-gon.

of the more than 57,000 randomly generated polygons[5] (see Figs 2-3 for sample output). While every one of the 57,000 polygons used for testing was fully illuminated, there are polygons that will not be fully illuminated by the current version of our algorithm. The reason is that the condition in step 3 of the recursive step is not strong enough. Currently, when one of the other two vertices (say v_a and v_b) of an unlit triangle t incident on v_m has a light, we force t to be lit by a light at the third vertex. This condition should be tightened. Instead, we need to illuminate t by a light at one of v_a or v_b whenever there are one or fewer vertices that want to light t (even though there may not yet be a light at that vertex). The proposed fix to the algorithm has been implemented and we are now re-testing the code on random polygons.

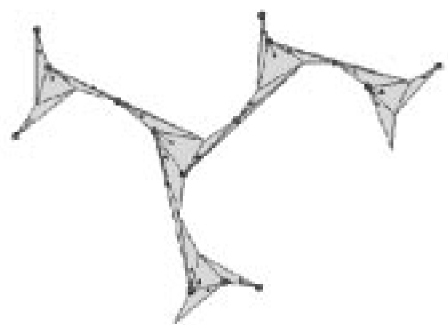

Figure 2: A Santos polygon: $n = 26, g = 15, g/n = 0.58$.

3 Outward Facing Lights

A *fan* is a polygon that admits a triangulation in which all triangles have a common vertex, called the fan *center*. The other vertices of the polygon are *boundary vertices*. A t-fan is a fan of t triangles. By the art gallery theorem [3, 4], it is known that any simple polygon P of n vertices can be decomposed into $h \leq \lfloor n/3 \rfloor$ fans. Let $f_1, f_2, \ldots, f_h$ be the fans, and let $t_i, 1 \leq i \leq h$ be the number of triangles in fan f_i. Let F_P be the graph representing the fan decomposition of P: each node of F_P is a fan, and there is an edge between two nodes when the corresponding fans share an edge. Observe that F_P is a tree. Assume that F_P is rooted at an arbitrary node. Let Δ_i be the triangle of fan f_i that is adjacent to its parent. We call this the *front triangle* of f_i, and the two fan boundary vertices of Δ_i are called *front vertices*.

[5] No known polynomial time algorithm generates all random simple polygons with uniform distribution [1]. We chose the 2-opt moves technique that does generate all possible polygons, but not with uniform distribution [2].

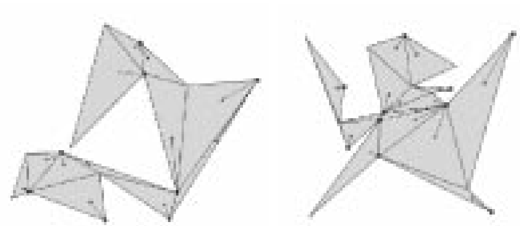

Figure 3: Polygons with large values of g/n (a) $n = 14$, $g = 8$, $g/n = 0.57$. (b) $n = 18$, $g = 10$, $g/n = 0.56$.

Lemma 1 *A t-fan of the fan decomposition of P can be illuminated with at most $\lceil (t+1)/2 \rceil$ π-lights, none of which are placed at the front vertices.*

Call a fan an odd fan if it has an odd number of triangles, and call it an even fan if it has an even number of triangles. Let $O = \{i | 1 \leq i \leq h, f_i \text{ is an odd fan}\}$, and let $E = \{i | 1 \leq i \leq h, f_i \text{ is an even fan}\}$. Let $h_o = |O|$, and $h_e = |E|$.

Theorem 1 *A simple n-gon can be illuminated with at most $\lfloor 5n/6 \rfloor$ π-lights.*

Proof: It follows from Lemma 1 that the total number of π-lights is $\leq \sum_{1 \leq i \leq h} \lceil (t_i + 1)/2 \rceil \leq \sum_{i \in O} (t_i + 1)/2 + \sum_{i \in E} (t_i + 2)/2 \leq (\sum_{1 \leq i \leq h} t_i + h_o + 2h_e)/2 = (n - 2 + h + h_e)/2 \leq (n - 2 + 2h)/2 = \lceil (n-2)/2 \rceil + h \leq \lfloor n/2 \rfloor + \lfloor n/3 \rfloor \leq \lfloor 5n/6 \rfloor$. ■

4 Ongoing Work

We are currently re-testing the modified code on randomly generated polygons. We hope that amortized analysis (using an appropriate charging scheme) could prove that our proposed algorithm places at most $\lfloor \frac{3}{5} n \rfloor$ π-lights. For outward facing lights, since $\lceil t/2 \rceil$ lights are in fact sufficient to illuminate a t-fan if $t \neq 2$, we hope to attain improved upper bound by lighting 2-fans with fewer than 2 lights per fan.

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